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Synchronization of hyperchaotic system

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In front of the jury

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ملخص

تهدف هذه المذكرة إلى دراسة وتحليل نظام فائق الفوضوية رباعي الأبعاد مشتق من نظام لورينتز بإضافة متغير جديد، مما أدى إلى ظهور سلوك فائق الفوضوية يتميز بوجود أكثر من أس موجب لياپونوف. تم التطرق إلى المفاهيم الأساسية للأنظمة الديناميكية والفوضوية، وأهم خصائص الفوضوية وفائقة الفوضوية، مع دراسة أبعادها وأسّيات لياپونوف ودورها في توصيف السلوك الديناميكي للنظام. كما تم إجراء تحليل ديناميكي للنظام المقترح من خلال دراسة نقاط التوازن، والتبدد، ومخططات التشعب، وإثبات وجود جاذب فائق الفوضوية. إضافة إلى ذلك، تمت دراسة مزامنة النظام باستعمال التحكم الفعال وتحليل الاستقرار وفق منهج لياپونوف، حيث أثبتت النتائج العددية قدرة المتحكم المقترح على تحقيق التزامن الكامل والتزامن الإسقاطي وضمان تقارب أخطاء المزامنة نحو الصفر، مما يؤكد فعالية الطريقة المقترحة في التحكم ومزامنة الأنظمة فائقة الفوضوية.

الكلمات المفتاحية: الأنظمة الديناميكية، الفوضوية، فائقة الفوضوية، نظام لورينتز المعدل، أسّيات لياپونوف، التشعب، التزامن الكامل، التزامن الإسقاطي، التحكم الفعال، استقرار لياپونوف.

Abstract

This dissertation focuses on the study and analysis of a four-dimensional hyperchaotic system derived from the classical Lorenz system through the introduction of an additional state variable. The proposed modification generates hyperchaotic behavior characterized by the existence of multiple positive Lyapunov exponents. The work reviews the fundamental concepts of dynamical systems, chaos, and hyperchaos, together with dimensions and Lyapunov exponents used to describe complex nonlinear dynamics. A comprehensive dynamical analysis of the proposed system is carried out through the investigation of equilibrium points, dissipativity, bifurcation diagrams, and the existence of a hyperchaotic attractor. Furthermore, synchronization of the system is achieved using active control and Lyapunov stability theory. Numerical simulations confirm the effectiveness of the proposed controller in achieving both complete synchronization and projective synchronization, while ensuring the asymptotic convergence of synchronization errors to zero. These results demonstrate the efficiency of the proposed approach for the control and synchronization of hyperchaotic systems.

Keywords: systèmes dynamiques, chaos, hyperchaos, système de Lorenz modifié, exposants de Lyapunov, bifurcation, synchronisation complète, synchronisation projective, commande active, stabilité de Lyapunov.

Résumé

Ce mémoire porte sur l'étude et l'analyse d'un système hyperchaotique à quatre dimensions dérivé du système de Lorenz par l'introduction d'une variable supplémentaire. Cette modification conduit à un comportement hyperchaotique caractérisé par la présence de plusieurs exposants de Lyapunov positifs. Le travail présente les notions fondamentales des systèmes dynamiques, du chaos et de l'hyperchaos, ainsi que l'étude des dimensions et des exposants de Lyapunov utilisés pour caractériser la dynamique du système. Une analyse dynamique approfondie du système proposé est réalisée à travers l'étude des points d'équilibre, de la dissipativité, des diagrammes de bifurcation et de l'existence d'un attracteur hyperchaotique. En outre, la synchronisation du système est étudiée à l'aide de la commande active et de l'analyse de stabilité de Lyapunov. Les simulations numériques montrent que la stratégie de contrôle proposée permet d'assurer la synchronisation complète ainsi que la synchronisation projective, avec une convergence asymptotique des erreurs vers zéro, démontrant ainsi l'efficacité de l'approche développée pour le contrôle et la synchronisation des systèmes hyperchaotiques.

Mots clés : systèmes dynamiques, chaos, hyperchaos, système de Lorenz modifié, exposants de Lyapunov, bifurcation, synchronisation complète, synchronisation projective, commande active, stabilité de Lyapunov.

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Rihab and Soundous

Dedication

*I dedicate this dissertation to my beloved mother and father,
whose unwavering support and encouragement accompanied me
throughout my academic journey.*

*I also dedicate it to my two sisters, my brother,
my nephews and nieces, and to all my friends who
contributed to the completion of this work,*

with special thanks to EL Moatazz Ferradji.

Rihab

Dedication

To the one whose prayers are the secret of my success, and whose fulfillment remains the ultimate aspiration of my life; to the one who sacrificed endlessly and stayed awake through the nights to witness this moment: My beloved mother, all my gratitude and devotion are yours.

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To my noble family, the Doula family, distinguished across generations for their devotion to literature and knowledge. Today, by the grace of God, I carry the banner of learning as a continuation of your honorable legacy, and I dedicate this achievement to you with immense pride.

This accomplishment is dedicated to you all.

Soundous

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Introduction

The roots of chaos theory can be traced back to the late 19th century through the work of the French mathematician Henri Poincaré, who was the first to observe irregular behavior in simple deterministic systems. However, chaos theory gained significant scientific recognition during the 1960s with the development of computational tools and the advent of computers. Among the most prominent scientists who contributed to the foundation of this theory, we mention Edward Lorenz, who discovered in 1963 that his simple atmospheric system, consisting of three ordinary differential equations, exhibits extreme sensitivity to initial conditions, a phenomenon later known as the “butterfly effect”. James Yorke, Mitchell Feigenbaum, and Robert May also contributed to the development of this field [5].

As for the phenomenon of hyperchaos, it was formally formulated by Rössler in 1979. It is characterized by the presence of at least two positive Lyapunov exponents, making the behavior more complex and difficult to predict compared to classical chaos. This phenomenon requires a phase space of at least four dimensions, and its applications are found in various fields such as secure communications, chemical reactions, nonlinear physics, and biology [4].

Concerning synchronization, this phenomenon has attracted considerable interest in the study of dynamical systems. The real breakthrough in the synchronization of chaotic systems came with the work of Pecora and Carroll in 1990, who demonstrated the possibility of synchronizing two identical chaotic systems with different initial conditions, provided they are properly coupled. Among other researchers who con-

tributed to this field, we mention Yamada and Fujisaka, Afraimovich, as well as Brown and Kocarev [6].

Synchronization is of paramount importance in several applied fields, most notably secure communications, where chaotic signals are used to mask transmitted information, making it difficult to intercept or detect. It is also used in cryptography to generate strong encryption keys, in neuroengineering to understand the synchronization of neurons, and in biological and economic systems [7].

In this context, the present thesis focuses on a novel four-dimensional hyperchaotic system characterized by an infinite number of equilibrium points[10].

The thesis is organized as follows:

- **Chapter One:** Recalls the basic concepts of deterministic dynamical systems, chaos theory, and the foundations of synchronization.
- **Chapter Two:** Provides a detailed synthesis of hyperchaotic systems, their properties, and their generation criteria.
- **Chapter Three:** Is devoted to the dynamic analysis of the proposed system (symmetry, dissipativity, equilibrium points, bifurcation diagram, Lyapunov exponents).
- **Chapter Four:** Is dedicated to the study of the synchronization of the proposed system using active control and projective synchronization, along with Lyapunov stability analysis.

Theoretical foundation of dynamical systems

1.1 Dynamical System

A **dynamical system** is a mathematical model that describes the time evolution of a set of interacting elements . It provides a rigorous framework for predicting the future (and sometimes past) states of a system based on its current condition [1]. Formally, a dynamical system is defined by a triplet (D, I, F) , consisting of :

- **State Space (D)**: The set of all possible states of the system.
- **Time Domain (I)**: The set over which time is measured (e.g., $\mathbb{R}^+$ for continuous time, $\mathbb{Z}$ for discrete time).
- **State Transition Function ($F : I \times D \rightarrow D$)**: A function that, given an initial state and a time increment, returns the state of the system at that later time [2].

Definition 1.1.1 *A dynamical system on $\mathbb{R}^n$ is a mapping:*

$$F : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$$

defined for all $(t, X) \in \mathbb{R}^+ \times \mathbb{R}^n$, such that :

1. For a fixed time, $F(t, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous.
2. For a fixed initial state, $F(\cdot, X) : \mathbb{R}^+ \rightarrow \mathbb{R}^n$ is continuous.
3. The initial condition is satisfied: $F(0, X) = X$ for all $X \in \mathbb{R}^n$.
4. The evolution is consistent over time (the semigroup property) :

$$F(t + s, X) = F(t, F(s, X)) \quad \text{for all } t, s \in \mathbb{R}^+, X \in \mathbb{R}^n.$$

The final property ensures that evolving the system for a total time $t + s$ is equivalent to evolving it first for time s and then for time t from the new state [3].

1.1.1 Continuous Dynamical Systems

In the general case, a **continuous-time dynamical system** can be represented by a system of ordinary differential equations [1] :

$$\frac{dx(t)}{dt} = f(x(t), t), \quad \text{where } x \in X \tag{1.1}$$

Here, the state x evolves smoothly and continuously over time t .

1.2 Phase Space

To analyze and describe dynamical systems, a specific vocabulary is used to characterize their structure and behavior [2].

Definition 1.2.1 (Phase Space) *The **phase space** (often denoted M) is a (potentially multi-dimensional) space where each point represents a unique state of the system. It allows for the geometrical interpretation of the motion described by the system's governing equations [3].*

Definition 1.2.2 (State Space) *The **state space** is the set of all coordinates (or variables) necessary for a complete description of the system at any given instant. In many contexts, the terms “state space” and “phase space” are used interchangeably [1].*

Definition 1.2.3 (Flow) Let M be a set (the phase space) and G an additive group, typically $\mathbb{R}$ for continuous time or $\mathbb{Z}$ for discrete time. A **flow** is a family of one-parameter applications $\{\varphi_t\}_{t \in G}$ mapping M onto itself, indexed by the time group G . The pair $(M, \{\varphi_t\}_{t \in G})$ constitutes the flow. For a given $x \in M$, $\varphi_t(x)$ represents the state of the system after evolving for a time t [2].

Definition 1.2.4 (Orbit and Trajectory) The **trajectory** of a point $x \in M$ is the application defined on G and taking values in M :

$$\begin{aligned} \varphi(x) : G &\rightarrow M \\ t &\mapsto \varphi_t(x) \end{aligned}$$

The image of this application, i.e., the subset of the phase space visited by the system starting from x , is called the **orbit** of x . It is formally defined as [3] :

$$O(x) = \{\varphi_t(x), t \in G\} \subseteq M.$$

In the context of a differential equation $\dot{x} = f(x)$, the orbit through a point x_0 is the set of all points that lie on the solution curve passing through x_0 :

$$O(x_0) = \{x(t), -\infty < t < +\infty\}.$$

Definition 1.2.5 (Phase Portrait) Each initial condition gives rise to a distinct orbit. The **phase portrait** of a dynamical system is the geometric representation of all its orbits in the phase space. It provides a complete picture of the system's possible behaviors [1].

The qualitative analysis of dynamical systems focuses on identifying specific types of orbits and invariant sets that structure the phase portrait [3].

Definition 1.2.6 (Periodic Point and Limit Cycle) 1. A point x is said to be **T -periodic** if its orbit satisfies $x(t + T) = x(t)$ for all $t \in \mathbb{R}$. The smallest positive such T is called the **period** of the solution. The orbit of a periodic point is a closed curve called a **periodic orbit** [3].

2. Consider a two-dimensional system:

$$\begin{cases} \dot{x} = P(x, y), \\ \dot{y} = Q(x, y), \end{cases}$$

where P and Q are polynomials. A **limit cycle** C is a closed trajectory (periodic orbit) that is isolated, meaning there are no other closed trajectories in its immediate neighborhood. Neighboring trajectories spiral either towards or away from it, making it a key feature in understanding oscillatory behavior [3].

Remark 1.2.1 A periodic orbit $O(x)$ is always composed of periodic points. All these points are called **periodic points** of period T [2].

Definition 1.2.7 (Invariant Set) Let A be a subset of the phase space M . A is said to be **invariant** under the flow φ_t if for all $t \in \mathbb{R}$, $\varphi_t(A) \subseteq A$ [1]. If this property holds only for all $t \geq 0$, then A is **positively invariant** [2].

Remark 1.2.2 The trajectory of an autonomous system in state space is always an invariant set. This means that if a solution starts in a particular set (like an orbit), it remains within that set for all time [1].

1.3 Equilibrium Points of a Dynamical System

A fundamental concept in the study of dynamical systems is that of an **equilibrium point** (also called a fixed point, stationary point, or steady state). If it is possible that the trajectory of a system corresponds, from a certain moment onward, to a single point, such a point is called an equilibrium point. As we shall see, stability problems are naturally formulated with respect to these equilibrium points.

Consider the continuous dynamical system [1]:

$$\begin{cases} \dot{X}(t) = F(X(t), t), \\ X(0) = X_0 \in \mathbb{R}^n, \end{cases} \quad (1.2)$$

where $F : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ is a function such that (1.2) admits at least one solution .

Definition 1.3.1 (Equilibrium Point) *The vector $\bar{X} \in \mathbb{R}^n$ is called an **equilibrium point** (or **fixed point**) of system (1.2) if:*

$$F(\bar{X}, t) = 0 \quad \text{for all } t \geq 0. \quad (1.3)$$

In other words, if the system starts exactly at an equilibrium point $\bar{X}$, it remains there for all future time . This is why equilibrium points are also called stationary solutions: $X(t) = \bar{X}$ is a constant solution of the differential equation [3].

1.4 Stability of Linear Systems

We now turn our attention to the study of stability for continuous linear systems governed by the equation [1]:

$$\begin{cases} X'(t) = AX(t), \\ X(0) = X_0 \in \mathbb{R}^n, \end{cases} \quad (1.4)$$

where A is an $n \times n$ square matrix .

Definition 1.4.1 *The continuous system (1.4) is said to be **stable** if the origin is stable . It is said to be **asymptotically stable** if the origin is asymptotically stable. In this case, the system matrix A is itself called **stable** or **asymptotically stable** [2].*

The stability of continuous linear systems is characterized by the following fundamental result, which relates the stability properties to the eigenvalues of the matrix A .

Proposition 1.4.1 (Stability Criterion for Linear Systems) *The system (1.4) is stable if and only if the following two conditions hold:*

1. $\Re(\lambda) \leq 0$ for every eigenvalue λ of A [2].
2. If there exists an eigenvalue λ of multiplicity k such that $\Re(\lambda) = 0$, then $\dim E_\lambda = k$, where E_λ is the eigenspace associated with λ .

1.5 Stability in the Sense of Lyapunov

We consider the class of nonlinear systems described by the following dynamical equation [3]:

$$\begin{cases} \dot{x}(t) = f(x(t), t), \\ x(t_0) = x_0, \end{cases} \quad (1.5)$$

where $x(t) \in \mathbb{R}^n$ and $f : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ is continuous .

Definition 1.5.1 (Lyapunov Stability) *An equilibrium point x^* of (1.5) is said to be **stable** in the sense of Lyapunov if:*

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \|x_0 - x^*\| < \delta \implies \|x(t) - x^*\| < \varepsilon, \text{ for all } t > 0. \quad (1.6)$$

Otherwise, x^* is said to be **unstable** [2].

Lyapunov Function Method

In addition to the definition of stability, we introduce the concept of a Lyapunov function, which is a powerful tool for proving stability without explicitly solving the differential equations. This method will be employed in Chapter 4 to analyze the convergence of synchronization errors.

Definition 1.5.2 (Lyapunov Function) *Let $\dot{x} = f(x)$ be an autonomous dynamical system with an equilibrium point at the origin, i.e., $f(0) = 0$. A continuously differentiable scalar function*

$$V : \mathbb{R}^n \rightarrow \mathbb{R}$$

*is called a **Lyapunov function** if there exists a neighbourhood $\Omega \subset \mathbb{R}^n$ of the origin such that the following conditions hold :*

1. **Positive definiteness :**

$$V(0) = 0 \quad \text{and} \quad V(x) > 0 \quad \text{for all } x \in \Omega \setminus \{0\}.$$

2. Negative semi-definite derivative :

$$\dot{V}(x) = \nabla V(x) \cdot f(x) \leq 0 \quad \text{for all } x \in \Omega.$$

If the derivative is strictly negative for all $x \neq 0$, i.e., $\dot{V}(x) < 0$, then V is called a **strict Lyapunov function**, and the origin is said to be **asymptotically stable** [5]. The existence of a Lyapunov function provides a sufficient condition for stability; it guarantees that the energy-like function V decreases along the trajectories of the system, forcing the state to converge to the equilibrium point.

This definition is essential for the active control and synchronization schemes presented later, where the error dynamics are shown to be asymptotically stable by constructing a suitable Lyapunov function of the form $V(e) = \frac{1}{2}e^T e$.

1.6 Properties of Chaos

Deterministic chaos refers to complex dynamic behavior appearing in deterministic nonlinear systems. Despite the absence of randomness in the equations, solutions may become unpredictable in the long term. Fundamental features of chaos include sensitivity to initial conditions, the existence of a strange attractor, and aperiodicity[5].

1.6.1 Sensitivity to Initial Conditions

Sensitivity to initial conditions means that two trajectories starting from very close points diverge exponentially over time. Mathematically, if $\delta(0)$ is a small initial perturbation, then

$$\|\delta(t)\| \approx \|\delta(0)\|e^{\lambda t},$$

where λ is a positive Lyapunov exponent. This exponential growth implies rapid amplification of initial errors[4].

In the study of the Lorenz system, it has been shown that extremely small differences in initial conditions can lead to completely different trajectories after some time. This

property explains the difficulty of long-term weather prediction[5].

Sensitivity to initial conditions does not imply that trajectories diverge to infinity. On the contrary, they remain confined within a bounded region of phase space, but their detailed evolution becomes unpredictable [4].

1.6.2 Strange Attractors

An attractor is an invariant set toward which trajectories converge for a non-negligible set of initial conditions . we consider two types of attractors regular ones as the fixed point the cycle limit and quasi peroidic cycle limit those are in the case that there isn't any choas,In the case of chaos, the attractor has a complex geometric structure called a *strange attractor*.

Strange attractors characterize the evolution of chaotic systems. After a certain time, all the points of the phase space that belong to the basin of attraction generate trajectories that tend to form a strange attractor [6].

A strange attractor is characterized by the following properties:

1. **Sensitivity to initial conditions:** two trajectories that are initially very close diverge exponentially with time, reflecting the chaotic behavior of the system .
2. **Fractal dimension:** the dimension of the attractor is fractal and non-integer, which justifies the term "strange" .
3. **Zero volume in phase space:** the attractor occupies a set of zero volume in the phase space .

Strange attractors generally have a fractal structure, characterized by a non-integer dimension . This property reflects the self-similar organization of trajectories at different scales.

In the Lorenz system, the attractor has a butterfly shape, composed of two lobes around which trajectories oscillate irregularly. Trajectories never exactly repeat but remain confined within this invariant set.

The presence of a strange attractor therefore implies behavior that is both bounded and non-periodic, combined with a complex geometry of the limiting set [4].

1.6.3 Dissipativity

A dynamical system is said to be dissipative if it contracts volumes in phase space as time evolves. Equivalently, the divergence of its vector field is negative on average, causing trajectories to converge toward a bounded attracting set (attractor) rather than spreading indefinitely[5].

Hyperchaotic Systems

2.1 Chaos

With the development of science during the twentieth century, scientists became convinced that many natural systems exhibit not only periodic or quasi-periodic motion, but also chaotic and unpredictable behavior over long periods of time. This type of behavior attracted considerable attention because of its complex nature.

The physicist Albert Einstein believed in the existence of complete order in the universe. However, modern studies have shown that deterministic systems themselves may display unpredictable behavior. Chaotic systems often appear random, yet they are governed by precise deterministic laws. This behavior arises from a fundamental property known as *sensitive dependence on initial conditions*, where a very small change in the initial state may lead to significant differences in the long-term evolution of the system.

With the advancement of computational capabilities, it became clear that even simple systems may generate highly complex dynamics due to their internal structure rather than external influences. The mathematician *James Alan Yorke* introduced the term “chaos” to describe this irregular behavior in deterministic systems.

Although chaos has been extensively studied, no universally accepted definition exists. Nevertheless, chaos is generally associated with two fundamental properties:

- sensitive dependence on initial conditions,

- topological mixing.

Chaotic trajectories are also generally non-periodic.

In 1971, *David Ruelle* and *Floris Takens* described certain chaotic attractors as “strange attractors” because of their unusual geometrical structures. These attractors are closely related to fractals, which are geometric objects characterized by self-similarity across multiple scales.

Nonlinearity and high system dimension are considered essential factors for the emergence of chaos in continuous systems, whereas discrete systems may exhibit chaotic behavior even in one-dimensional cases under certain conditions.

The foundations of chaos theory can be traced back to the pioneering work of the French mathematician *Henri Poincaré*, whose studies transformed the classical deterministic viewpoint. Several mathematicians later contributed significantly to the mathematical development of chaos theory, including *George David Birkhoff*, *Aleksandr Lyapunov*, *Andrey Kolmogorov*, and *Stephen Smale*.

In 1963, *Edward Lorenz* provided the first clear description of chaos in atmospheric dynamics. He observed that the solutions of his dynamical system did not converge to fixed points or periodic orbits, but instead evolved in an irregular manner with extreme sensitivity to initial conditions. When represented in phase space, these trajectories formed a geometric structure now known as the *Lorenz attractor*, associated with the so-called *butterfly effect*.

Several mathematical definitions of chaos have been proposed, among which the most important are the following[6] :

Chaos in the Sense of Devaney

A dynamical system is said to be chaotic if :

- it is topologically transitive,
- it possesses a dense set of periodic orbits,
- it exhibits sensitive dependence on initial conditions[6].

Chaos in the Sense of Li–Yorke

Tien-Yien Li and James Alan Yorke introduced the first mathematical definition of chaos and established the famous criterion :

“Period three implies chaos.”

This criterion is considered one of the fundamental results in the study of chaotic dynamical systems[6].

2.2 Hyperchaotic Systems

2.2.1 Definition and Concept of Hyperchaotic Systems

Hyperchaotic systems are considered an advanced class of chaotic systems, characterized by the presence of more than one positive Lyapunov exponent. This mathematical indicator reflects the system’s ability to expand dynamically in several independent directions simultaneously, making its behavior more complex and intricate than traditional chaotic systems [4].

Practically, the presence of multiple positive Lyapunov exponents means that the system exhibits instability in more than one direction, generating more random and unpredictable motion. Prominent examples of these systems include the hyperchaotic Lorenz version, the hyperchaotic Chen system, the hyperchaotic Li system, in addition to Vaidyanathan systems .

2.2.2 Applied Importance in Secure Communications

Recent years have witnessed increasing interest in employing chaotic systems in data encryption and communication protection. The idea of secure chaotic communication relies on using chaotic signals as a cover to hide transmitted information [4].

A study by Perez and Cerdeira revealed the inadequacy of traditional chaotic systems (with a single positive Lyapunov exponent) in providing sufficient information protection, as hidden messages could be detected. In contrast, Pecora demonstrated

that using higher-dimensional systems (with multiple positive exponents) overcomes this challenge and enhances communication security.

2.2.3 Conditions for Generating Hyperchaotic Behavior

Not every chaotic system can transform into a hyperchaotic system except by meeting two fundamental conditions[6]:

- **Dimensional condition:** The system dimension must not be less than 4, with the necessity that the order of each equation in the system is greater than or equal to 2.
- **Dynamic condition:** At least two positive Lyapunov exponents are required, while ensuring that the total sum of all exponents is negative to guarantee overall system stability.

2.3 Hyperchaos Phenomenon

2.3.1 Concept and Origin

Hyperchaos represents an extreme case of chaotic behavior characterized by a high degree of unpredictability. This concept was first formulated by Rössler in 1979 , and it refers to the presence of a minimum of two positive Lyapunov exponents in the system, leading to multiple directions of trajectory divergence.

The continuous nonlinear system requires at least a four-dimensional phase space for this phenomenon to appear, while three dimensions suffice in the case of discrete maps. Applications of hyperchaotic systems extend to multiple fields such as secure communications, chemical reactions, nonlinear physics, and biology[4].

2.3.2 Rössler Model

This phenomenon can be illustrated through the Rössler model of four-variable oscillators, given by[4]:

$$\begin{cases} \dot{x} = -(y + z) \\ \dot{y} = x + 0.25y + w \\ \dot{z} = xz + 3 \\ \dot{w} = -0.5z + 0.049w \end{cases} \quad (2.1)$$

This system is characterized by a single nonlinear term in the third equation (for the variable z). Analysis of the four Lyapunov exponents showed the following results:

$$\lambda_1 = 0.112 > 0, \quad \lambda_2 = 0.02 > 0, \quad \lambda_3 = 0, \quad \lambda_4 = -22.303 < 0 \quad (2.2)$$

The Lyapunov dimension of the hyperchaotic attractor can be calculated as:

$$D_L = 3 + \frac{1}{|\lambda_4|} \sum_{i=1}^3 \lambda_i \approx 3.0059 \quad (2.3)$$

2.3.3 Characteristics and Implications

Figure 2.1 shows the shape of the hyperchaotic attractor when applying the initial conditions $(-10, -6, 0, 10)$, in addition to the corresponding Poincaré section. This section is characterized by irregular point clusters reflecting high randomness and noise, which is the distinctive feature of hyperchaos .

The study of this phenomenon gains particular importance in the context of understanding complex turbulent flows, which represent infinite-dimensional dynamic processes not yet fully understood[4].

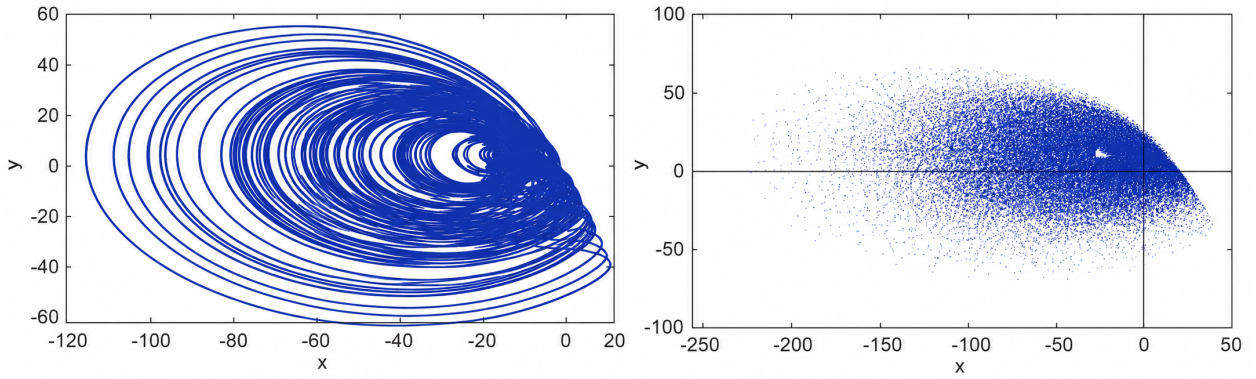


Figure 2.1: Phase portrait and Poincaré section of hyperchaotic attractor

2.3.4 Fractal Dimensions

The beginning of the last century witnessed the emergence of several definitions of fractal dimension . Some of these definitions are purely theoretical in nature, and others are more suitable for numerical and computational applications .

We will briefly review here the concept of the Hausdorff dimension, which is the most common in theoretical mathematics, and then move on to other definitions that are numerically computable.

The concept of traditional dimension in mathematics, known as topological dimension, accepts only integer values . Isolated points or finite sets are considered to have dimension zero, curves (similar to the real number line) have dimension one, surfaces (similar to the Euclidean plane) have dimension two [6] .

Classification of Fractal Dimensions

Definitions of fractal dimension can be divided into two main categories :

1. **Definitions that depend only on the set :** They are concerned with describing the set regardless of its dynamics, whether it is an attractor or not. The most prominent among these are the Hausdorff dimension and the capacity or Kolmogorov dimension.
2. **Definitions that take dynamics into account :** These definitions are related to

the behavior of the dynamical system that generates the set. The most important among these are the information dimension and the correlation dimension .

All these dimensions converge in their numerical values for most strange attractors and satisfy the following general properties [6]:

$$A \subset B \Rightarrow d(A) < d(B),$$

$$A = \emptyset \Rightarrow d(A) = 0,$$

$$d(A \times B) = d(A) + d(B),$$

$$f \text{ differentiable on } A \Rightarrow d(f(A)) = d(A).$$

Kaplan–Yorke Dimension (Lyapunov Dimension)

This dimension, proposed by Kaplan and Yorke, depends on the values of the system's Lyapunov exponents . If we arrange these exponents in descending order

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n,$$

we look for the largest integer k that satisfies the following conditions :

- The sum of the first k exponents is positive or equal to zero:

$$\sum_{i=1}^k \lambda_i \geq 0.$$

- The sum of the first $k + 1$ exponents is negative:

$$\sum_{i=1}^{k+1} \lambda_i < 0.$$

The Lyapunov dimension, denoted D_L , is calculated using the following formula :

$$D_L = k + \frac{\sum_{i=1}^k \lambda_i}{|\lambda_{k+1}|}. \quad (2.4)$$

In special cases, if the system tends towards a stable equilibrium point, the dimension equals zero. If it tends towards a stable limit cycle, the dimension equals one. In the case of a strange chaotic attractor, the dimension value is fractional (non-integer)[6].

Hausdorff Dimension

This dimension is considered the cornerstone of the mathematical theory of fractals . It is based on the idea of measuring a set using covers of very small balls or objects.

Suppose we have a set M in a Euclidean space $\mathbb{R}^n$, and we cover it with subsets A_i whose diameters do not exceed a value ε . The d -dimensional Hausdorff measure is defined as

$$\mu_d(M) = \liminf_{\varepsilon \rightarrow 0^+} \left\{ \sum_{i=1}^{\infty} [\text{diam}(A_i)]^d \right\}. \quad (2.5)$$

From this definition, we deduce that there exists a critical value d_{Haus} where the behavior of the measure changes . If $d > d_{\text{Haus}}$, then $\mu_d(M) = 0$; if $d < d_{\text{Haus}}$, then $\mu_d(M) = \infty$. This critical value is precisely the Hausdorff dimension of the set M [6].

Capacity or Kolmogorov Dimension

The Kolmogorov dimension is one of the simplest practical definitions of fractal dimension . To calculate it, the fractal set S is covered with a grid of squares (or cubes) of side length ε . Let $N(\varepsilon)$ be the number of such squares needed to cover the set.

The capacity dimension is then defined by

$$d_C = \lim_{\varepsilon \rightarrow 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}. \quad (2.6)$$

Despite theoretical differences, the Hausdorff and Kolmogorov dimensions are equal in many cases, especially for self-similar fractals [6]. In general the following inequality holds :

$$d_{\text{Haus}} \leq d_C. \quad (2.7)$$

2.4 Lyapunov Exponent and Chaos

The **Lyapunov exponent** is used to measure whether a system (or a function) exhibits chaotic behavior [4, 7]. It is based on the fact that the rate of convergence or divergence between nearby trajectories around a fixed point (or any other point) occurs exponentially fast.

Thus, it provides an averaged measure of the exponential rate at which near by orbits separate.

Intuitive Example

Consider a dynamical system with two points initially very close together. We study how the distance between these points evolves over time [4].

1. **Non-chaotic system:** Trajectories starting from nearby points remain close or diverge very slowly.
2. **Chaotic system:** Trajectories starting from nearby points separate rapidly, exponentially over time.

The presence of a positive average exponential rate indicates chaos. This average rate is called the **Lyapunov exponent** and quantifies the system's sensitivity to initial conditions.

2.4.1 Lyapunov Exponent for One-Dimensional Maps

Consider a one-dimensional map :

$$x_{n+1} = f(x_n), \quad x_0, x_0 + \varepsilon \text{ (neighboring points)}, \quad \varepsilon \ll 1.$$

After N iterations :

$$x_0 \rightarrow f^N(x_0), \quad x_0 + \varepsilon \rightarrow f^N(x_0 + \varepsilon).$$

The Lyapunov exponent is defined as :

$$\lambda = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \left| \frac{f^N(x_0 + \varepsilon) - f^N(x_0)}{\varepsilon} \right|.$$

Using the chain rule :

$$(f^N)'(x_0) = \prod_{i=0}^{N-1} f'(x_i), \quad x_i = f^i(x_0),$$

so that the Lyapunov exponent can be written as :

$$\lambda = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^{N-1} \ln |f'(x_i)|. \quad (2.8)$$

A system is :

- **Chaotic** if $\lambda > 0$,
- **Non-chaotic (regular)** if $\lambda \leq 0$.

Example : Logistic Map

The logistic map is defined as[4] :

$$f_r(x) = rx(1 - x), \quad 0 \leq r \leq 4, \quad x \in [0, 1].$$

Its derivative is :

$$f'_r(x) = r(1 - 2x),$$

yielding the Lyapunov exponent :

$$\lambda = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^{N-1} \ln |r(1 - 2x_i)|.$$

For $r = 4$, the Lyapunov exponent is $\lambda = \ln 2$, because the tent map is conjugate to $f_4(x)$ via :

$$h(x) = \sin^2\left(\frac{\pi x}{2}\right), \quad x \in [0, 1].$$

2.4.2 Lyapunov Exponents for Continuous Systems

For an n -dimensional system[4]:

$$\dot{\tilde{x}} = f(\tilde{x}), \quad \tilde{x} \in \mathbb{R}^n, \quad \tilde{x}(0) = \tilde{x}_0, \quad (2.9)$$

let $\phi_t(\tilde{x}_0)$ denote the flow. For two nearby points $\tilde{x}_0$ and $\tilde{x}_0 + \delta\tilde{x}_0$:

$$\delta\tilde{x}(t) = \phi_t(\tilde{x}_0 + \delta\tilde{x}_0) - \phi_t(\tilde{x}_0) \approx D\phi_t(\tilde{x}_0) \delta\tilde{x}_0,$$

where $D\phi_t(\tilde{x}_0)$ is the Jacobian of f .

The evolution of perturbations follows :

$$\frac{d\tilde{y}}{dt} = Df(\tilde{x}) \tilde{y}. \quad (2.10)$$

To compute all Lyapunov exponents, integrate the system with orthonormalization (Gram–Schmidt process) to avoid divergence :

$$u_1 = \frac{y_1}{\|y_1\|},$$

$$u_j = \frac{y_j - \sum_{l=1}^{j-1} \langle y_j, u_l \rangle u_l}{\|y_j - \sum_{l=1}^{j-1} \langle y_j, u_l \rangle u_l\|}, \quad j = 2, \dots, n.$$

The Lyapunov exponent is :

$$\lambda_j = \lim_{t \rightarrow \infty} \frac{1}{t} \sum_i \ln \|u_i\|, \quad j = 1, \dots, n.$$

Typical Dynamical Behavior from Lyapunov Spectrum

For a four-dimensional system [4] :

1. **Steady-state or equilibrium point** : if $\lambda_i < 0$ for each $i = 1, 2, 3, 4$.
2. **Periodic motion**: if $\lambda_1 = 0$ and $\lambda_i < 0$ for $i = 2, 3, 4$.
3. **Quasiperiodic motion**:
 - (a) $\lambda_1 = \lambda_2 = 0$ and $\lambda_3, \lambda_4 < 0$ for a 2-torus,
 - (b) $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and $\lambda_4 < 0$ for a 3-torus.
4. **Chaos** : if $\lambda_1 > 0$, $\lambda_2 = 0$, and $\lambda_3, \lambda_4 < 0$.
5. **Hyperchaos** : if $\lambda_1, \lambda_2 > 0$, $\lambda_3 = 0$, and $\lambda_4 < 0$.

Synchronization Of Systems

3.1 Introduction

The synchronization of chaotic systems emerged as a natural extension of developments in nonlinear control theory. Early work on chaos stabilization showed that it is possible not only to stabilize a chaotic system onto a periodic orbit but also to make it track a periodic trajectory. From this perspective, such control schemes can be interpreted as a form of synchronization between the chaotic system and an external periodic signal.

When the periodic signal is replaced by a chaotic one, we enter the domain of chaotic synchronization. Given the well-known sensitivity to initial conditions a hallmark of chaotic behavior this type of synchronization might seem difficult to achieve at first glance. Unlike classical synchronization, which aims to reproduce a single oscillation period, chaotic synchronization imposes more stringent constraints, as it requires a deeper correspondence between the dynamical behaviors of the two systems.

In recent years, radiocommunications have become a rapidly growing field, and many researchers have been exploring the possibility of using chaotic signals for data transmission. Synchronization plays a fundamental role in this context, as it is key to ensuring successful communication between transmitter and receiver.

In conventional telecommunication systems, synchronization is primarily concerned with reproducing the periodic carrier signal. In contrast, chaotic synchronization at the receiver aims to reconstruct the chaotic signal generated by the transmitter. Two chaotic

systems are said to be synchronized if their states become asymptotically identical over time that is, if the difference between them tends to zero as t approaches infinity.

Several concepts of chaotic synchronization have been proposed in the literature. Early contributions came from Yamada and Fujisaka, who adopted a local approach. Later, important developments were made by Afraimovich et al., and subsequently Pecora and Carroll introduced the concept of identical synchronization using coupled chaotic circuits in a master-slave configuration. Their work opened new avenues for applying chaos in telecommunications.

Over the past few years, synchronization of chaotic systems has attracted considerable attention due to its wide-ranging applications in physics and engineering. Various methods have been successfully applied to synchronize chaotic and hyperchaotic systems. These include active control, which is applicable when system parameters are known; adaptive control, used when parameters are unknown; as well as feedback control, backstepping, sliding mode control, and others[6].

3.2 Definition of Synchronization

3.2.1 General Definition

Definition 4.1 The word “synchronization” is derived from the Greek words: *syn* meaning “together,” and *chronos* meaning “time.” The term refers to the process of coordinating operations so that they occur simultaneously [6].

Definition 4.2 Synchronization can be defined as the way in which a periodic or chaotic motion is maintained. To say that two dynamical systems are synchronized means that the evolution of each follows the behavior of the other [6].

3.2.2 Mathematical Definition

After numerous attempts to formulate a precise mathematical framework for the concept of synchronized motion, Brown and Kocarev proposed a mathematical definition

of synchronization that has become a reference in this field [6]. In formulating this definition, the two researchers considered two subsystems :

$$\begin{cases} \dot{X} = F(X(t)), \\ \dot{Y} = G(Y(t)), \end{cases} \quad (3.1)$$

where $X(t) \in \mathbb{R}^n$ and $Y(t) \in \mathbb{R}^m$ are state vectors, which may have different dimensions (i.e., $n \neq m$).

Definition 4.3 The two subsystems in equations (3.1) are said to be synchronized along the trajectory $\varphi(\omega_0)$ with respect to the properties g_x and g_y if there exists a continuous mapping h , independent of time, such that:

$$\|h(g_x(X(t)), g_y(Y(t)))\| = 0 \quad \text{as } t \rightarrow \infty. \quad (3.2)$$

This means that, as time passes, the difference between the states of the two systems gradually approaches zero. By appropriately choosing the functions g_x , g_y , and h , different types of synchronization between the two systems can be distinguished, such as identical synchronization, generalized synchronization, and phase synchronization.

Theorem 4.1 For synchronization to be achieved between the master system (transmitter) and the slave system (receiver), all conditional Lyapunov exponents of the slave system must be negative. These exponents reflect the rate at which the trajectory of the slave system converges to that of the master system. If the largest conditional exponent is positive, the two systems remain unsynchronized; if it is negative, synchronization is gradually achieved over time.

3.2.3 Complete Synchronization

Complete synchronization is one of the most important forms of synchronization in nonlinear dynamical systems, and it is considered the foundation, as it refers to the case where the state variables of two coupled dynamical systems become identical over time, such that their behavior becomes completely identical after a transient period,

despite the difference in initial conditions [6].

As mentioned, this type of synchronization is the most common due to its simplicity, especially in the case of unidirectional coupling, where the master system affects the slave system without being influenced.

Mathematically, complete synchronization between two systems can be expressed as:

$$\lim_{t \rightarrow \infty} \|X(t) - Y(t)\| = 0, \quad (3.3)$$

which means that the difference between the state vectors of the master system $X(t)$ and the slave system $Y(t)$ tends to zero as time approaches infinity.

3.3 Mathematical Formulation of Complete Synchronization

Let us consider two nonlinear dynamical systems represented by the equations [6]:

$$\dot{X}(t) = F(X(t)), \quad (3.4)$$

$$\dot{Y}(t) = G(Y(t)) + U(X, Y, t), \quad (3.5)$$

where:

- $X(t) \in \mathbb{R}^n$ is the state vector of the master system,
- $Y(t) \in \mathbb{R}^n$ is the state vector of the slave system (dimensions unified),
- F and G are nonlinear vector fields,
- $U(X, Y, t)$ is the control or coupling input.

3.3.1 Definition of the Synchronization Error

The synchronization error is defined as:

$$e(t) = Y(t) - X(t). \quad (3.6)$$

Complete synchronization is achieved if:

$$\lim_{t \rightarrow +\infty} \|e(t)\| = 0, \quad (3.7)$$

where $\|\cdot\|$ denotes the Euclidean norm, meaning that the difference between the two systems vanishes over time.

3.4 Types of Synchronization

3.4.1 Identical Complete Synchronization (ICS)

This type of synchronization occurs when the two dynamical systems are **completely identical** in terms of their mathematical structure and parameters, i.e., they follow the same nonlinear function F with the same numerical parameter values [6].

In this case, the synchronization condition becomes:

$$\lim_{t \rightarrow +\infty} \|Y(t) - X(t)\| = 0. \quad (3.8)$$

Scientific Properties:

- Considered the simplest and most analytically clear form of synchronization.
- Often requires only a direct coupling.
- Features fast convergence, typically exponential.
- Relies on Lyapunov stability analysis to ensure convergence.

Physical Interpretation: After a short transient period, the trajectories of the two systems in phase space become **completely identical**, meaning the slave system reproduces the behavior of the master system exactly.

3.4.2 Non-Identical Complete Synchronization (NICS)

This type occurs when the two systems are **not identical**, i.e., there are differences in parameters or structure, reflecting real physical or engineering systems [6].

The synchronization error is defined as:

$$e(t) = Y(t) - X(t). \quad (3.9)$$

Complete synchronization is achieved if:

$$\lim_{t \rightarrow +\infty} \|e(t)\| = 0. \quad (3.10)$$

Scientific Properties:

- More complex due to mismatch between systems.
- Requires advanced control strategies such as:
 - Adaptive Control
 - Nonlinear Control
- Heavily relies on designing appropriate Lyapunov functions to guarantee stability.
- Used in practical applications such as secure communications and biological system synchronization.

Physical Interpretation: The slave system is “forced” to follow the master system’s behavior despite differences, making this type more realistic in practical applications.

Stability Analysis in Complete Synchronization

In the context of complete synchronization between coupled nonlinear systems, stability analysis refers to the ability of the synchronized state to persist under small perturbations. That is, once two systems achieve a synchronized trajectory, any slight

deviation from this trajectory must decay over time, returning the system to the synchronized manifold. This condition ensures that synchronization is robust and not fragile in the presence of disturbances. In chaotic systems, achieving a stable synchronized state requires sufficiently strong coupling; weak coupling may be unable to suppress divergence originating from sensitive dependence on initial conditions. The regions in parameter space where synchronization remains stable are referred to as *synchronization regions*, and their boundaries depend on the coupling strength and the characteristics of the individual systems [8].

3.4.3 Projective Synchronization (PS)

Projective synchronization (PS) is a particular form of chaos synchronization in which the state variables of the slave (response) system asymptotically evolve as constant scalar multiples of the corresponding state variables of the master (drive) system. In other words, instead of requiring the trajectories of the two systems to become identical, projective synchronization only requires the response system to follow the drive system proportionally through a constant scaling factor [9].

Let

$$X(t) = (x_1(t), x_2(t), \dots, x_n(t))^T,$$

be the state vector of the master chaotic system, and

$$Y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T,$$

be the state vector of the slave chaotic system. Then, projective synchronization is said to be achieved if there exist nonzero constant scaling factors $\alpha_i, i = 1, 2, \dots, n$, such that

$$\lim_{t \rightarrow \infty} \|y_i(t) - \alpha_i x_i(t)\| = 0, \quad i = 1, 2, \dots, n.$$

This means that, as time tends to infinity, each state variable of the slave system becomes proportional to the corresponding state variable of the master system. Hence, the synchronization error between the slave states and the scaled master states tends

asymptotically to zero .

Projective synchronization can be viewed as a more general synchronization scheme than complete synchronization. Indeed, when all scaling factors satisfy $\alpha_i = 1$, PS reduces to **complete synchronization**, whereas when all $\alpha_i = -1$, it corresponds to **complete anti-synchronization** [9].

3.4.4 Anti-Synchronization of Hyperchaotic Systems

Anti-synchronization in hyperchaotic systems refers to a dynamical behavior where two coupled systems, namely a master system and a slave system, evolve such that their state vectors have the same amplitudes but opposite signs. In other words, if $x(t)$ and $y(t)$ denote the state vectors of the master and slave systems, respectively, anti-synchronization is achieved when $y(t) = -x(t)$ as time tends to infinity . Hyperchaotic systems are characterized by having more than one positive Lyapunov exponent, which leads to highly complex and sensitive dynamics. The anti-synchronization error is defined as:

$$e(t) = x(t) + y(t), \quad (3.11)$$

and the systems are said to be anti-synchronized if:

$$\lim_{t \rightarrow \infty} \|e(t)\| = 0. \quad (3.12)$$

Due to the complexity of hyperchaotic systems, achieving anti-synchronization requires appropriate control strategies to ensure stability and convergence. This concept has important applications in secure communication and nonlinear system analysis [10].

3.4.5 Generalized Synchronization

In the early stages, the study of synchronization in chaotic systems was limited to the case where the two systems (the drive and the response) are completely identical. This is known as identical synchronization. In this case, the following condition holds [6]:

$$\lim_{t \rightarrow \infty} \|x(t) - y(t)\| = 0. \quad (3.13)$$

However, this assumption is largely idealized, since real-world systems often differ in structure, parameters, or even dimensions. This leads to a fundamental issue in synchronization theory, namely the possibility of achieving a form of dynamical agreement between non-identical systems. Extending the concept of synchronization to include such realistic situations leads to what is known as **generalized synchronization**.

Theoretical Foundations

To gain a deeper understanding of this type of synchronization, several theoretical and geometrical concepts are employed to describe the behavior of systems in phase space. Among these concepts are consistent systems theory and the notion of the *synchronization manifold*.

In the framework of generalized synchronization, the objective is no longer the exact matching of trajectories, but rather their convergence toward a specific manifold in the joint phase space, typically defined by a functional relationship of the form:

$$y = \phi(x). \quad (3.14)$$

From a topological perspective, this reflects the existence of a functional mapping between the phase spaces of the two systems, even when their dimensions differ. This property provides generalized synchronization with a high degree of flexibility in handling complex and multiscale systems[6].

Mathematical Representation

Consider two systems coupled in a unidirectional manner (from the drive to the response):

Drive System:

$$\dot{X} = F(X), \quad X \in \mathbb{R}^n$$

Response System:

$$\dot{Y} = G(Y) + H(X, Y), \quad Y \in \mathbb{R}^m$$

where $H(X, Y)$ represents the coupling term. In the particular case where both systems have the same dimension, the coupling can be expressed linearly as:

$$H(X, Y) = K \cdot (X - Y)$$

Formal Definition

The two systems are said to be in a state of generalized synchronization if there exists a function:

$$\phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

such that:

$$\lim_{t \rightarrow \infty} \|Y(t) - \phi(X(t))\| = 0$$

regardless of the initial conditions. This function is referred to as the *synchronization function* or *coordination function*[6].

Existence and Stability Conditions

To ensure the existence and stability of generalized synchronization, several key conditions must be satisfied[6]:

1. **Lipschitz Condition:** The functions F , G , and H must satisfy the Lipschitz condition to guarantee the existence and uniqueness of solutions.
2. **Critical Coupling Strength:** There exists a threshold value of the coupling parameter beyond which synchronization occurs.
3. **Conditional Lyapunov Exponents:** All conditional Lyapunov exponents of the response system must be negative, ensuring convergence of trajectories toward the synchronization manifold.

Properties and Importance

- **High Flexibility:** Allows synchronization even when the systems differ in structure or signals are distorted during transmission, thanks to the compensating functional mapping.
- **Wide Range of Applications:** Used in chaotic cryptography, secure communications, and neural modeling.
- **Open Research Field:** The study of generalized synchronization, particularly with differing dimensions or bidirectional coupling, remains an active and evolving research area.

Synchronization of hyperchaotic system

The celebrated Lorenz system is described by the following equations [11]:

$$\begin{cases} \dot{x} = -a(y - x), \\ \dot{y} = -xz + cx - y, \\ \dot{z} = xy - bz, \end{cases} \quad (4.1)$$

where a, b, c are real parameters. When $a = 10, b = 8/3, c = 28$, it shows chaotic behavior. According to this system, a new 4D system is obtained by introducing an Nonlinear Dynamics and systems theory additional dimension with a linear state feedback controller to the third equation, and our new 4D system is given by the following dynamics[12]:

$$\begin{cases} \dot{x} = -a(x - y), \\ \dot{y} = -xz + cx - y, \\ \dot{z} = xy - bz - u, \\ \dot{u} = -dy. \end{cases} \quad (4.2)$$

We represent the state of the 4D system (4.2) by $X = (x, y, z, u)$, and a, b, c, d are positive parameters. When the parameters of system (4.2) are taken as $(a, b, c, d) = (10, 5, 30, 20)$ and the initial conditions $(x_0, y_0, z_0, u_0) = (0.1, 0.1, 0.1, 0.1)$, the system has a hyper-chaotic attractor, and the corresponding four Lyapunov exponents can be

calculated: $L_1 = 0.720177, L_2 = 0.628404, L_3 = -1.05541, L_4 = -15.5262$. The phase portraits of the new hyperchaotic system (4.2) are shown in Figure (4.1) The Lyapunov dimension, commonly known as the Kaplan-Yorke dimension D_{KY} , of this system is

$$D_{KY} = j + \frac{1}{|L_{j+1}|} \sum_{i=1}^j L_i = 3 + \frac{L_1 + L_2 + L_3}{|L_4|},$$

$$D_{KY} = 3 + \frac{0.720177 + 0.628404 - 1.05541}{|-15.5262|} = 3 + \frac{0.293171}{15.5262} \approx 3.0189.$$

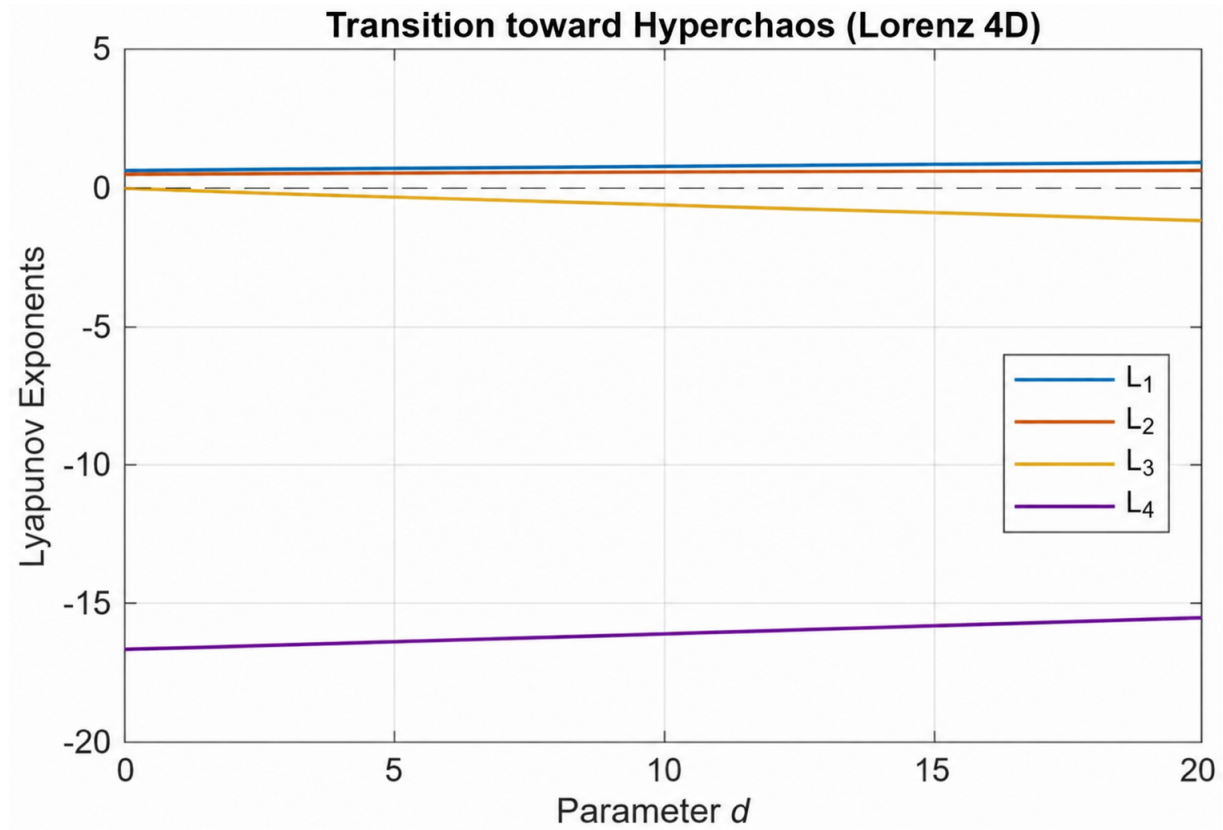


Figure 4.1: Lyapunov Spectrum of the 4D Lorenz System

4.1 Dynamical Analysis

4.1.1 Symmetry

The system (4.2) is symmetrical with respect to the z -axis for its invariance under the coordinate transformation $(x, y, z, u) \longrightarrow (-x, -y, z, -u)$.

4.1.2 Dissipativity and existence of attractor

For a dynamical system, the divergence of the system (4.2) is defined by

$$\nabla V = \frac{d\dot{x}}{dx} + \frac{d\dot{y}}{dy} + \frac{d\dot{z}}{dz} + \frac{d\dot{u}}{du} = -(a + b + 1) = -16 < 0. \quad (4.3)$$

Therefore, the above analysis proves that our system is dissipative. The exponential contraction rate is calculated as follows:

$$V(t) = V(0)e^{-(a+b+1)t}. \quad (4.4)$$

It shows that each volume containing the system trajectories shrinks to zero as $t \longrightarrow \infty$ at an exponential rate $-(a + b + 1)$. There exists an attractor in system (4.2).

4.1.3 Equilibrium points

The equilibria of the system (4.2) can be found by setting $\dot{x} = \dot{y} = \dot{z} = \dot{u} = 0$ and $a, b, c, d > 0$,

$$\begin{aligned} a(y - x) &= 0, \\ -xz + cx - y &= 0, \\ xy - bz - u &= 0, \\ -dy &= 0. \end{aligned}$$

We found that that $x = 0$, and $y = 0$, so we have

$$-bz - u = 0.$$

In other words, system (4.2) has an infinite number of equilibrium points

$$E = \{(x, y, z, u) \in R^4 | x = 0, y = 0, u = -bz\}.$$

For the equilibrium E , the Jacobian matrix of system (4.2) is given by

$$J_E = \begin{pmatrix} -a & a & 0 & 0 \\ c - z & -1 & 0 & 0 \\ 0 & 0 & -b & -1 \\ 0 & -d & 0 & 0 \end{pmatrix}.$$

The characteristic equation of the system evaluated at the equilibrium point E is obtained from

$$\det(\lambda I - J_E) = 0.$$

Thus,

$$\begin{vmatrix} \lambda + a & -a & 0 & 0 \\ -(c - z) & \lambda + 1 & 0 & 0 \\ 0 & 0 & \lambda + b & 1 \\ 0 & d & 0 & \lambda \end{vmatrix} = 0.$$

After computation, we obtain

$$\lambda(\lambda + b) [\lambda^2 + (a + 1)\lambda + a - a(c - z)] = 0.$$

Hence, the characteristic polynomial becomes

$$\lambda^4 + (a + b + 1)\lambda^3 + (ab + a + b - a(c - z))\lambda^2 + ab(1 - (c - z))\lambda = 0.$$

The eigenvalues are given by

$$\lambda_1 = 0,$$

$$\lambda_2 = -b,$$

and

$$\lambda_{3,4} = \frac{-(a+1) \pm \sqrt{(a-1)^2 + 4a(c-z)}}{2}.$$

Since

$$\lambda_1 = 0,$$

the equilibrium point is not asymptotically stable.

Moreover, if

$$b > 0,$$

then

$$\lambda_2 < 0.$$

For the remaining eigenvalues, the condition for negative real parts is

$$\sqrt{(a-1)^2 + 4a(c-z)} < a+1.$$

After simplification, this yields

$$z > c-1.$$

Therefore:

- if

$$z > c-1,$$

all nonzero eigenvalues have negative real parts, and the equilibrium point E is marginally stable;

- if

$$z < c - 1,$$

then one eigenvalue becomes positive, hence the equilibrium point E is unstable;

- if

$$z = c - 1,$$

the system possesses an additional zero eigenvalue and a bifurcation may occur.

4.1.4 Bifurcation diagram and Lyapunov exponents

In this subsection, to explore the effect of the parameters a, b, c and d on the behavior of a new 4D system, we fixed the parameters $(a, b, c) = (10, 5, 30)$ and varied d in $[0, 10]$. Now we carry out the dynamic analysis of the system (4.2) numerically by its Lyapunov exponent spectrum and bifurcation diagram. Figure (4.1) shows the bifurcation diagram of system (4.2) with different values of the parameter d and the initial conditions $(x_0, y, z_0, u_0) = (1.8, 0.1, 0.79, 2.8)$.

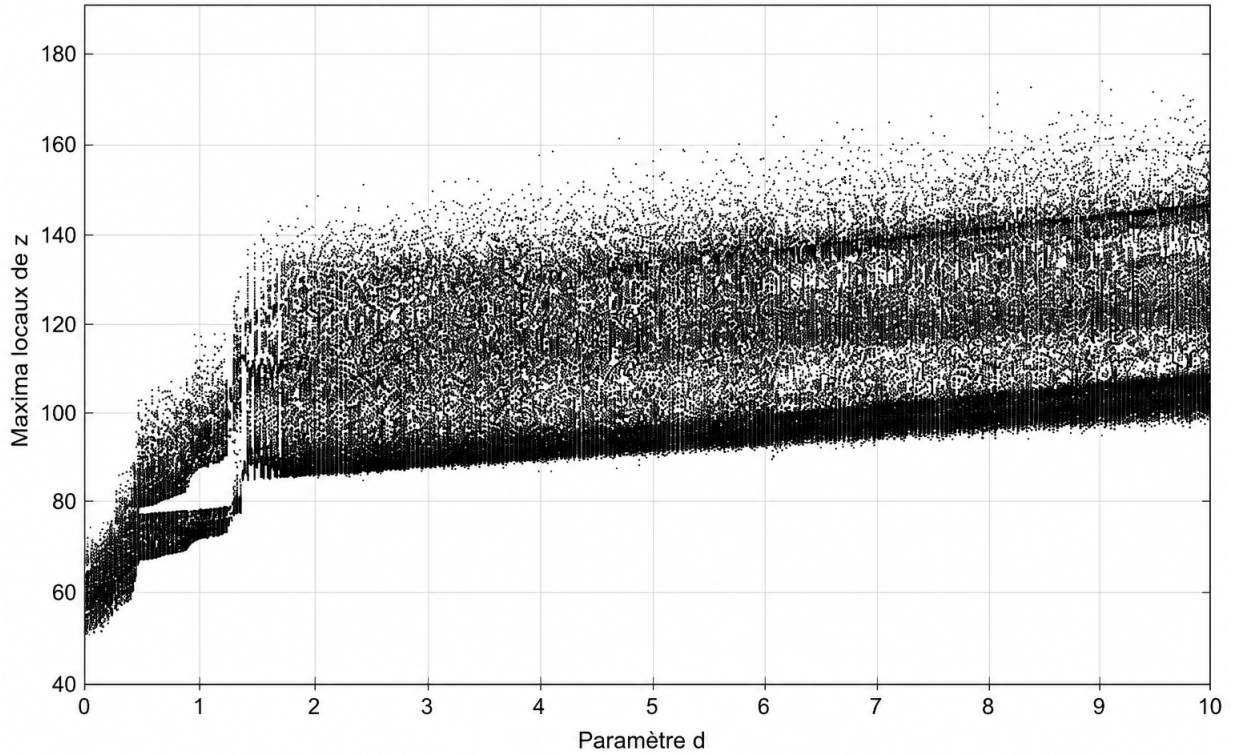


Figure 4.2: Bifurcation diagrams of hyperchaotic system (4.2) when $a = 10, b = 5, c = 30$ and d varies in $[0, 10]$

System (4.2) can display a variety of complex dynamical behaviors such as hyperchaos, chaos, and periodic behavior. For certain parameter values, multiple attractors may coexist simultaneously. This makes the system very sensitive to parameter changes and initial conditions, since even a slight disturbance may drive the system from one state to another. In addition, different initial values may produce completely different dynamical behaviors due to the coexistence of attractors.

4.2 Synchronization of the Modified Hyperchaotic System

In this section, the synchronization of the modified 4D hyperchaotic system is discussed. Synchronization is achieved between a master system and a slave system using the

active control method. The objective is to force the synchronization errors to converge asymptotically to zero.

As the master system, we consider the modified 4D hyperchaotic system given by

$$\begin{cases} \dot{y}_1 = -a(y_1 - y_2), \\ \dot{y}_2 = -y_1 y_3 + c y_1 - y_2, \\ \dot{y}_3 = y_1 y_2 - b y_3 - y_4, \\ \dot{y}_4 = -d y_2. \end{cases} \quad (4.5)$$

The slave system is defined as

$$\begin{cases} \dot{x}_1 = -a(x_1 - x_2) + u_1, \\ \dot{x}_2 = -x_1 x_3 + c x_1 - x_2 + u_2, \\ \dot{x}_3 = x_1 x_2 - b x_3 - x_4 + u_3, \\ \dot{x}_4 = -d x_2 + u_4, \end{cases} \quad (4.6)$$

where $U = \{u_1, u_2, u_3, u_4\}$ denotes the control input vector.

The synchronization errors are defined by

$$\begin{cases} e_1 = x_1 - y_1, \\ e_2 = x_2 - y_2, \\ e_3 = x_3 - y_3, \\ e_4 = x_4 - y_4. \end{cases} \quad (4.7)$$

Then, the error dynamics can be written as

$$\begin{cases} \dot{e}_1 = a(e_2 - e_1) + u_1, \\ \dot{e}_2 = c e_1 - e_2 - x_1 x_3 + y_1 y_3 + u_2, \\ \dot{e}_3 = -b e_3 - e_4 + x_1 x_2 - y_1 y_2 + u_3, \\ \dot{e}_4 = -d e_2 + u_4. \end{cases} \quad (4.8)$$

To eliminate the nonlinear terms, we choose the active control functions as follows:

$$\begin{cases} u_1 = v_1, \\ u_2 = x_1x_3 - y_1y_3 + v_2, \\ u_3 = -x_1x_2 + y_1y_2 + v_3, \\ u_4 = v_4. \end{cases} \quad (4.9)$$

Substituting the above controllers into the error system yields the following linear system:

$$\begin{cases} \dot{e}_1 = a(e_2 - e_1) + v_1, \\ \dot{e}_2 = ce_1 - e_2 + v_2, \\ \dot{e}_3 = -be_3 - e_4 + v_3, \\ \dot{e}_4 = -de_2 + v_4. \end{cases} \quad (4.10)$$

The controlled system can be expressed in matrix form as

$$[\dot{e}_1, \dot{e}_2, \dot{e}_3, \dot{e}_4]^T = C[e_1, e_2, e_3, e_4]^T, \quad (4.11)$$

where the constant matrix C is chosen as

$$C = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (4.12)$$

The eigenvalues of matrix C are

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = -1,$$

which are all negative. Therefore, the synchronization errors converge asymptotically to zero and the synchronization between the master and slave systems is achieved.

Using the previous equations, we obtain

$$\begin{cases} a(e_2 - e_1) + v_1 = -e_1, \\ ce_1 - e_2 + v_2 = -e_2, \\ -be_3 - e_4 + v_3 = -e_3, \\ -de_2 + v_4 = -e_4. \end{cases} \quad (4.13)$$

Hence, the control laws are given by

$$\begin{cases} v_1 = (a - 1)e_1 - ae_2, \\ v_2 = -ce_1, \\ v_3 = (b - 1)e_3 + e_4, \\ v_4 = de_2 - e_4. \end{cases} \quad (4.14)$$

Finally, the active controllers ensuring synchronization are obtained as

$$\begin{cases} u_1 = (a - 1)e_1 - ae_2, \\ u_2 = x_1x_3 - y_1y_3 - ce_1, \\ u_3 = -x_1x_2 + y_1y_2 + (b - 1)e_3 + e_4, \\ u_4 = de_2 - e_4. \end{cases} \quad (4.15)$$

The results of simulations are depicted in Figure 4.3 and Figure 4.4, these figures show that the synchronization occurs and the errors will converge to zero exponentially after applying the active control.

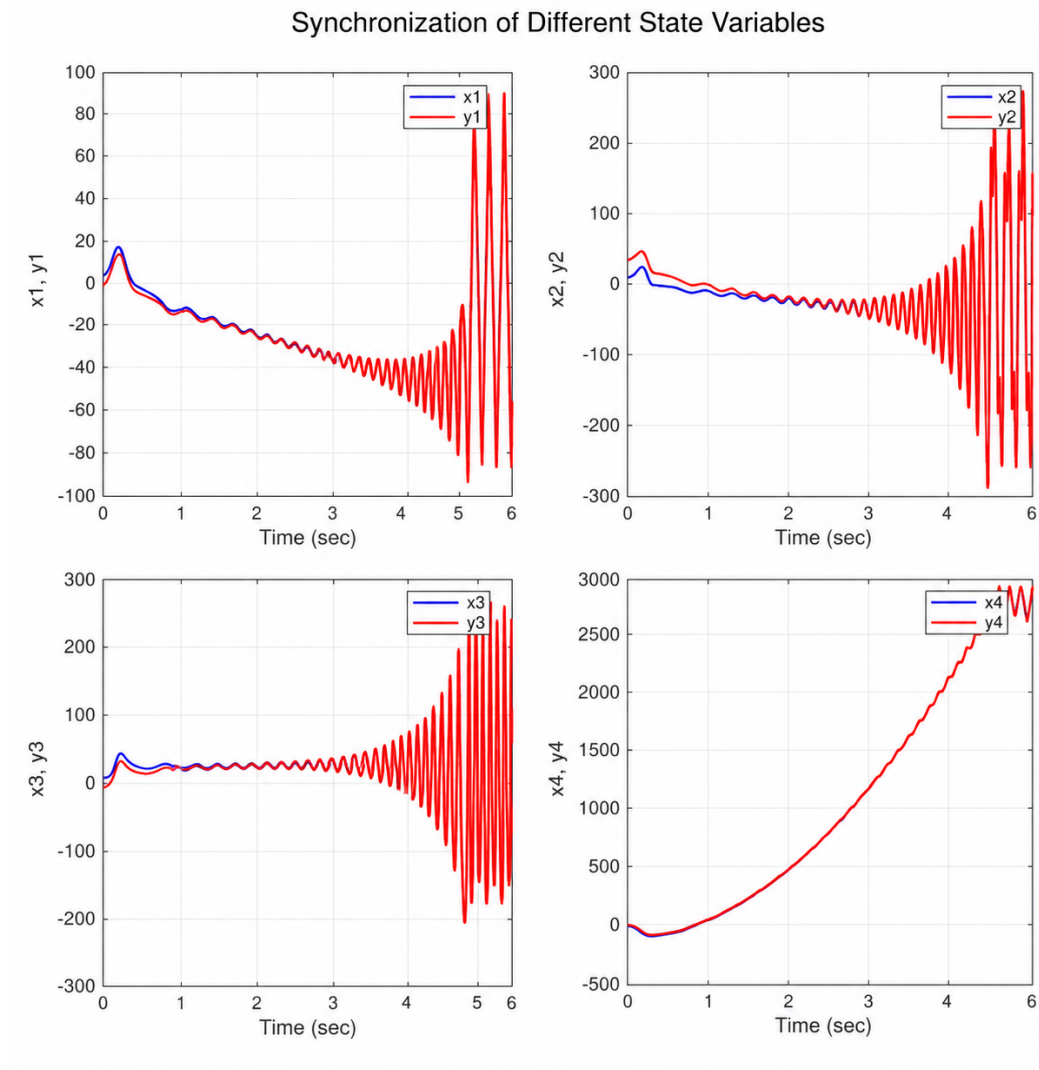


Figure 4.3: Synchronization of different state variables.

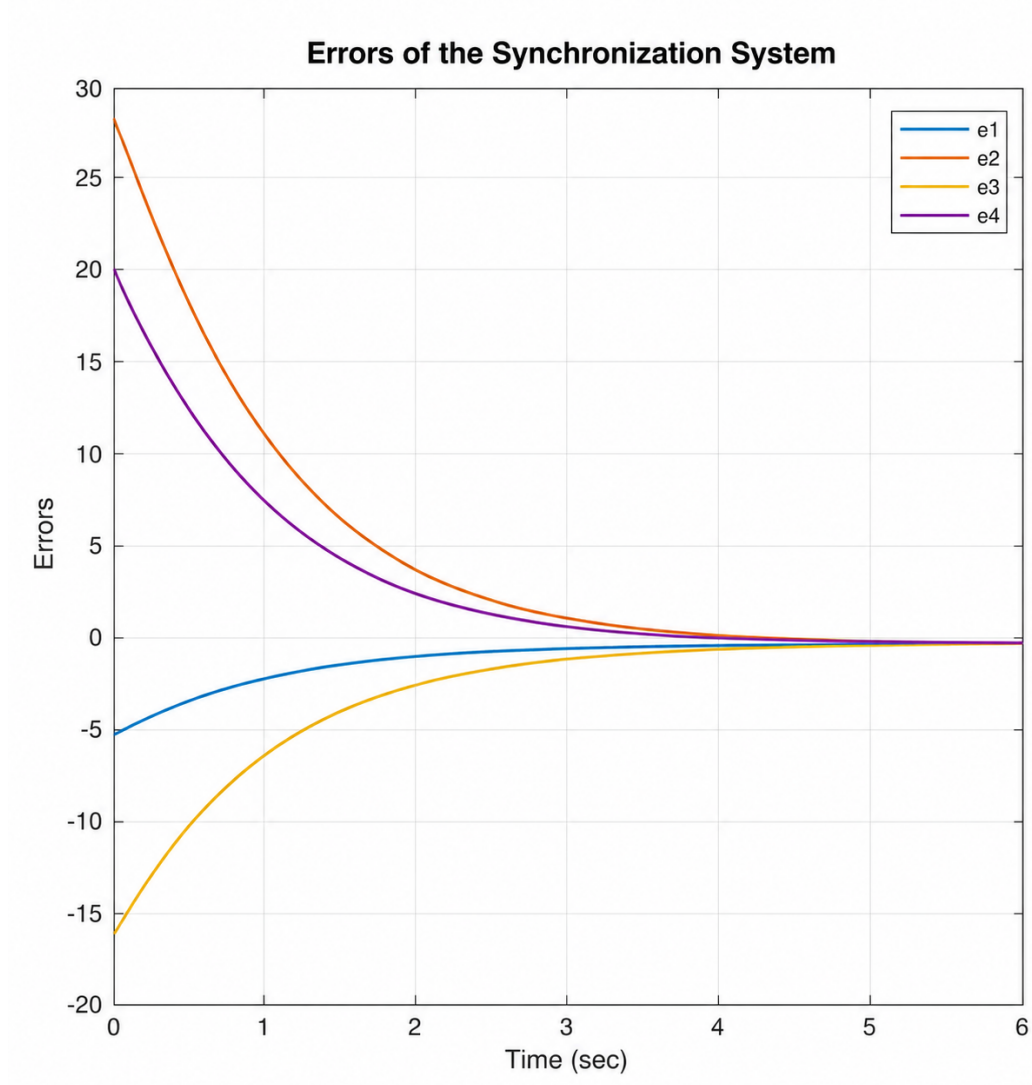


Figure 4.4: Errors of the synchronization system.

4.3 Projective Synchronization of the Hyperchaotic System

We consider the same 4D hyperchaotic system as the master system:

$$\begin{cases} \dot{y}_1 = -a(y_1 - y_2), \\ \dot{y}_2 = -y_1 y_3 + c y_1 - y_2, \\ \dot{y}_3 = y_1 y_2 - b y_3 - y_4, \\ \dot{y}_4 = -d y_2, \end{cases} \quad (4.16)$$

with parameters $a = 10$, $b = 5$, $c = 30$, and $d = 20$. The controlled slave system is defined as

$$\begin{cases} \dot{x}_1 = -a(x_1 - x_2) + u_1, \\ \dot{x}_2 = -x_1x_3 + cx_1 - x_2 + u_2, \\ \dot{x}_3 = x_1x_2 - bx_3 - x_4 + u_3, \\ \dot{x}_4 = -dx_2 + u_4, \end{cases} \quad (4.17)$$

where u_1, u_2, u_3, u_4 are active controllers. The objective is to achieve *projective synchronization* with a constant scaling factor $\alpha \in \mathbb{R}$; that is,

$$\lim_{t \rightarrow \infty} \|(x_1, x_2, x_3, x_4)^\top - \alpha(y_1, y_2, y_3, y_4)^\top\| = 0. \quad (4.18)$$

Define the synchronization errors $e_i = x_i - \alpha y_i$ for $i = 1, \dots, 4$. Substituting the system equations and simplifying yields

$$\begin{cases} \dot{e}_1 = -a(e_1 - e_2) + u_1, \\ \dot{e}_2 = -[e_1e_3 + \alpha y_1e_3 + \alpha y_3e_1 + (\alpha^2 - \alpha)y_1y_3] + ce_1 - e_2 + u_2, \\ \dot{e}_3 = e_1e_2 + \alpha y_1e_2 + \alpha y_2e_1 + (\alpha^2 - \alpha)y_1y_2 - be_3 - e_4 + u_3, \\ \dot{e}_4 = -de_2 + u_4. \end{cases} \quad (4.19)$$

We choose the controllers so as to cancel all nonlinear terms:

$$\begin{cases} u_1 = v_1, \\ u_2 = e_1e_3 + \alpha y_1e_3 + \alpha y_3e_1 + (\alpha^2 - \alpha)y_1y_3 + v_2, \\ u_3 = -e_1e_2 - \alpha y_1e_2 - \alpha y_2e_1 - (\alpha^2 - \alpha)y_1y_2 + v_3, \\ u_4 = v_4. \end{cases} \quad (4.20)$$

Substituting (4.20) into (4.19) gives the linear error system

$$\begin{cases} \dot{e}_1 = -a(e_1 - e_2) + v_1, \\ \dot{e}_2 = ce_1 - e_2 + v_2, \\ \dot{e}_3 = -be_3 - e_4 + v_3, \\ \dot{e}_4 = -de_2 + v_4. \end{cases} \quad (4.21)$$

We now set $v_i = k_i e_i$ with constant gains k_i . Then

$$\begin{cases} \dot{e}_1 = (k_1 - a)e_1 + ae_2, \\ \dot{e}_2 = ce_1 + (k_2 - 1)e_2, \\ \dot{e}_3 = (k_3 - b)e_3 - e_4, \\ \dot{e}_4 = -de_2 + k_4 e_4. \end{cases} \quad (4.22)$$

Consider the Lyapunov function candidate:

$$V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2).$$

Its time derivative along the trajectories of (4.22) is:

$$\dot{V} = (k_1 - a)e_1^2 + (k_2 - 1)e_2^2 + (k_3 - b)e_3^2 + k_4 e_4^2 + (a + c)e_1 e_2 - e_3 e_4 - de_2 e_4.$$

Using Young's inequality $2|uv| \leq \alpha u^2 + \alpha^{-1}v^2$, with $\alpha_1, \alpha_2, \alpha_3 > 0$, we bound the cross terms:

$$\begin{aligned} (a + c)e_1 e_2 &\leq \frac{\alpha_1}{2} e_1^2 + \frac{(a + c)^2}{2\alpha_1} e_2^2, \\ -e_3 e_4 &\leq \frac{\alpha_2}{2} e_3^2 + \frac{1}{2\alpha_2} e_4^2, \\ -de_2 e_4 &\leq \frac{\alpha_3}{2} e_2^2 + \frac{d^2}{2\alpha_3} e_4^2. \end{aligned}$$

Substituting these bounds into $\dot{V}$ yields:

$$\dot{V} \leq -A_1 e_1^2 - A_2 e_2^2 - A_3 e_3^2 - A_4 e_4^2,$$

where

$$\begin{aligned} A_1 &= a - k_1 - \frac{\alpha_1}{2}, \\ A_2 &= 1 - k_2 - \frac{(a+c)^2}{2\alpha_1} - \frac{\alpha_3}{2}, \\ A_3 &= b - k_3 - \frac{\alpha_2}{2}, \\ A_4 &= -k_4 - \frac{1}{2\alpha_2} - \frac{d^2}{2\alpha_3}. \end{aligned}$$

We choose the control gains k_i such that $A_i > 0$ for $i = 1, 2, 3, 4$. This is always possible by taking k_i sufficiently large negative. Hence, $\dot{V} < 0$ for all non-zero error vectors, ensuring asymptotic stability. Therefore, projective synchronization is achieved:

$$(x_1, x_2, x_3, x_4) \rightarrow \alpha(y_1, y_2, y_3, y_4) \quad \text{as } t \rightarrow \infty.$$

For numerical implementation, we select:

$$k_1 = -20, \quad k_2 = -20, \quad k_3 = -10, \quad k_4 = -20.$$

Substituting $a = 10, b = 5, c = 30, d = 20$ and the chosen k_i into the control law (4.20) gives the explicit controllers:

$$\left\{ \begin{aligned} u_1 &= -20e_1, \\ u_2 &= e_1e_3 + \alpha y_1e_3 + \alpha y_3e_1 + (\alpha^2 - \alpha)y_1y_3 - 20e_2, \\ u_3 &= -e_1e_2 - \alpha y_1e_2 - \alpha y_2e_1 - (\alpha^2 - \alpha)y_1y_2 - 10e_3, \\ u_4 &= -20e_4. \end{aligned} \right. \quad (4.23)$$

These controllers guarantee convergence of the synchronization errors, as confirmed

by the numerical simulations shown in Figures (4.5) and (4.6).

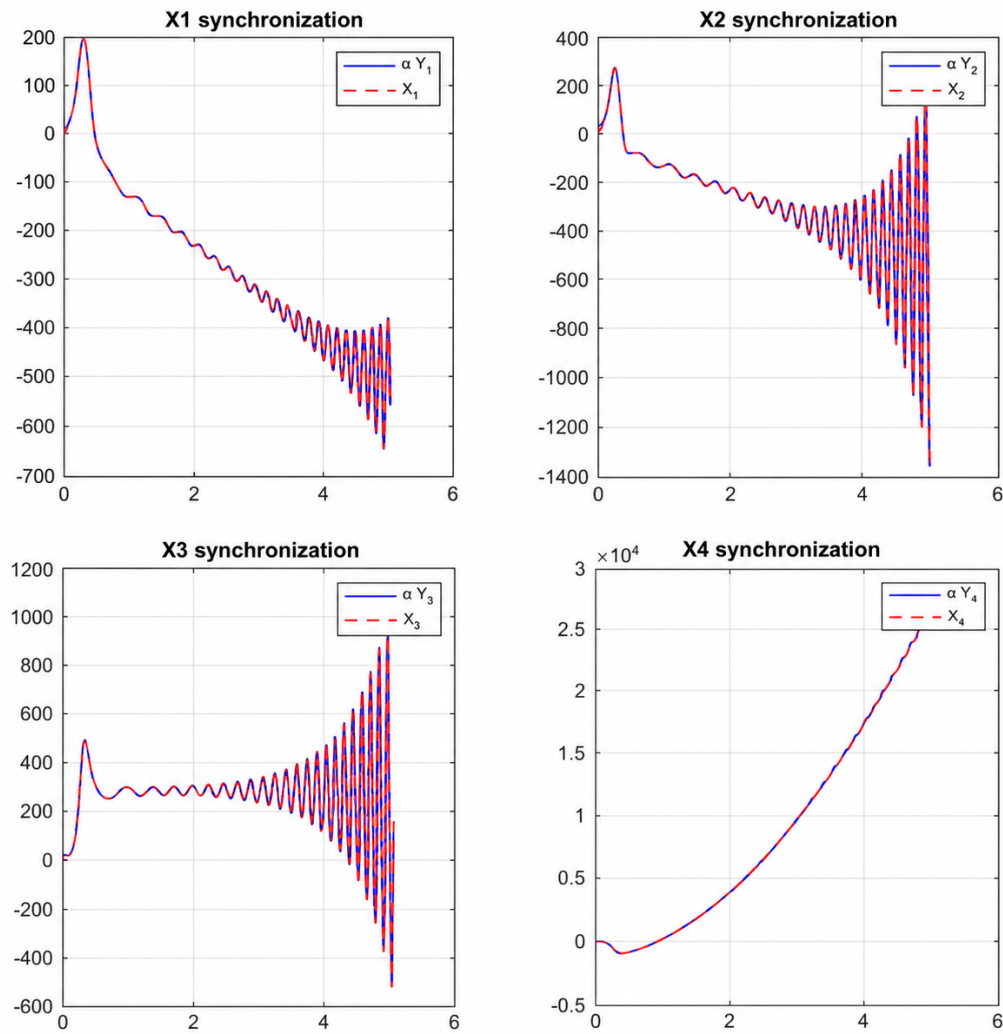


Figure 4.5: The simulation results demonstrate the successful achievement of projective synchronization, where the curves x_s, y_s, z_s, u_s perfectly match $\alpha x, \alpha y, \alpha z, \alpha u$, respectively, after a short convergence.

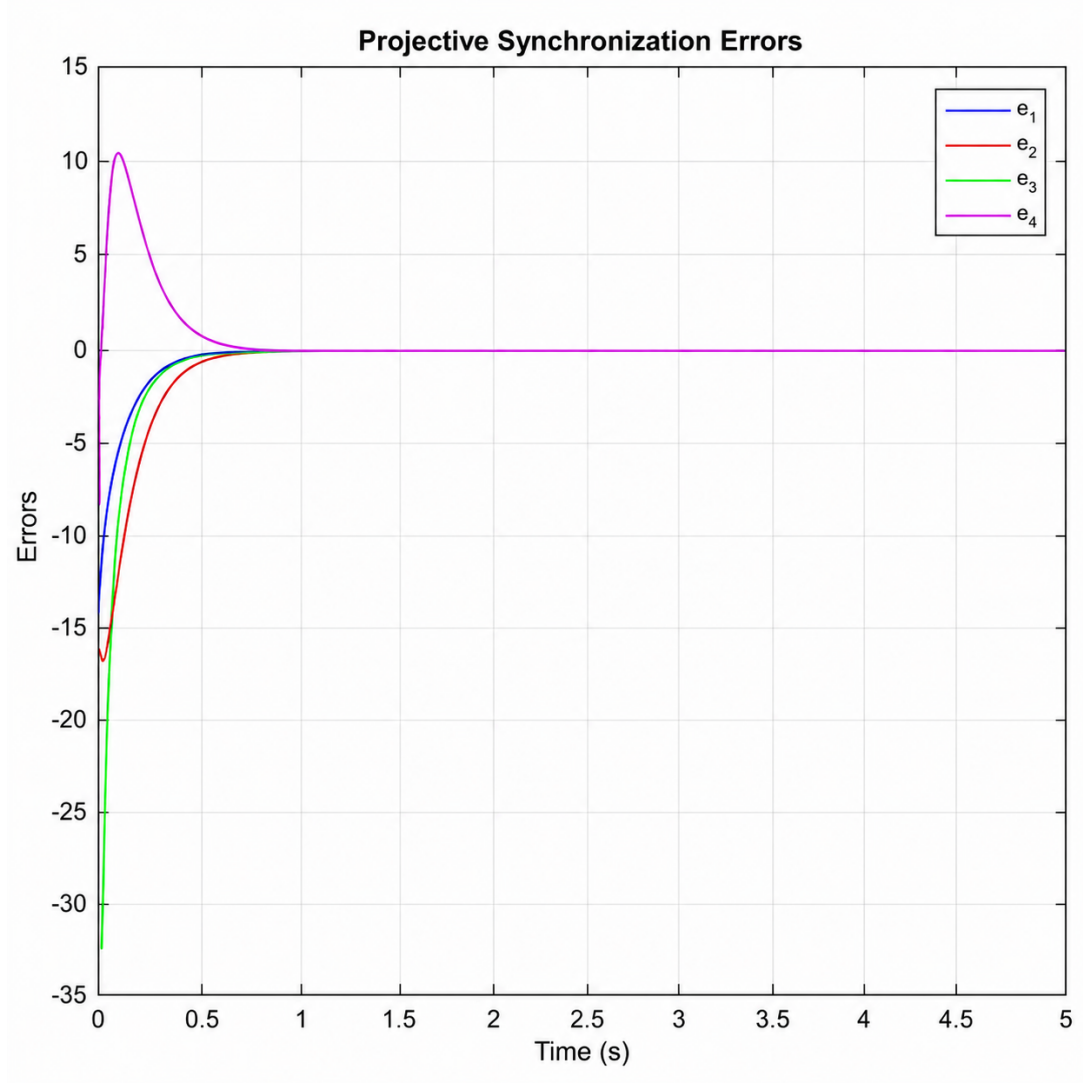


Figure 4.6: The simulation results show that the projective synchronization errors e_1, e_2, e_3, e_4 converge rapidly to zero, confirming the success of the synchronization process and the achievement of the desired objective.

Conclusion

The main objective of this dissertation was to study and analyze hyperchaotic systems and the synchronization of nonlinear dynamical systems. We first introduced the fundamental concepts of dynamical systems and Lyapunov stability theory, followed by a discussion of the main properties of chaos, including sensitivity to initial conditions and the role of Lyapunov exponents in characterizing chaotic behavior.

Subsequently, we presented the concept of hyperchaotic systems and their dynamical characteristics. We also reviewed the different types of synchronization in chaotic systems, with special emphasis on complete synchronization, projective synchronization, and generalized synchronization.

In the practical part of this work, a novel 4D hyperchaotic system with an infinite number of equilibrium points was investigated. Its dynamical properties were analyzed through the study of equilibrium points, dissipativity, Lyapunov exponents, and bifurcation diagrams. Furthermore, an appropriate active control strategy was designed to achieve synchronization between the master and slave systems.

Finally, the stability analysis based on Lyapunov theory demonstrated that the synchronization errors converge asymptotically to zero, ensuring the successful achievement of synchronization. Numerical simulations carried out using MATLAB confirmed the effectiveness and accuracy of the proposed control approach and validated the theoretical results. These findings demonstrate the potential of hyperchaotic synchronization techniques for a wide range of engineering and scientific applications involving complex nonlinear dynamical systems.

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