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Stability and bifurcation investigation of fractional discrete dynamical models using the conformable derivative

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In front of the jury

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DEDICATION

"In the middle of chaos, there is always a point of stability."

To those who taught me that even the most unstable systems can find their equilibrium with a little control.

To my **father**, the fixed point around which my world has always revolved.

To my **mother**, whose love is the stability that no bifurcation can disturb.

To my **family**, my silent supporters and my source of strength.

To my **professors**, the guides who illuminated the path of knowledge.

To my **friends and colleagues**, the companions who made this journey meaningful.

To the class of **2025–2026** a class that will always be remembered.

This work is not mine alone; it belongs to all of you.

MIYADA

DEDICATION

First, I thank Almighty Allah, who illuminated my path and granted me the strength and success to complete this work. All praise and gratitude are due to Him, always and forever.

To my father and mother, the greatest blessing for which I thank Allah every single day, to the ones whose prayers were my support and whose patience was the reason for my perseverance, I dedicate to you the fruit of this humble effort. No words can ever truly express my gratitude, for you have always been the source of strength, safety, and comfort in my life.

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Abstract

This thesis addresses the mathematical modeling of selected biological phenomena and provides a qualitative analysis of a conformable fractional-order ratio-dependent Holling-Tanner predator-prey model. This thesis is organized into three chapters:

Chapter 1 introduces the fundamental concepts of discrete dynamical systems, including definitions, fixed points, periodic orbits, and stability theory. It also presents the main analytical tools required for the study of nonlinear dynamical systems.

Chapter 2 introduces population dynamics through classical models such as the Lotka-Volterra and predator-prey systems. It also discusses various functional responses to provide a more realistic description of species interactions.

Chapter 3 investigates a discretized conformable fractional-order Holling-Tanner predator-prey model with fear effects. The existence and stability of fixed points are studied, together with the influence of the fractional-order parameter on the system dynamics. Various bifurcation phenomena leading to periodic, quasi-periodic, and chaotic behaviors are analyzed. Chaotic dynamics are confirmed using the 0–1 test for chaos and illustrated through numerical simulations. Furthermore, chaos control techniques are applied to stabilize unstable and chaotic behaviors.

Keywords: discrete dynamical systems, predator-prey model, conformable fractional derivative, fixed points, stability, bifurcations, 0–1 test, chaos control.

Résumé

Ce mémoire est consacré à la modélisation mathématique de certains phénomènes biologiques, ainsi qu'à l'étude qualitative d'un modèle proie-prédateur de Holling-Tanner à ordre fractionnaire conforme et dépendant du rapport. Il est organisée en trois chapitres: **Le Chapitre 1** introduit les concepts fondamentaux des systèmes dynamiques discrets, notamment les définitions, les points fixes, les orbites périodiques et la théorie de la stabilité. Il présente également les principaux outils analytiques nécessaires à l'étude des systèmes dynamiques non linéaires. **Le chapitre 2** présente la dynamique des populations à travers des modèles classiques tels que le modèle de Lotka-Volterra et les systèmes proie-prédateur. Il aborde également différents types de réponses fonctionnelles afin de fournir une description plus réaliste des interactions entre espèces. **Le Chapitre 3** étudie un modèle proie-prédateur de Holling-Tanner discrétisé d'ordre fractionnaire conforme avec effet de peur. L'existence et la stabilité des points fixes sont analysées, ainsi que l'influence du paramètre d'ordre fractionnaire sur la dynamique du système. Divers phénomènes de bifurcation conduisant à des comportements périodiques, quasi-périodiques et chaotiques sont étudiés. Les dynamiques chaotiques sont confirmées à l'aide du test 0–1 du chaos et illustrées par des simulations numériques. En outre, des techniques de contrôle du chaos sont appliquées afin de stabiliser les comportements instables et chaotiques.

Mots-clés : systèmes dynamiques discrets , modèle proie-prédateur , dérivée fractionnaire conforme , points fixes , stabilité , bifurcations , test 0–1 du chaos , contrôle du chaos.

ملخص

يتعلق هذا العمل بالنمذجة الرياضية لبعض الظواهر البيولوجية، و الدراسة النوعية لنموذج هولينغ-تانر للمفترس والفريسة المعتمد على النسبة وذي الرتبة الكسرية المطابقة. وقد تم تنظيم هذه المذكرة في ثلاثة فصول:

الفصل الأول يقدم المفاهيم الأساسية للأنظمة الديناميكية المتقطعة، بما في ذلك التعاريف، ونقاط التوازن، والمدارات الدورية، ونظرية الاستقرار. كما يعرض الأدوات التحليلية الرئيسية اللازمة لدراسة الأنظمة الديناميكية غير الخطية.

الفصل الثاني يقدم ديناميكية السكان من خلال النماذج الكلاسيكية مثل نموذج لوتكا-فولتيرا ونماذج المفترس والفريسة. كما يناقش مختلف الاستجابات الوظيفية لتقديم وصف أكثر واقعية للتفاعلات بين الأنواع.

الفصل الثالث يدرس نموذج هولينغ-تانر للمفترس والفريسة من الرتبة الكسرية المطابقة بعد تحويله الى متقطع، مع تأثيرات الخوف. تتم دراسة وجود نقاط التوازن واستقرارها، بالإضافة إلى تأثير معامل الرتبة الكسرية على ديناميكية النظام. كما يتم تحليل مختلف ظواهر التشعب المؤدية إلى سلوكيات دورية، وشبه دورية، وفوضوية. ويتم تأكيد الديناميكية الفوضوية باستخدام اختبار الفوضى 0-1 مع توضيحها بواسطة محاكاة عددية. علاوة على ذلك، تُطبّق تقنيات التحكم في الفوضى من أجل تثبيت السلوكيات غير المستقرة والفوضوية.

الكلمات الأساسية: الأنظمة الديناميكية المتقطعة، نموذج المفترس-الفريسة، المشتقة الكسرية المطابقة، نقاط التوازن، الاستقرار، التشعبات، اختبار 0-1، التحكم في الفوضى.

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General introduction

Dynamical systems theory is a branch of mathematics that studies systems evolving over time according to fixed rules. Originating with Newton (around 1665), it provides a framework for understanding complex behaviors in natural and applied sciences. A key phenomenon in this field is chaos, characterized by strong sensitivity to initial conditions and long-term unpredictability. Such systems may also exhibit attractors, including strange attractors, which describe their asymptotic behavior.

The study of nonlinear systems relies strongly on linear theory. As shown in [1–4], linear systems theory provides fundamental results such as existence and uniqueness of solutions, equilibrium classification via eigenvalues, and matrix-based representations in $\mathbb{R}^n$. Nonlinear systems are often analyzed through local linearization near fixed points, allowing the use of linear methods to approximate local dynamics when exact solutions are not available.

Recently, conformable fractional-order predator–prey models have gained attention due to their ability to describe ecological interactions more precisely than classical integer-order models. Several studies [5–7] show that the fractional order significantly affects stability, bifurcation behavior, and population dynamics. Other works highlight its influence on singularities, wave propagation, disease spread, and chaotic behavior in ecological systems [8–14].

Overall, these results demonstrate that fractional-order models provide a flexible

framework for capturing complex ecological dynamics, where the fractional parameter plays a crucial role in determining system behavior.

Motivated by this, the present thesis studies a fractional-order discrete-time predator–prey system based on the generalized conformable derivative. The model is analyzed both analytically and numerically for different fractional-order values.

Objectives of the Thesis

This work aims to analyze a discretized generalized conformable fractional-order ratio-dependent Holling–Tanner predator–prey model with fear effect. The main objectives are:

1. Derive the discrete model via piecewise constant approximation.
2. Determine fixed points.
3. Study local stability using Jury conditions.
4. Investigate flip and Neimark–Sacker bifurcations.
5. Detect chaos using the 0–1 test and Lyapunov exponents.
6. Apply hybrid chaos control.

Organization of the Thesis

Chapter 1 introduces discrete dynamical systems, stability theory, bifurcation analysis, chaos indicators, and fractional calculus. Chapter 2 presents population dynamics models and functional responses. Chapter 3 focuses on the qualitative analysis of the proposed Holling–Tanner model, including stability, bifurcations, chaos detection, and control, supported by numerical simulations.

Finally, the stability of fixed points is analyzed within the framework of fractional discrete dynamical systems to study the influence of the fractional order on system behavior.

Basic concepts and preliminaries

1.1 Introduction

A dynamical system is defined as a collection of all possible states together with a deterministic rule that specifies how the state evolves over time. Depending on the description of time, this evolution may be governed by continuous or discrete models. When differential equations are used, the system is referred to as a *continuous* dynamical system, whereas a system described by difference equations is called a *discrete* dynamical system.

In general, these two types of systems can be represented as

$$\dot{x} = \frac{dx}{dt} = \mathcal{G}(x, t, \varphi), \quad (1.1)$$

$$x \mapsto \mathcal{G}(x, \varphi), \quad (1.2)$$

where $x \in \mathbb{R}^n$ denotes the state vector, $t \in \mathbb{R}$ is time, and $\varphi \in \mathbb{R}^p$ is a vector of parameters. Equation (1.1) describes an ordinary differential equation, while equation (1.2) defines a discrete map. Both forms are encompassed by the general theory of dynamical systems, and in subsequent sections we outline the main analytical tools used to study models of this type.

1.2 Basic definitions and concepts in discrete dynamical system

In this section, we present several fundamental properties of dynamical systems that will be used throughout the remainder of the thesis.

Definition 1.2.1 ([15]) *A dynamical system is a mathematical model that describes the evolution of a set of interacting components over time. It is defined by a triplet $(X, T, \mathcal{G})$, where:*

- X is the state space,
- T is the time domain,
- $\mathcal{G} : X \times T \rightarrow X$ is the state transition map,

which determines the state of the system at any time, given an initial state vector.

Definition 1.2.2 ([15]) *A discrete dynamical system is described by*

$$x_{n+1} = \mathcal{G}(x_n), \quad n \in \mathbb{N}, \quad (1.3)$$

where $x_n \in \mathbb{R}^k$ and $\mathcal{G} : \mathbb{R}^k \rightarrow \mathbb{R}^k$ is a continuous map. A solution of (1.3) is a sequence $\{x_n\}_{n=0}^{\infty}$ that satisfies (1.3) for all $n \geq 0$.

Discrete dynamical systems are classified as:

- **linear**, if $\mathcal{G}$ is a linear map,
- **nonlinear**, if $\mathcal{G}$ is nonlinear.

Given an initial condition $x_0 \in \mathbb{R}^k$, the successive iterates of the system are obtained as

$$x_1 = \mathcal{G}(x_0), \quad x_2 = \mathcal{G}(x_1) = \mathcal{G}(\mathcal{G}(x_0)) = \mathcal{G}^2(x_0), \quad x_3 = \mathcal{G}^3(x_0), \quad \dots, \quad x_n = \mathcal{G}^n(x_0),$$

where $\mathcal{G}^n$ denotes the n -fold composition of the map $\mathcal{G}$. Thus, a discrete dynamical system can be viewed as the repeated iteration of the function $\mathcal{G}$.

Definition 1.2.3 (Autonomous and non-autonomous discrete dynamical systems[16])

The system (1.3) is called **non-autonomous** if the function $\mathcal{G}$ depends explicitly on n (or time), that is,

$$x_{n+1} = \mathcal{G}(n, x_n).$$

If $\mathcal{G}$ does not depend on n and the evolution rule is the same at every step,

$$x_{n+1} = \mathcal{G}(x_n),$$

then the system is said to be **autonomous**.

Definition 1.2.4 (Trajectories[16]) Given an initial point x_0 , the **orbit** (or **trajectory**) of the system (1.3) is the sequence

$$O(x_0) = \{ x_n = \mathcal{G}^n(x_0) : n \geq 0 \},$$

where $\mathcal{G}^0(x_0) = x_0$ and $\mathcal{G}^n$ denotes the n -fold iteration of the map $\mathcal{G}$.

1.2.1 Classification of dynamical systems

Dynamical systems can generally be classified into three principal categories:

- **Random Systems:** These systems evolve in an unpredictable manner without being governed by explicit mathematical rules. Their behavior cannot be accurately forecasted over time, as the evolution is essentially stochastic.
- **Deterministic Systems:** These systems follow well-defined mathematical laws that uniquely determine their evolution. Given an initial state, the future behavior of the system can be predicted exactly.

- **Chaotic Systems:** These are deterministic systems that exhibit extremely sensitive dependence on initial conditions, resulting in highly complex and seemingly random behavior. Their trajectories tend to be confined to intricate geometric structures called attractors, on which they evolve without repetition. Although governed by deterministic laws, their long-term behavior becomes practically unpredictable.

1.2.2 Stability theory and fixed points

In many applications, dynamical systems and difference equations arising in various fields are often difficult or impossible to solve explicitly. Consequently, much of the analysis focuses on identifying their fixed points, since these points provide significant insight into the qualitative behavior of solutions. Knowledge of fixed points allows researchers to predict long-term dynamics, detect stability properties, and understand how systems respond to perturbations. We now present the formal definition of a fixed point.

Definition 1.2.5 (Fixed point [3, 17]) *A point x^* in the domain of $\mathcal{G}$ is called an **fixed point** (or **equilibrium point**) of the discrete system (1.3) if it satisfies*

$$\mathcal{G}(x^*) = x^*.$$

*Such a point is also referred to as a **critical point**.*

Graphically, a **fixed point** corresponds to an intersection point of the curves $y = \mathcal{G}(x)$ with $y = x$.

Definition 1.2.6 (Periodic Points and p -Cycles [3]) *Let x be a point in the domain of $\mathcal{G}$. If there exists a fixed point x^* of (1.3) and a positive integer r such that*

$$\mathcal{G}^r(x^*) = x^* \quad \text{and} \quad \mathcal{G}^{r-1}(x^*) \neq x^*,$$

then x is called an *eventually equilibrium* (or *eventually fixed*) point.

Definition 1.2.7 (Periodic Points and p -Cycles [2, 3]) A point $x \in \mathbb{R}^k$ is called a **periodic point** if there exists an integer $p \geq 1$ such that

$$\mathcal{G}^p(x) = x.$$

The smallest integer p satisfying this relation is called the **period** of x . The **orbit** of a p -periodic point is the set

$$O(x) = \{x, \mathcal{G}(x), \mathcal{G}^2(x), \dots, \mathcal{G}^{p-1}(x)\},$$

which is also referred to as a **p -periodic cycle**.

Remark 1.2.1

- A fixed point is a periodic point of period 1.
- A point x is said p -periodic point of a transformation $\mathcal{G}$, if it is a fixed point of $\mathcal{G}^p$, without being a fixed point of $\mathcal{G}^l$, $1 \leq l < p$, (l and p are integers).

Definition 1.2.8 ([2, 3]) Let x^* be an fixed point.

- The fixed point x^* of (1.3) is stable if given $\epsilon > 0$ there exists $\delta > 0$ such that $|x_0 - x^*| < \delta$ implies $|\mathcal{G}^n(x_0) - x^*| < \epsilon$ for all $n > 0$. If x^* is not stable, then it is called unstable.
- The fixed point x^* is said to be attracting if there exists $\eta > 0$ such that

$$|x_0 - x^*| < \eta \quad \text{implies} \quad \lim_{n \rightarrow \infty} \mathcal{G}^n(x_0) = x^*.$$

If $\eta = \infty$, x^* is called a globally attracting.

- The point x^* is an asymptotically stable fixed point if it is stable and attracting.
If $\eta = \infty$, x^* is said to be globally asymptotically stable.

Suppose that the discrete system (1.3) has a power series. Then, it can be written in the following linear form:

$$x_{n+1} = Ax_n + \mathcal{R}(|x_n|), \quad (1.4)$$

where A is a Jacobian matrix, and $\mathcal{R}(|x_n|)$ is negligible.

Definition 1.2.9 ([18]) Let A be $(k \times k)$ matrix, then:

- The scalar λ is called an eigenvalue of A if and only if $\det(A - \lambda I_k) = 0$.
- Let x be a non-zero column vector. Then x is called an eigenvector associated with the eigenvalue λ if and only if $\det(A - \lambda I_k)x = 0$.

Definition 1.2.10 ([2]) A periodic orbit $\{x_0, x_1, \dots, x_{p-1}\}$ of a dynamical system governed by a map $\mathcal{G}$ is stable if every point x_i ($i = 0, 1, \dots, p$) is a stable fixed point of the dynamical system governed by the map $\mathcal{G}^p$.

Definition 1.2.11 ([2]) Let A be the Jacobian matrix of system (1.3) and x^* is a fixed point of the system (1.3), then:

- If all the eigenvalues of A are modules strictly less than unity circle, then the fixed point x^* is asymptotically stable.
- If the matrix A admits at least one eigenvalue of modulus strictly greater than unity circle, then the fixed point x^* is unstable.
- If some eigenvalues of A are on the circle of the radius of the unit and the others on inside, we can not conclude the local stability of the fixed point x^* (non-hyperbolic).

For two-dimensional discrete-time dynamical system we introduce the following lemma.

Lemma 1.2.1 ([2]) Let (x^*, y^*) be a fixed point of the system (1.3) and λ_1, λ_2 are two eigenvalues of the Jacobian matrix of the system (1.3).

1. If $|\lambda_1| < 1$ and $|\lambda_2| < 1$ a fixed point (x^*, y^*) is a sink and is locally asymptotically stable.
2. If $|\lambda_1| > 1$ and $|\lambda_2| > 1$ a fixed point (x^*, y^*) is called source, so a source is unstable.
3. If $|\lambda_1| < 1$ and $|\lambda_2| > 1$ or $(|\lambda_1| > 1$ and $|\lambda_2| < 1)$ a fixed point (x^*, y^*) is called saddle.
4. If $|\lambda_1| = 1$ or $|\lambda_2| = 1$ a fixed point (x^*, y^*) is called to be non-hyperbolic.

Lemma 1.2.2 ([19]) Assume that α_1, α_2 , and α_3 are real numbers. Then, a necessary and sufficient condition for all roots of the equation

$$\lambda^3 + \alpha_1\lambda^2 + \alpha_2\lambda + \alpha_3 = 0,$$

to lie inside the unit disk is

$$|\alpha_1 + \alpha_3| < 1 + \alpha_2, \quad |\alpha_1 - 3\alpha_3| < 3 - \alpha_2, \quad \text{and} \quad \alpha_3^2 + \alpha_2 - \alpha_3\alpha_1 < 1. \quad (1.5)$$

Theorem 1.2.2 ([2, 20]) Let $A = (a_{ij})$ be a 2×2 matrix. Then $|\lambda_{1,2}| < 1$ if and only if

$$|\text{Tr}(A)| < 1 + \text{Det}(A) < 2. \quad (1.6)$$

This condition can also be expressed as the following three inequalities (Jury conditions):

$$\text{Det}(A) > \text{Tr}(A) - 1, \quad \text{Det}(A) > -\text{Tr}(A) - 1, \quad \text{Det}(A) < 1.$$

Lemma 1.2.3 ([21]) Let $\Psi(\lambda) = \lambda^2 - \text{Tr}(J)\lambda + \det(J)$ be a quadratic polynomial satisfying $\Psi(1) > 0$, and let λ_1 and λ_2 denote the roots of $\Psi(\lambda) = 0$. Then the following results hold:

1. $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $\Psi(-1) > 0$ and $\det(J) < 1$.
2. $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $\Psi(-1) > 0$ and $\det(J) > 1$.
3. $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or vice versa) if and only if $\Psi(-1) < 0$.
4. $\lambda_1 = -1$ and $|\lambda_2| \neq 1$ if and only if $\Psi(-1) = 0$ and $\text{Tr}(J) \neq 0$.

5. The roots λ_1 and λ_2 are complex with $|\lambda_1| = |\lambda_2| = 1$ if and only if

$$\text{Tr}(J)^2 - 4 \det(J) < 0 \quad \text{and} \quad \det(J) = 1.$$

Theorem 1.2.3 ([2]) *The following statements hold for any 2×2 matrix A*

- *If $|\text{Tr}(A)| - 1 = \text{Det}(A)$, then we have*
 - *the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = \text{Det}(A)$ if $\text{Tr}(A) > 0$,*
 - *the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = -\text{Det}(A)$ if $\text{Tr}(A) < 0$,*
- *If $|\text{Tr}(A)| - 1 < \text{Det}(A)$, and $\text{Det}(A) = 1$, then the eigenvalues of A are $e^{\pm i\theta}$, where $\theta = \cos^{-1}\left(\frac{\text{Tr}(A)}{2}\right)$.*

The violation of one of these conditions of Theorem 1.2.2 (Jury conditions), with the other two being simultaneously fulfilled leads to:

- A **transcritical bifurcation** (a real eigenvalue that passes through 1), i.e., $1 - \text{Tr}(A) + \text{Det}(A) = 0$.
- A **period doubling bifurcation** (a real eigenvalue that passes through -1), i.e., $1 + \text{Tr}(A) + \text{Det}(A) = 0$.
- A **Neimark-Sacker bifurcation** (the modulus of a complex eigenvalue pair that passes through 1), i.e., $\text{Det}(A) = 1$.

1.3 Bifurcations and chaos

Chaos theory focuses on the study of dynamical systems whose behavior is extremely sensitive to initial conditions. Such behavior can be analyzed through mathematical modeling or by using analytical recurrence methods and Poincaré maps. Meanwhile, bifurcation theory aims to examine the qualitative changes that dynamical systems undergo as their governing parameters vary. This section presents some insights into both of these topics.

1.3.1 Bifurcations theory

Consider the nonlinear discrete system

$$x_{n+1} = \mathcal{G}(x_n, \mu), \quad (1.7)$$

where $\mu \in \mathbb{R}$. Any change in the parameter μ (**bifurcation parameter**), will change the dynamic behaviors of the system (1.7). This change is known as a bifurcation. A bifurcation occurs when even a minor change to the bifurcation parameter leads to a qualitative or topological change in the systems behavior. Additionally, bifurcation diagrams depict certain characteristic aspects of the asymptotic solution of a dynamical system as a function of a bifurcation parameter. This allows one to observe the qualitative changes in the behavior of the states of the system [22]. There are two classes of bifurcation:

1. **Global bifurcations:** These typically occur when large invariant sets of the system interact or collide with one another, or with the systems equilibria. They cannot be detected solely through an analysis of the stability of fixed points.
2. **Local bifurcations:** These can be analyzed entirely through changes in the stability properties of local equilibria, periodic orbits, or other invariant sets. Such bifurcations occur when a variation in a parameter causes a change in the stability of a fixed point. In continuous-time systems, this corresponds to the real part of an eigenvalue of an equilibrium crossing zero. In discrete-time systems (difference equations), it corresponds to a fixed point having a Floquet multiplier (eigenvalue) with modulus equal to 1. In both cases, the equilibrium is non-hyperbolic at the bifurcation point. Examples of local bifurcations include:
 - **Saddle-Node (Fold) Bifurcation:** this bifurcation occurs when one of the two eigenvalues of the Jacobian matrix is equal to ± 1 . On the diagram of the bifurcations we observe, in this case, a continuous curve of fixed points tangent to the vertical straight line. Two points of fixed exist (one stable and one unstable)

before the bifurcation. After the bifurcation, no more equilibrium exists.

- **Transcritical Bifurcation:** on the diagram of bifurcations this results in two branches different fixed points which intersect at a point and by the change of stability of the two branches passing through the point of intersection.
- **Flip (Period Doubling) Bifurcation:** this bifurcation takes place when one of the two eigenvalues of the Jacobian matrix is equal to -1 . A stable fixed point of order 1, becomes unstable at the same time as the appearance of a cycle of order 2 stable.
- **Neimark-Sacker Bifurcation:** this bifurcation is characterized by the birth (when the bifurcation parameter μ exceeds the bifurcation point μ_0) of a closed invariant curve from a fixed point, when the fixed point changes stability via a pair of complex eigenvalues (equal to $e^{i\theta}$) with a unitary module.

The Main Steps of Studying Bifurcation in Discrete Dynamical Systems

The study of bifurcations in a discrete dynamical system generally involves the following steps:

1. **Finding the Fixed points:** Solve for x^* such that $\mathcal{G}(x^*) = x^*$.
2. **Local linearization:** Linearize the discrete system near an equilibrium point by computing the Jacobian matrix

$$A = \left. \frac{\partial \mathcal{G}}{\partial x} \right|_{x=x^*}.$$

3. **Eigenvalue analysis:** Determine the eigenvalues of the Jacobian matrix to classify the type and stability of the fixed point.
4. **Studying local bifurcations:** Analyze how the system's behavior changes as parameters vary. Common bifurcations in discrete systems include:

- *Fold (saddle-node) bifurcation:* Creation or annihilation of fixed points.
- *Flip (period-doubling) bifurcation:* a fixed point loses stability and a 2-period orbit emerges.
- *Neimark-Sacker (Hopf) bifurcation:* a fixed point loses stability and a closed invariant curve appears.

Remark 1.3.1

- *Bifurcation occurs when a fixed point loses its stability.*
- *The condition for the stability of a fixed point is $|\lambda| < 1$, meaning that all eigenvalues must lie inside the unit circle.*
- *A **Hopf bifurcation**, also called a **Neimark–Sacker bifurcation** in discrete systems, occurs when a pair of complex conjugate eigenvalues cross the unit circle. In this case, a periodic orbit can change into a quasi-periodic orbit.*
- *The **Neimark–Sacker bifurcation** in discrete systems is the discrete-time analogue of the continuous-time **Hopf bifurcation**.*

Lemma 1.3.1 (Existence NaimarkSacker bifurcation [23–25]) Assume that $x_{n+1} = \mathcal{G}_\mu(x_n, \mu)$ is a k -dimensional discrete dynamical system where $\mu \in \mathbb{R}$ is a bifurcation parameter. Let x^* be an fixed point of $\mathcal{G}_\mu$ and suppose that the characteristic polynomial of the Jacobian matrix $J(x^*) = (b_{ij})_{n \times n}$ of k -dimensional map $\mathcal{G}_\mu(x_n, \mu)$ is given by

$$P_\mu(\lambda) = \lambda^k + b_1\lambda^{k-1} + \cdots + b_{k-1}\lambda + b_k, \quad (1.8)$$

where $b_i = b_i(\mu, u)$, $i = 1, 2, 3, \dots, k$ and u is a control parameter or another parameter to be deduced. Let $\Delta_0^\pm(\mu, u) = 1, \Delta_1^\pm(\mu, u), \dots, \Delta_k^\pm(\mu, u)$ be a sequence of the determinants defined by

$$\Delta_i^\pm(\mu, u) = \det(A_1 \pm A_2), \quad i = 1, 2, \dots, k, \quad (1.9)$$

where

$$A_1 = \begin{pmatrix} 1 & b_1 & b_2 & \cdots & b_{i-1} \\ 0 & 1 & b_1 & \cdots & b_{i-2} \\ 0 & 0 & 1 & \cdots & b_{i-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, A_2 = \begin{pmatrix} b_{k-i+1} & b_{k-i+2} & \cdots & b_{k-1} & b_k \\ b_{k-i+2} & b_{k-i+3} & \cdots & b_k & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{k-1} & b_k & \cdots & 0 & 0 \\ b_k & 0 & 0 & \cdots & 0 \end{pmatrix}. \quad (1.10)$$

Moreover, suppose that the following conditions hold

H1- Eigenvalue assignment $\Delta_{k-1}^-(\mu_0, u) = 0$, $\Delta_{k-1}^+(\mu_0, u) > 0$, $P_{\mu_0}(1) > 0$, $(-1)^k P_{\mu_0}(-1) > 0$, $\Delta_i^\pm(\mu, u) > 0$, for $i = k-3, k-5, \dots, 2$ (or 1), when k is odd (or even), respectively.

H2- Transversality condition: $\left[\frac{d\Delta_{k-1}^-(\mu, u)}{d\mu} \right]_{\mu=\mu_0} \neq 0$.

H3- Non-resonance condition: $\cos(2\pi/j) \neq \phi$, or resonance condition $\cos(2\pi/j) = \phi$, where $j = 3, 4, 5, \dots$, and $\phi = 1 - 0.5P_{\mu_0}(1)\Delta_{k-3}^-(\mu_0, u)/\Delta_{k-2}^+(\mu_0, u)$. Then, a Neimark-Sacker bifurcation occurs at μ_0 .

Lemma 1.3.2 (Existence period-doubling bifurcation [23, 26]) Consider the following k -dimensional discrete system:

$$x_{n+1} = \mathcal{G}_\mu(x_n, \mu),$$

where $x_n \in \mathbb{R}^k$ and $\mu \in \mathbb{R}$ represents the bifurcation parameter. Moreover, assume that conditions (1.8)–(1.10) of Lemma 1.3.1 are satisfied and suppose that the following conditions hold:

C1- Eigenvalue criterion: $P_{\mu_0}(-1) = 0$, $\Delta_{k-1}^\pm(\mu_0, u) > 0$, $P_{\mu_0}(1) > 0$, $\Delta_i^\pm(\mu_0, u) > 0$, $i = k-2, k-4, \dots, 1$ (or 1), when k is even (or odd), respectively.

C2- Transversality criterion:

$$\frac{\sum_{i=1}^k (-1)^{k-i} b'_i}{\sum_{i=1}^k (-1)^{k-i} (k-i+1) b_{i-1}} \neq 0,$$

where b'_i denotes derivative of $b(\mu)$ at $\mu = \mu_0$. Then, a period-doubling bifurcation occurs at critical value μ_0 .

1.3.2 Chaos theory

The world around us often seems unpredictable, messy and chaotic. A chaotic system is a simple or complex system, sensitive to the initial conditions and which presents a repetitive character, a strong recurrence. A small disturbance can cause instability or a gigantic unpredictable imbalance in the long term. Thus simple devices can give rise to complex phenomena. A chaotic system is the opposite of a perfectly regular system.

Definition of chaos

Before we state the definition of chaos, some necessary concepts need to be defined [2, 27]. Let $\mathcal{G} : (X, d) \rightarrow (X, d)$, where (X, d) is a metric space and

$$x_{n+1} = \mathcal{G}(x_n), \quad n = 1, 2, 3, \dots, \quad (1.11)$$

Definition 1.3.1 ([20]) The map G is said to have sensitive dependence on the initial conditions, if there exists $\epsilon > 0$ such that for any $x_0 \in X$ and any open U containing x_0 there exists $y_0 \in U$ and $n \in \mathbb{Z}^+$ satisfying

$$d(\mathcal{G}^n(x_0), \mathcal{G}^n(y_0)) > \epsilon.$$

Definition 1.3.2 ([20]) Let $\mathcal{G}$ be a map on a metric space (X, d) . Then f is said to be (topologically) transitive if for any nonempty open sets U and V , there exists a positive integer n such that

$$\mathcal{G}^n(U) \cap U \neq \emptyset.$$

Definition 1.3.3 ([20]) Let X_1 be a subset of an arbitrary set X . Then, X_1 is said to be dense in X if for any $x \in X$, there exists a sequence $(x'_n)_{n \in \mathbb{N}} \in X_1$ such that $\lim_{n \rightarrow \infty} x'_n = x$. i.e, the closure

of X_1 is X .

The following definition of chaos is due to Devaney [27].

Definition 1.3.4 ([20]) *A map $\mathcal{G} : X \rightarrow X$, where X is a metric space, is said to be chaotic if*

- $\mathcal{G}$ is topologically transitive,
- $\mathcal{G}$ possesses sensitivity to initial values,
- The set of periodic points is dense in X .

Despite the fact that there is no commonly recognized concept of chaos, this definition remains the most intriguing since the notions upon which it is found are immediately observed.

1.3.3 Lyapunov Exponents

In the analysis of a dynamical system, the Lyapunov exponent [28–30] allows us to quantify the stability or instability of its trajectories. A Lyapunov exponent is either a finite real number or can take the values $+\infty$ or $-\infty$.

- A motion is **unstable** if it has a positive Lyapunov exponent.
- A motion is **stable** if it has a negative Lyapunov exponent.
- Bounded motions of a linear system have Lyapunov exponents that are either negative or zero.

Lyapunov exponents can also be used to study the stability of fixed points in non-linear systems. In this subsection, we define the notion of the Lyapunov exponent and illustrate how it can be employed to analyze chaotic behavior and even detect the presence of chaos in dynamical systems.

Consider the nonlinear dynamical system

$$x_{n+1} = \mathcal{G}(x_n), \quad x_n \in \mathbb{R}^k. \quad (1.12)$$

We assume that the trajectory originating from an initial state x_0 reaches an attractor, so x_n remains bounded within the attractor.

Let us choose two very close initial conditions, x_0 and $\bar{x}_0$, and observe the evolution of the resulting trajectories. Assuming that the trajectories x_k and $\bar{x}_k$ diverge exponentially, after k iterations we have:

$$|\bar{x}_n - x_n| = |\bar{x}_0 - x_0| e^{\lambda n}, \quad (1.13)$$

where λ indicates the rate of divergence. It can be expressed as

$$\lambda = \frac{1}{n} \ln \left| \frac{\bar{x}_n - x_n}{\bar{x}_0 - x_0} \right|. \quad (1.14)$$

For x_0 and $\bar{x}_0$ very close, letting $\varepsilon = |\bar{x}_0 - x_0| \rightarrow 0$, we define the Lyapunov exponent as

$$\lambda = \lim_{n \rightarrow \infty} \frac{1}{n} \lim_{\varepsilon \rightarrow 0} \ln \left| \frac{\bar{x}_n - x_n}{\bar{x}_0 - x_0} \right|. \quad (1.15)$$

Expanding recursively, we have

$$\lambda = \lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| \frac{\bar{x}_{i+1} - x_{i+1}}{\bar{x}_i - x_i} \right| = \lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| \frac{\mathcal{G}(\bar{x}_i) - \mathcal{G}(x_i)}{\bar{x}_i - x_i} \right|.$$

Finally, using the derivative for infinitesimal differences, we get

$$\lambda = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| \frac{d\mathcal{G}(x_i)}{dx_i} \right|. \quad (1.16)$$

λ , the Lyapunov exponent, measures the average rate of divergence of two trajectories starting from very close initial conditions.

For a system of dimension $k > 1$, there exist k Lyapunov exponents $\lambda^{(j)}$, ($j = 1, 2, \dots, k$), each measuring the rate of divergence along one axis of the phase space. Starting from an initial point $x_0 \in \mathbb{R}^k$, we define the derivative matrix

$$D\mathcal{G}(x_i) = \begin{pmatrix} \frac{\partial \mathcal{G}_1(x_i)}{\partial x_i^1} & \dots & \frac{\partial \mathcal{G}_1(x_i)}{\partial x_i^k} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{G}_n(x_i)}{\partial x_i^1} & \dots & \frac{\partial \mathcal{G}_n(x_i)}{\partial x_i^k} \end{pmatrix}.$$

Define

$$J_n = D\mathcal{G}(x_{n-1}) \cdots D\mathcal{G}(x_0), \text{ with } J_0 = D\mathcal{G}(x_0).$$

Definition 1.3.5 ([20]) For $k = 1$ (one dimensional system), the Lyapunov exponent is defined as

$$\lambda = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n-1} \ln |\mathcal{G}'(x_i)|.$$

For higher-dimensional systems, the Lyapunov exponents are

$$\lambda^{(j)} = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \lambda_j (J_n \cdots J_1), \quad j = 1, 2, \dots, k,$$

where λ_j denotes the j -th eigenvalue of the product of derivative matrices.

The sign of the Lyapunov exponents indicates the system behavior:

- If $\lambda_k \leq \dots \leq \lambda_1 < 0$, the system has asymptotically stable fixed points.
- If $\lambda_1 = 0$ and $\lambda_n \leq \dots \leq \lambda_2 < 0$, the attractor is a stable limit cycle.
- If $\lambda_1 > 0$, the system exhibits chaotic behavior.
- If $\lambda_1 < 0$, the system shows periodic behavior.
- If $\lambda_1 = 0$, a bifurcation occurs.

1.3.4 0-1 test method

The 0–1 test, proposed by Gottwald and Melbourne [31], is an easy and reliable method used to detect chaos in dynamical systems. It can be applied to both continuous and discrete systems, as shown in previous studies [32, 33]. The method analyzes a time series and returns a value between 0 and 1. A value close to 0 indicates regular (non-chaotic) behavior, while a value close to 1 indicates chaotic behavior. Consider a time series

$$\{x(n) \in \mathbb{R} \mid n = 1, 2, \dots, N\}$$

consisting of N data points. The transformation of this time series into the (V_s, W_s) -plane is defined by

$$\begin{aligned} V_s(n) &= \sum_{j=1}^n x(j) \cos(js), \\ W_s(n) &= \sum_{j=1}^n x(j) \sin(js), \end{aligned} \tag{1.17}$$

where $x(j)$ denotes the j -th element of the time series, $j \in \{1, 2, \dots, N\}$, and $s \in (0, \pi)$ is a fixed parameter.

To characterize the evolution of the trajectory in the (V_s, W_s) -plane, Gottwald and Melbourne [31] proposed the mean square displacement (MSD) as a quantitative measure. This indicator has proven to be highly effective in distinguishing between regular and chaotic dynamics. The MSD is defined as

$$M_s(n) = \frac{1}{N} \sum_{j=1}^N \left((V_s(j+n) - V_s(j))^2 + (W_s(j+n) - W_s(j))^2 \right), \quad n \leq \frac{N}{10}. \tag{1.18}$$

To further evaluate the dynamical behavior of the system, the asymptotic growth rate K is calculated as

$$K = \text{median}(K_s), \tag{1.19}$$

where each K_s is computed using the correlation coefficient

$$K_s = \frac{\text{cov}(\phi_1, \phi_2)}{\sqrt{\text{var}(\phi_1) \text{var}(\phi_2)}} \in [-1, 1]. \quad (1.20)$$

with $\phi_1 = \{1, 2, \dots, n_{\text{cut}}\}$ and $\phi_2 = \{M_s(1), M_s(2), \dots, M_s(n)\}$, where

$$n_{\text{cut}} = \text{round}\left(\frac{N}{10}\right),$$

represents a truncation threshold. The variance and covariance of the vectors x and y , each of length $\bar{n}$, are computed as

$$\text{cov}(y, z) = \frac{1}{\bar{n}} \sum_{i=1}^{\bar{n}} (y(i) - \bar{y})(z(i) - \bar{z}), \quad (1.21)$$

with

$$\bar{y} = \frac{1}{\bar{n}} \sum_{j=1}^{\bar{n}} y(j), \quad \text{var}(y) = \text{cov}(y, y). \quad (1.22)$$

When M_s increases, the system tends to exhibit chaotic behavior, with the corresponding the value of K approaches 1. On the other hand, for regular systems(periodic or quasi-periodic) K remains close to 0 as long as M_s remains within a finite range. In the plane (V_s, W_s) , a finite path indicates regular dynamics, while a path resembling Brownian motion indicates that the system is behaving chaotically.

1.4 Fractional derivatives

From a mathematical point of view, the fractional derivative of a function $\mathcal{G}(x)$ of order α , denoted by $D_t^\alpha \mathcal{G}(x)$ or $\frac{d^\alpha \mathcal{G}(x)}{dx^\alpha}$, is defined within the framework of fractional calculus through several operators, such as the Riemann–Liouville, Caputo, and Grünwald–Letnikov derivatives. These operators extend the classical concept of differentiation to non-integer orders by involving integration and differentiation over fractional intervals.

In this section, we introduce some standard definitions of fractional-order derivatives, including the *Riemann–Liouville (R–L)* and *Caputo fractional differential operators*, as well as a recently proposed definition known as *conformable fractional calculus*. In addition, we present several essential theorems that will be employed in the subsequent sections to support our analysis and results.

1.4.1 Caputo fractional derivatives

Definition 1.4.1 ([34]) *Let $\mathcal{G} : (0, \infty) \rightarrow \mathbb{R}$ be a continuous function. The Caputo fractional derivative of order α is defined as*

$${}^C D_t^\alpha \mathcal{G}(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t \frac{\mathcal{G}^{(n)}(\tau)}{(t - \tau)^{\alpha - n + 1}} d\tau, \quad (1.23)$$

where $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ satisfy $n - 1 \leq \alpha < n$. Here, $\Gamma(\cdot)$ denotes the Euler Gamma function.

Proposition 1.4.1 [15] *Let $\mathcal{G}$ and $\mathcal{H}$ be functions whose derivatives of order n are continuous on $(0, \infty)$, and let $a, b \in \mathbb{R}$. Suppose that the Caputo fractional derivatives of order α , ${}^C D_*^\alpha \mathcal{G}$ and ${}^C D_*^\alpha \mathcal{H}$, exist. Then the following properties are satisfied:*

1. Linearity property:

$${}^C D_*^\alpha (a\mathcal{G}(x) + b\mathcal{H}(x)) = a {}^C D_*^\alpha \mathcal{G}(x) + b {}^C D_*^\alpha \mathcal{H}(x).$$

2. Left inverse of the fractional integral:

$${}^C D_*^\alpha J^\alpha \mathcal{G}(x) = \mathcal{G}(x).$$

3. Reduction to the classical derivative: If $\alpha = n \in \mathbb{N}$, then

$${}^C D_*^n (J^{n-\alpha} \mathcal{G}(x)) = \frac{d^n}{dt^n} \mathcal{G}(x).$$

4. Composition with integer-order derivatives: For all $n, m \in \mathbb{N}$,

$${}^C D_*^\alpha D^m(\mathcal{G}(t)) = {}^C D_*^{\alpha+m}(\mathcal{G}(t)) \neq {}^C D_*^{m+\alpha}(\mathcal{G}(t)).$$

1.4.2 Riemann–Liouville fractional derivatives

Definition 1.4.2 ([35]) *The Riemann–Liouville fractional derivative of order α is defined by*

$${}_{RL}D_t^\alpha \mathcal{G}(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \mathcal{G}(\tau) (t-\tau)^{n-\alpha-1} d\tau, \quad (1.24)$$

where $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ satisfy $n-1 \leq \alpha < n$.

Proposition 1.4.2 *Let $\mathcal{G}$ and $\mathcal{H}$ be two functions that are continuous on the interval $(0, b)$, and let $a, b \in \mathbb{R}$. Assume that the Riemann–Liouville fractional derivatives of order α , denoted by ${}_{RL}D^\alpha(\mathcal{G})$ and ${}_{RL}D^\alpha(\mathcal{H})$, exist. Then the following properties hold:*

1. **Linearity:**

$${}_{RL}D^\alpha(a\mathcal{G}(x) + b\mathcal{H}(x)) = a {}_{RL}D^\alpha\mathcal{G}(x) + b {}_{RL}D^\alpha\mathcal{H}(x).$$

2. **Inverse property:**

$${}_{RL}D^\alpha J^\alpha \mathcal{G}(x) = \mathcal{G}(x).$$

3. **Integer-order case:** If $\alpha = n \in \mathbb{N}$, then

$${}_{RL}D^n(J^{n-\alpha}\mathcal{G}(x)) = \frac{d^n}{dt^n}\mathcal{G}(x).$$

Remark 1.4.1 *The Riemann–Liouville and Caputo fractional integrals coincide in their action on sufficiently smooth functions. Therefore, throughout this work, the notation $J^\alpha \mathcal{G}(x)$ will be used to denote either fractional integral interchangeably.*

1.4.3 The conformable fractional derivative

The conformable fractional derivative is a modern formulation in fractional calculus that provides a simpler and more intuitive alternative to classical definitions such as the Riemann–Liouville and Caputo fractional derivatives. This approach was introduced to avoid some of the technical difficulties encountered in traditional fractional differentiation.

Khalil et al. [36] proposed a new definition of fractional calculus that is particularly effective for derivatives of order $\alpha \in (0, 1]$. Although this definition can be extended to arbitrary orders α , the interval $\alpha \in (0, 1]$ plays a fundamental role, since higher-order cases can be derived directly once this case is established.

Definition 1.4.3 ([36]) Let $\mathcal{G} : [0, \infty) \rightarrow \mathbb{R}$ be a given function. The conformable fractional derivative of order $\alpha \in (0, 1]$ of $\mathcal{G}$ at $t > 0$ is defined by

$$T_\alpha \mathcal{G}(t) = \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{G}(t + \varepsilon t^{1-\alpha}) - \mathcal{G}(t)}{\varepsilon}, \quad 0 < \alpha \leq 1. \quad (1.25)$$

If $\mathcal{G}$ is α -differentiable on $(0, a)$ for some $a > 0$ and the limit

$$\lim_{t \rightarrow 0^+} \mathcal{G}^{(\alpha)}(t)$$

exists, then the conformable fractional derivative at $t = 0$ is defined by

$$\mathcal{G}^{(\alpha)}(0) = \lim_{t \rightarrow 0^+} \mathcal{G}^{(\alpha)}(t).$$

Furthermore, as shown in [37], the conformable derivative given in (1.25) admits the representation

$$T_\alpha \mathcal{G}(t) = (t - h)^{1-\alpha} \frac{d\mathcal{G}}{dt}(t), \quad (1.26)$$

where T_α denotes the conformable-type fractional derivative and $h > 0$ is a discretization parameter.

Remark 1.4.2 Whenever the conformable fractional derivative of a function $\mathcal{G}$ of order α exists, the function $\mathcal{G}$ is said to be α -differentiable.

Proposition 1.4.3 ([36]) Let $0 < \alpha \leq 1$ and $\mathcal{G}_1, \mathcal{G}_2$ be α -differentiable at a point $t > 0$. Then the following properties are satisfied:

1. $T_\alpha(a\mathcal{G}_1 + b\mathcal{G}_2) = aT_\alpha(\mathcal{G}_1) + bT_\alpha(\mathcal{G}_2)$, for all $a, b \in \mathbb{R}$.
2. $T_\alpha(t^p) = p t^{p-\alpha}$, for all $p \in \mathbb{R}$.
3. $T_\alpha(\lambda) = 0$, for any constant function $\mathcal{G}(t) = \lambda$.
4. $T_\alpha(\mathcal{G}_1\mathcal{G}_2) = \mathcal{G}_1T_\alpha(\mathcal{G}_2) + \mathcal{G}_2T_\alpha(\mathcal{G}_1)$.
5. $T_\alpha\left(\frac{\mathcal{G}_1}{\mathcal{G}_2}\right) = \frac{\mathcal{G}_1T_\alpha(\mathcal{G}_2) - \mathcal{G}_2T_\alpha(\mathcal{G}_1)}{\mathcal{G}_2^2}$, provided that $\mathcal{G}_2(t) \neq 0$.
6. If, in addition, $\mathcal{G}$ is differentiable, then

$$T_\alpha(\mathcal{G})(t) = t^{1-\alpha} \frac{d\mathcal{G}}{dt}(t).$$

1.4.4 The generalized conformable fractional derivative

All fractional derivative definitions introduced previously, including those given in Definitions (1.4.1) and (1.4.2), satisfy the linearity property. Although these operators possess several useful characteristics, they are not suitable for all applications. In particular, the Riemann–Liouville fractional derivative exhibits certain drawbacks when fractional differential equations are used to model physical or real-world phenomena. One major limitation is that the Riemann–Liouville derivative of a constant does not vanish. Moreover, if a function assumes a constant value at the origin, its fractional derivative may become singular at that point, as observed in functions such as the exponential and Mittag–Leffler functions. Consequently, the applicability of the Riemann–Liouville derivative is somewhat restricted.

On the other hand, the Caputo fractional derivative requires stronger regularity conditions. In particular, the computation of a Caputo derivative presupposes the existence of the classical derivative of the function. Hence, Caputo derivatives are defined only for functions that are at least once differentiable. In contrast, functions that are not classically differentiable may still admit fractional derivatives of order less than one in the sense of Riemann–Liouville (see [38]).

In [36], a relatively simple fractional derivative, known as the conformable derivative, was introduced based solely on the classical limit definition of differentiation. This operator satisfies several fundamental properties that are not shared by the Riemann–Liouville and Caputo derivatives. Nevertheless, it was shown in [39] that the conformable derivative does not always produce results consistent with those obtained using the Caputo definition for certain classes of functions. These observations motivated the introduction of a generalized version of the conformable fractional derivative.

Definition 1.4.4 Let $\mathcal{G}$ be a real-valued function defined on $(0, \infty)$. The generalized fractional derivative of $\mathcal{G}(t)$ of order α , with $0 < \alpha \leq 1$ and $\beta > -1$, $\beta \in \mathbb{R}^+$, is defined for all $t > 0$ by

$${}_{\text{GFD}}T^\alpha \mathcal{G}(t) = \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{G}\left(t + \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha + 1)} \varepsilon t^{1-\alpha}\right) - \mathcal{G}(t)}{\varepsilon}. \quad (1.27)$$

The generalized fractional derivative at $t = 0$ is defined by

$${}_{\text{GFD}}T^\alpha \mathcal{G}(0) = \lim_{t \rightarrow 0^+} {}_{\text{GFD}}T^\alpha \mathcal{G}(t).$$

Theorem 1.4.3 If $\mathcal{G}(t)$ is an α -differentiable function, then

$${}_{\text{GFD}}T^\alpha \mathcal{G}(t) = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha + 1)} t^{1-\alpha} \frac{d\mathcal{G}(t)}{dt}, \quad \beta > -1, \quad 0 < \alpha < 1, \quad \beta \in \mathbb{R}^+. \quad (1.28)$$

Theorem 1.4.4 For the power function $\mathcal{G}(t) = t^k$, where $k \in \mathbb{R}^+$, the following composition rule holds:

$${}_{\text{GFD}}T^\alpha {}_{\text{GFD}}T^\beta t^k = {}_{\text{GFD}}T^{\alpha+\beta} t^k.$$

Theorem 1.4.5 Let $\mathcal{G}(t)$ be a differentiable function admitting a power series expansion about the origin,

$$\mathcal{G}(t) = \sum_{k=0}^{\infty} \frac{\mathcal{G}^{(k)}(0)}{k!} t^k.$$

Then,

$${}_{\text{GFD}}T^{\alpha} {}_{\text{GFD}}T^{\beta} \mathcal{G}(t) = {}_{\text{GFD}}T^{\alpha+\beta} \mathcal{G}(t).$$

Theorem 1.4.6 Let $\alpha \in (0, 1]$ and let $\mathcal{G}$ and $\mathcal{H}$ be α -differentiable functions. Then:

1.

$${}_{\text{GFD}}T^{\alpha} (\mathcal{G}\mathcal{H}) = \mathcal{G} {}_{\text{GFD}}T^{\alpha} (\mathcal{H}) + \mathcal{H} {}_{\text{GFD}}T^{\alpha} (\mathcal{G}).$$

2.

$${}_{\text{GFD}}T^{\alpha} \left(\frac{\mathcal{G}}{\mathcal{H}} \right) = \frac{\mathcal{H} {}_{\text{GFD}}T^{\alpha} (\mathcal{G}) - \mathcal{G} {}_{\text{GFD}}T^{\alpha} (\mathcal{H})}{\mathcal{H}^2}, \quad \mathcal{H}(t) \neq 0.$$

Theorem 1.4.7 (Rolles Theorem for the Generalized Fractional Derivative[36]) Let $a > 0$ and let $\mathcal{G} : [a, b] \rightarrow \mathbb{R}$ satisfy:

1. $\mathcal{G}$ is continuous on $[a, b]$;
2. $\mathcal{G}$ is α -differentiable on (a, b) for some $\alpha \in (0, 1]$;
3. $\mathcal{G}(a) = \mathcal{G}(b)$.

Then there exists $c \in (a, b)$ such that

$${}_{\text{GFD}}T^{\alpha} \mathcal{G}(c) = 0.$$

Theorem 1.4.8 (Mean Value Theorem for the Generalized Fractional Derivative[36])

Let $a > 0$ and let $\mathcal{G} : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and α -differentiable on (a, b) for some $\alpha \in (0, 1)$. Then there exists $c \in (a, b)$ such that

$${}_{\text{GFD}}T^{\alpha} \mathcal{G}(c) = \frac{\mathcal{G}(b) - \mathcal{G}(a)}{h(b^{\alpha} - a^{\alpha})}.$$

Theorem 1.4.9 [36] *Let $\mathcal{G}$ be any continuous function. Then*

$${}_{\text{GFD}}T^\alpha I^\alpha \mathcal{G}(t) = \mathcal{G}(t), \quad t \geq 0.$$

Population dynamics

Population dynamics is a well-established area of biology in which mathematical methods have been extensively applied for many decades. The concept of population dynamics should be interpreted in a broad sense: it not only concerns the evolution of animal, plant, and microbial populations, but also extends to genetic and genomic processes at the cellular and molecular levels. In addition to describing changes in population size over time, population dynamics seeks to explain and predict the underlying mechanisms responsible for observed evolutionary behaviors.

In the following section, we introduce one of the most classical and widely studied models in population dynamics, namely the Lotka–Volterra model [40–42].

2.1 The Lotka–Volterra model

The Lotka–Volterra model is one of the earliest and most well-known mathematical descriptions of predator–prey interactions. It was independently proposed by the American ecologist Alfred J. Lotka in 1925 to describe the interaction between a plant population and a herbivorous species that feeds exclusively on it, and by the Italian mathematician Vito Volterra in 1926 to model the dynamics between sharks and fish in the Adriatic Sea. Owing to its simplicity, the model has become a classical framework

for studying predator–prey dynamics and for introducing several important theoretical concepts in population ecology.

Let $X = X(t)$ denote the density of the prey population and $Y = Y(t)$ denote the density of the predator population at time t . The Lotka–Volterra model can be written as the following system of differential equations:

$$\begin{cases} \frac{dX(t)}{dt} = \alpha X(t) - \gamma X(t)Y(t), \\ \frac{dY(t)}{dt} = Y(t)(e\gamma X(t) - \beta). \end{cases} \quad (2.1)$$

The parameters α, β, γ, e are positive constants. Their biological interpretations are given as follows:

- α denotes the intrinsic growth rate of the prey population in the absence of predators.
- β represents the natural mortality rate of the predator population.
- γ is the predation rate coefficient, describing the frequency of encounters between predators and prey.
- e is the conversion efficiency that measures how effectively consumed prey are transformed into new predators.

The simplicity of the Lotka–Volterra model relies on several idealized assumptions. First, the prey population is assumed to have unlimited resources and therefore grows exponentially in the absence of predators, which is represented by the term $\alpha X(t)$. Second, the predator population depends entirely on the prey for survival; consequently, in the absence of prey the predator population declines exponentially, as represented by the term $-\beta Y(t)$.

Moreover, the model assumes that both species move randomly in a homogeneous environment, leading to predator–prey encounters that are proportional to the product

$X(t)Y(t)$. Among these encounters, a constant fraction γ results in successful predation. Finally, it is assumed that the consumption of prey contributes to predator reproduction with a fixed conversion efficiency e , which leads to the term $e\gamma X(t)Y(t)$ in the predator equation.

Although the Lotka–Volterra model provides a simplified description of predator–prey interactions and does not capture all ecological complexities observed in real ecosystems, it remains a fundamental framework for understanding population dynamics and for developing more sophisticated ecological models.

2.2 Predator–prey model

Following the classical Lotka–Volterra formulation, many studies have attempted to improve the model in order to overcome some of the limitations discussed in Section 2.1. A more general framework for predator–prey interactions was proposed by Yodzis in 1994 (see [43]). The model can be written in the following form:

$$\begin{cases} \frac{dX(t)}{dt} = F(X(t)) - Y(t) G_1(X(t), Y(t)), \\ \frac{dY(t)}{dt} = Y(t) G_2(X(t), Y(t)). \end{cases} \quad (2.2)$$

In this formulation, the functions have the following biological interpretations:

- $F(X(t))$ represents the intrinsic growth of the prey population in the absence of predators.
- $G_1(X(t), Y(t))$ denotes the predator functional response, which describes the number of prey individuals consumed by a single predator per unit time and reflects how the consumption rate depends on prey density.
- $G_2(X(t), Y(t))$ represents the predator numerical response, corresponding to the per capita growth rate of the predator population as a function of prey availability.

Although model (2.2) provides a general description of predator–prey interactions, it remains quite abstract since the specific forms of the functions F , G_1 , and G_2 are not specified. To obtain a more tractable model capable of describing the biological interactions in a simpler way, particular functional forms are often chosen. One well-known example is the Rosenzweig–MacArthur model, given by

$$\begin{cases} \frac{dX(t)}{dt} = F(X(t)) - X(t) Y(t) G(X(t)), \\ \frac{dY(t)}{dt} = Y(t) (e X(t) G(X(t)) - \beta). \end{cases} \quad (2.3)$$

Although predator–prey models represent simplified descriptions of ecological interactions, they provide valuable insight into the mechanisms governing the dynamics and interdependence of predator and prey populations. Such models are widely used in ecology, population biology, and conservation science to investigate the impact of predation and to analyze the factors that influence the stability of ecosystems.

2.3 Functional response

The concept of functional response was introduced by Holling to describe the relationship between prey density and the rate at which predators consume prey. It plays an important role in predator–prey theory because it characterizes how the feeding rate of predators varies with the availability of prey. Functional responses are influenced by several biological and environmental factors, including prey abundance, predator density, and behavioral interactions between species.

Different types of functional responses have been proposed in the literature to capture various biological mechanisms in predator–prey systems. Some of the most commonly used forms are summarized below.

1. **Holling type I functional response** [44, 45]

$$G(X) = \begin{cases} \gamma X, & X \leq X_0, \\ \gamma X_0, & X > X_0, \end{cases}$$

where $\gamma > 0$. This response assumes a linear increase of the predation rate with prey density until a saturation level is reached.

2. **Holling type II functional response** (Michaelis–Menten form) [44, 46]

$$G(X) = \frac{\gamma X}{1 + wX},$$

where $\gamma, w > 0$. The predation rate increases with prey density but eventually approaches a maximum level due to handling time.

3. **Holling type III functional response**

$$G(X) = \frac{\gamma X^p}{b + X^p},$$

where $\gamma, b, p > 0$. When $p = 2$, the function represents the classical Holling type III response. This sigmoidal response reflects slow predation at low prey densities and rapid increase at intermediate densities.

4. **Holling type IV functional response** [44, 47]

$$G(X) = \frac{\gamma X}{a + bX + X^2},$$

where $\gamma, a, b > 0$. This response allows the predation rate to decrease when prey density becomes very large.

5. **Beddington–DeAngelis functional response** [47]

$$G(X, Y) = \frac{\gamma X}{a + bX + cY},$$

where $\gamma, a, b, c > 0$. This formulation incorporates predator interference and avoids the singular behavior observed in some ratio-dependent models.

6. Hassell–Varley functional response [47]

$$G(X, Y) = \frac{\gamma X}{X + mY^\sigma},$$

where $\gamma, m > 0$ and σ is the interference parameter describing predator competition.

7. Ivlev functional response

$$G(X) = \gamma(1 - e^{-aX}),$$

where $\gamma, a > 0$. This response approaches saturation exponentially as prey density increases.

8. Ratio-dependent functional response

$$G(X, Y) = \frac{\gamma X}{aY + X},$$

where $\gamma, a > 0$. In this case, the predation rate depends on the ratio between prey and predator densities.

Among these functional responses, the Holling type II response is one of the most frequently used in predator–prey models because of its biological realism and mathematical simplicity.

Qualitative analysis of a discretized generalized conformable fractional-order ratio-dependent Holling-Tanner predator-prey model incorporating the Fear effect in the prey population

3.1 Introduction

The analysis of nonlinear dynamical systems arising in science and engineering is essential for understanding complex real-world phenomena. In ecology, population dynamics models play a central role, particularly in describing prey–predator interactions, which fundamentally shape ecosystem structure and function [48]. Recent studies have shown that environmental changes, such as ocean acidification, significantly influence these interactions by impairing prey defense mechanisms, for example, reducing the ability of sea snails to escape sea star predation [49, 50].

The classical Lotka–Volterra model, independently proposed by Lotka (1925) and

Volterra (1926), remains one of the most fundamental frameworks for modeling predator-prey dynamics. Although simple, this system of ordinary differential equations captures essential ecological features and exhibits sensitivity to parameter variations, making it a cornerstone for the development of more advanced models [51].

In addition to direct predation, the fear effect has emerged as a crucial factor influencing prey populations. Fear-induced behavioral changes, such as reduced foraging, lower reproduction rates, and habitat avoidance, can significantly alter population dynamics and may even exceed the impact of direct predation [52, 53]. Incorporating fear into predator-prey models has been shown to modify stability conditions, suppress oscillations, and generate new bifurcation patterns, underscoring its ecological importance [54].

Despite substantial progress in the study of fractional-order systems, discrete-time models, and ratio-dependent Holling-Tanner dynamics, a comprehensive framework integrating generalized conformable incommensurate fractional-order dynamics, discretization, ratio-dependent predation, and fear effects remains underexplored. In particular, the combined influence of fear and fractional-order dynamics on the stability and bifurcation structure of discrete Holling-Tanner systems has not been fully addressed.

The Holling-Tanner model is widely recognized for its ability to describe complex predator-prey interactions, exhibiting both stable and unstable behaviors under varying ecological conditions [55]. Recent studies have extended this model to include fear effects, ratio dependence, and discrete-time formulations, revealing rich dynamical behaviors such as period-doubling and Neimark-Sacker bifurcations [56]. Additionally, the inclusion of the Allee effect has been shown to significantly influence system stability and bifurcation characteristics [56]. Further investigations into delayed and fractional-order systems have demonstrated that the interplay between memory effects and time delays can generate complex oscillatory and chaotic dynamics [57, 58].

Moreover, fractional-order Holling-Tanner models with harvesting have been proposed to better capture ecological realism. For instance, Suryanto et al. [59] established

the existence, uniqueness, and boundedness of solutions, demonstrating the effectiveness of fractional calculus in modeling ecological interactions and enhancing our understanding of predator-prey dynamics.

Motivated by the above results, this chapter proposes and analyzes a discretized generalized conformable fractional-order ratio-dependent Holling-Tanner predator-prey system incorporating fear effects in the prey population. The main objective is to investigate the emergence of complex dynamics under variations of key biological and fractional parameters, including the existence and stability of fixed points, as well as the occurrence of local bifurcations. Furthermore, the theoretical findings are validated through numerical simulations, which demonstrate a wide range of dynamical behaviors arising from the interplay among discretization, fractional-order dynamics, and fear-mediated ecological interactions.

3.2 Formulation of the model

We consider a ratio-dependent Holling-Tanner predator-prey model incorporating the **fear effect** in the prey population. The continuous-time fractional-order system is given by:

$$\begin{cases} {}^{\text{GFD}}T_S^\alpha P_1(t) = RP_1(t) \left(\frac{1}{1 + mP_2(t)} \right) \left(1 - \frac{P_1(t)}{\kappa} \right) - \frac{\epsilon P_2(t)P_1(t)}{P_2(t) + P_1(t)}, \\ {}^{\text{GFD}}T_S^\alpha P_2(t) = CP_2(t) \left(1 - \frac{DP_2(t)}{P_1(t)} \right). \end{cases} \quad (3.1)$$

All parameter values in the model are assumed to be positive, where:

- $P_1(t), P_2(t)$: population densities of prey and predator species at time t ,
- ${}^{\text{GFD}}T_S^\alpha$: Generalized Conformable Fractional Derivative of order α ,
- R : intrinsic growth rate of the prey,
- κ : carrying capacity of the prey environment,

- $m > 0$: intensity of the fear effect,
- ϵ : maximum per capita consumption rate of the predator,
- C : intrinsic growth rate of the predator,
- D : conversion efficiency parameter.

The predators functional response is described by a ratio-dependent Holling type II form:

$$\frac{\epsilon P_1}{P_1 + P_2},$$

where ϵ denotes the maximum per capita consumption rate.

The fear effect acting on the prey population is modeled by:

$$\frac{1}{1 + mP_2},$$

where $m > 0$ measures the intensity of fear induced by predators.

By applying the same procedure to the fractional-order model (3.1) under the method of piecewise constant argument [6, 60], we arrive at the following discrete-time formulation. Starting from the system

$$\begin{cases} {}^{GFD}T_S^\alpha P_1(t) = P_1(t) \left(R \left(\frac{1}{1 + mP_2\left(\left[\frac{t}{S}\right]S\right)} \right) \left(1 - \frac{P_1\left(\left[\frac{t}{S}\right]S\right)}{\kappa} \right) - \frac{\epsilon P_2\left(\left[\frac{t}{S}\right]S\right)}{P_2\left(\left[\frac{t}{S}\right]S\right) + P_1\left(\left[\frac{t}{S}\right]S\right)} \right), \\ {}^{GFD}T_S^\alpha P_2(t) = CP_2(t) \left(1 - \frac{DP_2\left(\left[\frac{t}{S}\right]S\right)}{P_1\left(\left[\frac{t}{S}\right]S\right)} \right), \end{cases} \quad (3.2)$$

where $\left[\frac{t}{S}\right]$ is the integer part of $t \in [nS, (n+1)S)$.

Substituting the definition of the fractional derivative (1.28) into the first equation yields a separable differential equation:

$$\frac{P_1'(t)}{P_1(t)} = \left(R \left(\frac{1}{1 + mP_2(nS)} \right) \left(1 - \frac{P_1(nS)}{\kappa} \right) - \frac{\epsilon P_2(nS)}{P_2(nS) + P_1(nS)} \right) \frac{\Gamma(\beta - \alpha + 1)}{\Gamma(\beta)(t - nS)^{1-\alpha}}.$$

Integrating from nS to t , we obtain:

$$\ln P_1(t) - \ln P_1(nS) = \left(R \left(\frac{1}{1 + mP_2(nS)} \right) \left(1 - \frac{P_1(nS)}{\kappa} \right) - \frac{\epsilon P_2(nS)}{P_2(nS) + P_1(nS)} \right) \frac{\Gamma(\beta - \alpha + 1)}{\alpha \Gamma(\beta)} (t - nS)^\alpha.$$

Hence,

$$P_1(t) = P_1(nS) \exp \left(\left(R \left(\frac{1}{1 + mP_2(nS)} \right) \left(1 - \frac{P_1(nS)}{\kappa} \right) - \frac{\epsilon P_2(nS)}{P_2(nS) + P_1(nS)} \right) \frac{\Gamma(\beta - \alpha + 1)}{\alpha \Gamma(\beta)} (t - nS)^\alpha \right).$$

Letting $t \rightarrow (n + 1)S$ and setting $P_1(nS) = P_{1,n}$ and $P_2(nS) = P_{2,n}$, we obtain:

$$P_{1,n+1} = P_{1,n} \exp \left(\left(R \left(\frac{1}{1 + mP_{2,n}} \right) \left(1 - \frac{P_{1,n}}{\kappa} \right) - \frac{\epsilon P_{2,n}}{P_{2,n} + P_{1,n}} \right) \frac{\Gamma(\beta - \alpha + 1)S^\alpha}{\alpha \Gamma(\beta)} \right). \quad (3.3)$$

Similarly, for the predator equation we have:

$$\frac{P_2'(t)}{P_2(t)} = C \left(1 - \frac{DP_2(nS)}{P_1(nS)} \right) \frac{\Gamma(\beta - \alpha + 1)}{\Gamma(\beta)} (t - nS)^{1-\alpha}.$$

Integrating from nS to t gives:

$$\ln \frac{P_2(t)}{P_2(nS)} = C \left(1 - \frac{DP_2(nS)}{P_1(nS)} \right) \frac{\Gamma(\beta - \alpha + 1)}{\alpha \Gamma(\beta)} (t - nS)^\alpha.$$

Thus,

$$P_2(t) = P_2(nS) \exp \left(C \left(1 - \frac{DP_2(nS)}{P_1(nS)} \right) \frac{\Gamma(\beta - \alpha + 1)}{\alpha \Gamma(\beta)} (t - nS)^\alpha \right).$$

Letting $t \rightarrow (n + 1)S$, we obtain:

$$P_{2,n+1} = P_{2,n} \exp \left(C \left(1 - \frac{DP_{2,n}}{P_{1,n}} \right) \frac{\Gamma(\beta - \alpha + 1)S^\alpha}{\alpha \Gamma(\beta)} \right). \quad (3.4)$$

Combining (3.3) and (3.4), the resulting discrete-time system is:

$$\begin{cases} P_{1,n+1} = P_{1,n} \exp \left(\left(R \left(\frac{1}{1 + mP_{2,n}} \right) \left(1 - \frac{P_{1,n}}{\kappa} \right) - \frac{\epsilon P_{2,n}}{P_{2,n} + P_{1,n}} \right) \frac{\Gamma(\beta - \alpha + 1)S^\alpha}{\alpha\Gamma(\beta)} \right), \\ P_{2,n+1} = P_{2,n} \exp \left(C \left(1 - \frac{DP_{2,n}}{P_{1,n}} \right) \frac{\Gamma(\beta - \alpha + 1)S^\alpha}{\alpha\Gamma(\beta)} \right), \end{cases} \quad (3.5)$$

where S is the discretization step size.

3.3 Fixed points

In this section, we investigate the fixed points of the discrete system (3.5), which play a fundamental role in understanding the long-term dynamical behavior of the model. Fixed points represent equilibrium states where both predator and prey populations remain constant over time. The existence and nature of these equilibria provide important insights into the persistence, extinction, or coexistence of species within the ecological system. Accordingly, we derive all possible fixed points and discuss the conditions under which they exist.

Let (P_1^*, P_2^*) be a fixed point of system (3.5). Then

$$P_{1,n+1} = P_1^*, \quad P_{2,n+1} = P_2^*.$$

From system (3.5), we obtain the fixed point equations:

$$\begin{cases} P_1^* \exp \left(\mathcal{A} \left[R \left(\frac{1}{1 + mP_2^*} \right) \left(1 - \frac{P_1^*}{\kappa} \right) - \frac{\epsilon P_2^*}{P_1^* + P_2^*} \right] \right) = P_1^*, \\ P_2^* \exp \left(\mathcal{A} C \left(1 - \frac{DP_2^*}{P_1^*} \right) \right) = P_2^*, \end{cases}$$

where

$$\mathcal{A} = \frac{\Gamma(\beta - \alpha + 1)S^\alpha}{\alpha\Gamma(\beta)}.$$

- **Predator-extinction fixed point:** Setting $P_2 = 0$, the first equation reduces to

$$P_1^* \exp\left(\mathcal{A}R\left(1 - \frac{P_1^*}{\kappa}\right)\right) = P_1^*,$$

which yields $P_1^* = \kappa$. Hence, the trivial fixed point is

$$O = (\kappa, 0).$$

- **Coexistence fixed point:** For $P_1^* > 0$ and $P_2^* > 0$, the exponential terms must equal one, leading to

$$\begin{cases} R\left(\frac{1}{1 + mP_2^*}\right)\left(1 - \frac{P_1^*}{\kappa}\right) - \frac{\epsilon P_2^*}{P_2^* + P_1^*} = 0, \\ 1 - \frac{DP_2^*}{P_1^*} = 0. \end{cases}$$

Solving this system yields the positive equilibrium point

$$E = (P_1^*, P_2^*) = \left(\frac{\kappa D(R(D+1) - \epsilon)}{RD(D+1) + \epsilon m \kappa}, \frac{\kappa(R(D+1) - \epsilon)}{RD(D+1) + \epsilon m \kappa} \right),$$

which exists whenever

$$R(D+1) > \epsilon.$$

3.4 Existence and stability of fixed points

In this section, we investigate the local stability of the fixed points of system (3.5). The stability analysis plays a crucial role in understanding the qualitative behavior of the dynamical system in the neighborhood of equilibrium states. In particular, it determines whether small perturbations around a fixed point will decay over time or lead to divergent or oscillatory dynamics.

To this end, we employ the Jacobian matrix method. By evaluating the Jacobian matrix at each fixed point, we derive the corresponding eigenvalues, which provide

sufficient conditions for local stability, instability, or bifurcation behavior. This analysis allows us to characterize the dynamical nature of both the extinction and coexistence equilibria of the system.

The Jacobian matrix of system (3.5) at a fixed point (P_1, P_2) is given by

$$J(P_1, P_2) = \begin{pmatrix} \frac{\partial \mathcal{F}}{\partial P_1} & \frac{\partial \mathcal{F}}{\partial P_2} \\ \frac{\partial \mathcal{G}}{\partial P_1} & \frac{\partial \mathcal{G}}{\partial P_2} \end{pmatrix},$$

where

$$\mathcal{F}(P_1, P_2) = P_1 \exp \left(\mathcal{A} \left[R \left(\frac{1}{1 + mP_2} \right) \left(1 - \frac{P_1}{\kappa} \right) - \frac{\epsilon P_2}{P_2 + P_1} \right] \right),$$

$$\mathcal{G}(P_1, P_2) = P_2 \exp \left(\mathcal{A} C \left(1 - \frac{DP_2}{P_1} \right) \right).$$

Let

$$\mathcal{A} = \frac{\Gamma(\beta - \alpha + 1) S^\alpha}{\alpha \Gamma(\beta)}.$$

The partial derivatives are computed as follows:

$$\frac{\partial \mathcal{F}}{\partial P_1} = \mathcal{F}(P_1, P_2) \left(\frac{1}{P_1} + \mathcal{A} \left(-\frac{R}{\kappa(1 + mP_2)} + \frac{\epsilon P_2}{(P_2 + P_1)^2} \right) \right),$$

$$\frac{\partial \mathcal{F}}{\partial P_2} = \mathcal{F}(P_1, P_2) \left(\mathcal{A} \left(-\frac{Rm \left(1 - \frac{P_1}{\kappa} \right)}{(1 + mP_2)^2} - \frac{\epsilon P_1}{(P_1 + P_2)^2} \right) \right),$$

$$\frac{\partial \mathcal{G}}{\partial P_1} = \mathcal{G}(P_1, P_2) \mathcal{A} C \left(\frac{DP_2}{P_1^2} \right),$$

$$\frac{\partial \mathcal{G}}{\partial P_2} = \mathcal{G}(P_1, P_2) \left(\frac{1}{P_2} - \mathcal{A} C \left(\frac{D}{P_1} \right) \right).$$

Hence, the Jacobian matrix can be written in compact form as

$$J(P_1, P_2) = \begin{pmatrix} \mathcal{F}\left(\frac{1}{P_1} + \mathcal{A}\left(-\frac{R}{\kappa(1+mP_2)} + \frac{\epsilon P_2}{(P_2+P_1)^2}\right)\right) & \mathcal{F}\left(\mathcal{A}\left(-\frac{Rm\left(1-\frac{P_1}{\kappa}\right)}{(1+mP_2)^2} - \frac{\epsilon P_1}{(P_2+P_1)^2}\right)\right) \\ \mathcal{G}\mathcal{A}\mathcal{C}\left(\frac{DP_2}{P_1^2}\right) & \mathcal{G}\left(\frac{1}{P_2} - \mathcal{A}\mathcal{C}\left(\frac{D}{P_1}\right)\right) \end{pmatrix}.$$

3.4.1 Stability of the trivial fixed point O

The Jacobian matrix of system (3.5) evaluated at the trivial fixed point $O = (\kappa, 0)$ is given by

$$J(\kappa, 0) = \begin{pmatrix} 1 - R\mathcal{A} & -\epsilon\mathcal{A} \\ 0 & \exp(C\mathcal{A}) \end{pmatrix},$$

where

$$\mathcal{A} = \frac{\Gamma(\beta - \alpha + 1) S^\alpha}{\alpha \Gamma(\beta)}.$$

Since the matrix is upper triangular, its eigenvalues are

$$\lambda_1 = 1 - R\mathcal{A}, \quad \lambda_2 = \exp(C\mathcal{A}) > 1.$$

Hence, the stability of O depends on λ_1 , since λ_2 is always outside the unit circle.

- **Case 1:** $|\lambda_1| < 1$

$$|1 - R\mathcal{A}| < 1$$

which gives

$$-1 < 1 - R\mathcal{A} < 1 \iff 0 < R\mathcal{A} < 2 \iff 0 < R < \frac{2}{\mathcal{A}}.$$

In this case, $|\lambda_1| < 1$ and $|\lambda_2| > 1$, so the fixed point O is a saddle point.

- **Case 2:** $|\lambda_1| > 1$

$$|1 - R\mathcal{A}| > 1 \iff R\mathcal{A} < 0 \text{ or } R\mathcal{A} > 2.$$

Since all parameters are positive, only $R\mathcal{A} > 2$ is admissible, i.e.,

$$R > \frac{2}{\mathcal{A}}.$$

In this case, $|\lambda_1| > 1$ and $|\lambda_2| > 1$, so O is a source.

- **Case 3:** $|\lambda_1| = 1$

$$|1 - R\mathcal{A}| = 1 \iff R\mathcal{A} = 2 \iff R = \frac{2}{\mathcal{A}}.$$

In this case, the fixed point O is non-hyperbolic.

From the above analysis, we obtain the following theorem.

Theorem 3.4.1 *Assume that all model parameters are positive. Then the extinction fixed point O of system (3.5) is classified as follows:*

1. O is a saddle point if $0 < R < \frac{2}{\mathcal{A}}$,
2. O is a source if $R > \frac{2}{\mathcal{A}}$,
3. O is non-hyperbolic if $R = \frac{2}{\mathcal{A}}$.

3.4.2 Stability of the coexistence fixed point E

The Jacobian matrix of system (3.5) evaluated at the coexistence fixed point $E = (P_1^*, P_2^*)$ is given by

$$J(P_1^*, P_2^*) = \begin{pmatrix} 1 + \mathcal{A}P_1^* \left(-\frac{R}{\kappa(1 + mP_2^*)} + \frac{\epsilon}{(D+1)^2 P_2^*} \right) & \mathcal{A}P_1^* \left(-\frac{Rm \left(1 - \frac{P_1^*}{\kappa}\right)}{(1 + mP_2^*)^2} - \frac{\epsilon D}{(D+1)^2 P_2^*} \right) \\ \frac{C\Gamma(\beta - \alpha + 1)S^\alpha}{D\alpha\Gamma(\beta)} & 1 - \frac{C\Gamma(\beta - \alpha + 1)S^\alpha}{\alpha\Gamma(\beta)} \end{pmatrix},$$

where

$$\mathcal{A} = \frac{\Gamma(\beta - \alpha + 1)S^\alpha}{\alpha\Gamma(\beta)}.$$

The characteristic equation of $J(P_1^*, P_2^*)$ is

$$\Psi(\lambda) = \lambda^2 - \text{Tr}(J)\lambda + \det(J) = 0, \quad (3.6)$$

where

$$\begin{aligned} \text{Tr}(J) &= 2 - \mathcal{A} \left(\frac{RDP_2^*}{\kappa(1 + mP_2^*)} - \frac{\epsilon D}{(D+1)^2} \right) - \frac{C\Gamma(\beta - \alpha + 1)S^\alpha}{\alpha\Gamma(\beta)}, \\ &= 2 - \frac{\Gamma(\beta - \alpha + 1)S^\alpha}{\alpha\Gamma(\beta)} \left(\frac{RDP_2^*}{\kappa(1 + mP_2^*)} - \frac{\epsilon D}{(D+1)^2} + C \right) \end{aligned}$$

and

$$\begin{aligned} \det(J) &= \left(1 - \mathcal{A} \left(\frac{RDP_2^*}{\kappa(1 + mP_2^*)} - \frac{\epsilon D}{(D+1)^2} \right) \right) \left(1 - \frac{C\Gamma(\beta - \alpha + 1)S^\alpha}{\alpha\Gamma(\beta)} \right) + \frac{C\Gamma(\beta - \alpha + 1)S^\alpha}{D\alpha\Gamma(\beta)} \\ &\quad \mathcal{A}\epsilon \left(\frac{mDP_2^*}{(D+1)(1 + mP_2^*)} + \frac{D^2}{(D+1)^2} \right), \\ &= 1 + \left(\frac{\Gamma(\beta - \alpha + 1)S^\alpha}{\alpha\Gamma(\beta)} \right)^2 \left(\frac{DCRP_2^*}{\kappa(1 + mP_2^*)} + \frac{m\epsilon CP_2^*}{(D+1)(1 + mP_2^*)} \right) - \left(\frac{\Gamma(\beta - \alpha + 1)S^\alpha}{\alpha\Gamma(\beta)} \right) \\ &\quad \left(\frac{RDP_2^*}{\kappa(1 + mP_2^*)} - \frac{\epsilon D}{(D+1)^2} + C \right). \end{aligned}$$

The discriminant of (3.6) is

$$\text{Disc} = \mathcal{A}^2 \mathbb{D}, \quad \text{where} \quad \mathbb{D} = \mathbb{Q}^2 - 4\mathbb{P},$$

with

$$\mathbb{Q} = \left(\frac{RDP_2^*}{\kappa(1 + mP_2^*)} - \frac{\epsilon D}{(D + 1)^2} + C \right), \quad \text{and} \quad \mathbb{P} = \left(\frac{DCRP_2^*}{\kappa(1 + mP_2^*)} + \frac{m\epsilon CP_2^*}{(D + 1)(1 + mP_2^*)} \right).$$

Let λ_1, λ_2 be the eigenvalues of J . The stability of E is determined using Lemma 1.2.3.

We compute:

$$\begin{aligned} \Psi(0) &= \text{Det}(J), \quad \Psi(1) = \left(\frac{\Gamma(\beta - \alpha + 1) S^\alpha}{\alpha \Gamma(\beta)} \right)^2 \left(\frac{DCRP_2^*}{\kappa(1 + mP_2^*)} + \frac{m\epsilon CP_2^*}{(D + 1)(1 + mP_2^*)} \right), \\ \Psi(-1) &= 4 + \left(\frac{\Gamma(\beta - \alpha + 1) S^\alpha}{\alpha \Gamma(\beta)} \right)^2 \left(\frac{DCRP_2^*}{\kappa(1 + mP_2^*)} + \frac{m\epsilon CP_2^*}{(D + 1)(1 + mP_2^*)} \right) - 2 \left(\frac{\Gamma(\beta - \alpha + 1) S^\alpha}{\alpha \Gamma(\beta)} \right) \\ &\quad \left(\frac{RDP_2^*}{\kappa(1 + mP_2^*)} - \frac{\epsilon D}{(D + 1)^2} + C \right). \end{aligned}$$

By applying Lemma 1.2.3, we obtain the following conditions:

$$\Psi(-1) = 0, \tag{*}$$

$$\Psi(0) - 1 = 0. \tag{**}$$

Equation (*) leads to the quadratic equation in $\mathcal{A}$:

$$\mathcal{A}^2 \mathbb{P} - 2\mathcal{A}\mathbb{Q} + 4 = 0.$$

Its discriminant is

$$\Delta = 4(Q^2 - 4P) = 4D.$$

- If $D > 0$: the equation admits two distinct real solutions

$$\mathcal{A}_{1,2} = \frac{Q \pm \sqrt{D}}{2P}.$$

Hence,

$$S_{1,2} = \left(\frac{\alpha \Gamma(\beta)(Q \mp \sqrt{D})}{P \Gamma(\beta - \alpha + 1)} \right)^{1/\alpha}.$$

- If $D = 0$: a double root exists

$$S_0 = \left(\frac{\alpha \Gamma(\beta)Q}{P \Gamma(\beta - \alpha + 1)} \right)^{1/\alpha}.$$

- If $D < 0$: no real roots exist, and $\Psi(-1)$ does not change sign.

From equation (**), we obtain

$$\mathcal{A}^2 P - \mathcal{A} Q = 0 \quad \Rightarrow \quad \mathcal{A} P = Q \quad \Rightarrow \quad \mathcal{A} = \frac{Q}{P}.$$

Thus,

$$S_0 = \left(\frac{\alpha \Gamma(\beta) Q}{P \Gamma(\beta - \alpha + 1)} \right)^{1/\alpha}.$$

Applying Lemma 1.2.3, we conclude the following result.

Theorem 3.4.2 *Consider the unique positive fixed point E of system (3.5). Suppose that $R(D + 1) > \epsilon$ and introduce*

$$\begin{aligned} D &= \left(\frac{RDP_2^*}{\kappa(1 + mP_2^*)} - \frac{\epsilon D}{(D + 1)^2} + C \right)^2 - 4 \left(\frac{DCRP_2^*}{\kappa(1 + mP_2^*)} + \frac{m\epsilon CP_2^*}{(D + 1)(1 + mP_2^*)} \right), \\ Q &= \left(\frac{RDP_2^*}{\kappa(1 + mP_2^*)} - \frac{\epsilon D}{(D + 1)^2} + C \right), \quad P = \left(\frac{DCRP_2^*}{\kappa(1 + mP_2^*)} + \frac{m\epsilon CP_2^*}{(D + 1)(1 + mP_2^*)} \right). \end{aligned}$$

Then the stability properties of E are described as follows.

i- The fixed E is locally asymptotically stable (that is, it behaves as a sink) in either of the following cases:

1. $\mathbb{D} < 0$ and

$$0 < S \leq S_0,$$

2. $\mathbb{D} \geq 0$ and

$$0 < S \leq S_1.$$

ii- The fixed E becomes unstable (source) if one of the following conditions is satisfied:

1. $\mathbb{D} < 0$ and

$$S > S_0,$$

2. $\mathbb{D} \geq 0$ and

$$S > S_2.$$

iii- When $\mathbb{D} \geq 0$ and the parameter S lies in the interval

$$S_1 < S \leq S_2,$$

the fixed E is unstable and takes the form of a saddle point.

iv- The fixed E is non-hyperbolic under either of the following situations:

1. $\mathbb{D} < 0$ and

$$S = S_0,$$

2. $\mathbb{D} \geq 0$ and

$$S = S_{1,2},$$

provided that

$$S \neq \left(\frac{2\alpha\Gamma(\beta)}{Q\Gamma(\beta - \alpha + 1)} \right)^{1/\alpha}.$$

3.5 Local bifurcation analysis

In this section, we investigate the existence and stability of period-doubling and Neimark–Sacker bifurcations occurring at the equilibrium point E . In particular, the discretization parameter S is considered as the bifurcation parameter throughout the analysis.

3.5.1 Flip bifurcation about E

Using Lemma 1.3.2 and the characteristic polynomial (3.6), we obtain the following conditions for a flip (period-doubling) bifurcation:

$$\begin{aligned}\Delta_0^\mp(S) &= 1 > 0, \\ \Delta_1^\pm(S) &= 1 \pm \det(J) > 0, \\ \Psi(1) &= 1 - \text{Tr}(J) + \det(J) > 0, \\ \Psi(-1) &= 1 + \text{Tr}(J) + \det(J) = 0.\end{aligned}$$

The condition $\Psi(-1) = 0$ yields

$$1 + \text{Tr}(J) + \det(J) = 0.$$

Substituting $\text{Tr}(J)$ and $\det(J)$, we obtain

$$\mathcal{A}^2 \mathbb{P} - 2\mathcal{A}\mathbb{Q} + 4 = 0,$$

which gives the critical values

$$S_{1,2} = \left(\frac{\alpha \Gamma(\beta) (\mathbb{Q} \mp \sqrt{\mathbb{D}})}{\mathbb{P} \Gamma(\beta - \alpha + 1)} \right)^{1/\alpha},$$

where $\mathbb{D} = \mathbb{Q}^2 - 4\mathbb{P}$.

From $\Delta_1^\pm(S) = 1 \pm \det(J) > 0$, we obtain

$$-1 < \det(J) < 1.$$

Hence,

$$-2 < \mathcal{A}^2 \mathbb{P} - \mathcal{A} \mathbb{Q} < 0.$$

The transversality condition is given by

$$\frac{\sum_{i=1}^k (-1)^{k-i} b'_i}{\sum_{i=1}^k (-1)^{k-i} (k-i+1) b_{i-1}} \neq 0$$

which can be written as

$$\frac{2\alpha \mathbb{P} \Gamma(\beta - \alpha + 1)^2 S^{2\alpha-1}}{\alpha^2 \Gamma(\beta)^2 \mathbb{Q}} \neq 0.$$

Let

$$b_1 = \text{Tr}(J), \quad b_2 = \det(J).$$

Then

$$b'_1 = -\frac{\mathbb{Q} \Gamma(\beta - \alpha + 1)}{\Gamma(\beta)} S^{\alpha-1},$$

and

$$b'_2 = \frac{2\mathbb{P} \Gamma(\beta - \alpha + 1)^2}{\Gamma(\beta)^2} S^{2\alpha-1} - \frac{\mathbb{Q} \Gamma(\beta - \alpha + 1)}{\Gamma(\beta)} S^{\alpha-1}.$$

Moreover,

$$-2b_0 + b_1 = -\mathcal{A} \mathbb{Q} \neq 0.$$

Therefore, all conditions of the flip bifurcation theorem are satisfied. Hence, a period-doubling bifurcation occurs at $S = S_{1,2}$ when the system parameters $(R, \kappa, m, \epsilon, C, D, \alpha, \beta)$

lie in a neighborhood of

$$\mathbb{B}_F = \left\{ (R, \kappa, m, \epsilon, C, D, \alpha, \beta) \in \mathbb{R}_+^8 \mid S = S_{1,2}, S \neq \left(\frac{2\alpha \Gamma(\beta)}{Q\Gamma(\beta - \alpha + 1)} \right)^{1/\alpha} \right\}.$$

3.5.2 Neimark–Sacker bifurcation about E

In the subsequent analysis, we establish the conditions under which system (3.5) exhibits a Neimark–Sacker bifurcation at the positive fixed point E , and we investigate the qualitative dynamics of the system near the bifurcation threshold. Specifically, the fixed point E undergoes a Neimark–Sacker bifurcation when the parameters $(R, \kappa, m, \epsilon, C, D, \beta, \alpha, S)$ lie in a neighbourhood of the set

$$\mathbb{N}_S = \left\{ \begin{array}{l} (R, \kappa, m, \epsilon, C, D, \beta, \alpha) \in \mathbb{R}_+^8 \mid \alpha \in (0, 1], \quad S = S_0 = \left(\frac{\alpha Q \Gamma(\beta)}{P \Gamma(\beta - \alpha + 1)} \right)^{1/\alpha}, \\ \mathbb{D} = \left(\frac{RDP_2^*}{\kappa(1 + mP_2^*)} - \frac{\epsilon D}{(D + 1)^2} + C \right)^2 - 4 \left(\frac{DCRP_2^*}{\kappa(1 + mP_2^*)} + \frac{m\epsilon CP_2^*}{(D + 1)(1 + mP_2^*)} \right), \\ \mathbb{Q} = \left(\frac{RDP_2^*}{\kappa(1 + mP_2^*)} - \frac{\epsilon D}{(D + 1)^2} + C \right), \text{ and } \mathbb{P} = \left(\frac{DCRP_2^*}{\kappa(1 + mP_2^*)} + \frac{m\epsilon CP_2^*}{(D + 1)(1 + mP_2^*)} \right). \end{array} \right\}.$$

Suppose that $(R, \kappa, m, \epsilon, C, D, \beta, \alpha, S) \in \mathbb{N}_S$. In this case, system (3.5) can be expressed as

$$\begin{cases} P_{1,n+1} = P_{1,n} \exp \left(\left(R \left(\frac{1}{1 + mP_{2,n}} \right) \left(1 - \frac{P_{1,n}}{\kappa} \right) - \frac{\epsilon P_{2,n}}{P_{2,n} + P_{1,n}} \right) \frac{\Gamma(\beta - \alpha + 1)S^\alpha}{\alpha \Gamma(\beta)} \right) = H_1(P_{1,n}, P_{2,n}, \bar{S}'), \\ P_{2,n+1} = P_{2,n} \exp \left(C \left(1 - \frac{DP_{2,n}}{P_{1,n}} \right) \frac{\Gamma(\beta - \alpha + 1)S^\alpha}{\alpha \Gamma(\beta)} \right) = H_2(P_{1,n}, P_{2,n}, \bar{S}'). \end{cases} \quad (3.7)$$

where $S = (S_0 + \bar{S}')$ and $\bar{S}'$ represents a small perturbation of S_0 such that $|\bar{S}'| \ll 1$.

Introducing the new variables

$$N_{1,n} = P_{1,n} - \left(\frac{\kappa D (R(D + 1) - \epsilon)}{RD(D + 1) + \epsilon m \kappa} \right), \quad N_{2,n} = P_{2,n} - \left(\frac{\kappa (R(D + 1) - \epsilon)}{RD(D + 1) + \epsilon m \kappa} \right),$$

the fixed point E is translated to the origin. Expanding the functions H_1 and H_2 in a Taylor series around the origin up to third order yields the following system:

$$\begin{cases} N_{1,n+1} = \wp_{11}N_{1,n} + \wp_{12}N_{2,n} + \wp_{13}N_{1,n}^2 + \wp_{14}N_{1,n}N_{2,n} + \wp_{15}N_{2,n}^2 \\ \quad + \wp_{16}N_{1,n}^3 + \wp_{17}N_{1,n}^2N_{2,n} + \wp_{18}N_{1,n}N_{2,n}^2 + \wp_{19}N_{2,n}^3 + \mathcal{R}_1(|N_{1,n}|, |N_{2,n}|)^4, \\ N_{2,n+1} = \wp_{21}N_{1,n} + \wp_{22}N_{2,n} + \wp_{23}N_{1,n}^2 + \wp_{24}N_{1,n}N_{2,n} + \wp_{25}N_{2,n}^2 \\ \quad + \wp_{26}N_{1,n}^3 + \wp_{27}N_{1,n}^2N_{2,n} + \wp_{28}N_{1,n}N_{2,n}^2 + \wp_{29}N_{2,n}^3 + \mathcal{R}_2(|N_{1,n}|, |N_{2,n}|)^4. \end{cases} \quad (3.8)$$

where the coefficients $\wp_{ij}$ are given by

$$\begin{aligned} \wp_{11} &= 1 - \left(\frac{\Gamma(\beta - \alpha + 1) S^\alpha}{\alpha \Gamma(\beta)} \right) \left(\frac{RDP_2^*}{\kappa(1 + mP_2^*)} - \frac{\epsilon D}{(D + 1)^2} \right), \quad \wp_{21} = \left(\frac{C\Gamma(\beta - \alpha + 1) S^\alpha}{\alpha D \Gamma(\beta)} \right), \\ \wp_{12} &= - \left(\frac{\epsilon \Gamma(\beta - \alpha + 1) S^\alpha}{\alpha \Gamma(\beta)} \right) \left(\frac{mDP_2^*}{(D + 1)(1 + mP_2^*)} + \frac{D^2}{(D + 1)^2} \right), \quad \wp_{22} = 1 - \left(\frac{C\Gamma(\beta - \alpha + 1) S^\alpha}{\alpha \Gamma(\beta)} \right), \\ \wp_{13} &= \left(\left(\frac{1}{P_1} + \mathcal{A} \left(-\frac{R}{\kappa(1 + mP_2)} + \frac{\epsilon P_2}{(P_2 + P_1)^2} \right) \right)^2 - \frac{1}{P_1^2} + \mathcal{A} \left(-\frac{2\epsilon P_2}{(P_2 + P_1)^3} \right) \right), \\ \wp_{14} &= \mathcal{A} \left(\left(\frac{1}{P_1} + \mathcal{A} \left(-\frac{R}{\kappa(1 + mP_2)} + \frac{\epsilon P_2}{(P_2 + P_1)^2} \right) \right) \left(-Rm \frac{1 - \frac{P_1}{\kappa}}{(1 + mP_2)^2} - \frac{\epsilon P_1}{(P_2 + P_1)^2} \right) - \frac{2\epsilon P_2}{(P_2 + P_1)^3} \right), \\ \wp_{15} &= \mathcal{A} \left(\mathcal{A} \left(-Rm \frac{1 - \frac{P_1}{\kappa}}{(1 + mP_2)^2} - \epsilon \frac{P_1}{(P_2 + P_1)^2} \right)^2 + \frac{2\epsilon P_2}{(P_2 + P_1)^3} \right), \quad \wp_{25} = \left(\left(\frac{1}{P_2} - \frac{\mathcal{A}CD}{P_1} \right)^2 - \frac{1}{P_2^2} \right), \\ \wp_{16} &= \left(\left(\frac{1}{P_1} + \mathcal{A} \left(-\frac{R}{\kappa(1 + mP_2)} + \frac{\epsilon P_2}{(P_2 + P_1)^2} \right) \right) \left(\mathcal{A} \frac{\alpha}{S} \right) + \mathcal{A} \left(-\frac{R}{\kappa(1 + mP_2)} + \frac{\epsilon P_2}{(P_2 + P_1)^2} \right) \frac{\alpha}{S} \right), \\ \wp_{19} &= \left(\mathcal{A} \left(-Rm \frac{1 - \frac{P_1}{\kappa}}{(1 + mP_2)^2} - \epsilon \frac{P_1}{(P_2 + P_1)^2} \right) \left(\mathcal{A} \frac{\alpha}{S} \right) + \mathcal{A} \left(-Rm \frac{1 - \frac{P_1}{\kappa}}{(1 + mP_2)^2} - \epsilon \frac{P_1}{(P_2 + P_1)^2} \right) \frac{\alpha}{S} \right), \\ \wp_{23} &= \left(\left(\frac{\mathcal{A}CDP_2}{P_1^2} \right)^2 + \frac{\mathcal{A}CD}{P_1^2} \right), \quad \wp_{24} = \left(\left(\frac{1}{P_2} - \frac{\mathcal{A}CD}{P_1} \right) \frac{\mathcal{A}CDP_2}{P_1^2} + B_{P_1P_2} \right). \end{aligned}$$

The characteristic polynomial of the Jacobian matrix of system (3.8), evaluated at the fixed point, takes the form

$$\lambda^2 - \text{Tr}(\bar{S}') \lambda + \det(\bar{S}') = 0, \quad (3.9)$$

with

$$\begin{aligned}\text{Tr}(\bar{S}') &= 2 - \frac{\Gamma(\beta - \alpha + 1)(S + \bar{S}')^\alpha}{\alpha\Gamma(\beta)} \left(\frac{RDP_2^*}{\kappa(1 + mP_2^*)} - \frac{\epsilon D}{(D + 1)^2} + C \right), \\ \det(\bar{S}') &= 1 + \left(\frac{\Gamma(\beta - \alpha + 1)(S + \bar{S}')^\alpha}{\alpha\Gamma(\beta)} \right)^2 \left(\frac{DCRP_2^*}{\kappa(1 + mP_2^*)} + \frac{m\epsilon CP_2^*}{(D + 1)(1 + mP_2^*)} \right) - \left(\frac{\Gamma(\beta - \alpha + 1)(S + \bar{S}')^\alpha}{\alpha\Gamma(\beta)} \right) \\ &\quad \left(\frac{RDP_2^*}{\kappa(1 + mP_2^*)} - \frac{\epsilon D}{(D + 1)^2} + C \right).\end{aligned}$$

Because $(R, \kappa, m, \epsilon, C, D, \beta, \alpha, S) \in \mathbb{N}_S$, Eq. (3.9) admits a pair of complex conjugate eigenvalues of the form

$$\lambda_{1,2}(\bar{S}') = \frac{\text{Tr}(\bar{S}')}{2} \pm \frac{i}{2} \sqrt{4 \det(\bar{S}') - \text{Tr}(\bar{S}')^2}.$$

Hence,

$$|\lambda_1(\bar{S}')| = |\lambda_2(\bar{S}')| = \sqrt{\det(\bar{S}')},$$

and

$$\left. \frac{d|\lambda_1(\bar{S}')|}{d\bar{S}'} \right|_{\bar{S}'=0} = \left. \frac{d|\lambda_2(\bar{S}')|}{d\bar{S}'} \right|_{\bar{S}'=0} = \frac{\mathbb{Q}\Gamma(\beta - \alpha + 1)}{\Gamma(\beta)} \left(\frac{\alpha\mathbb{Q}\Gamma(\beta)}{\mathbb{P}\Gamma(\beta - \alpha + 1)} \right)^{\frac{\alpha-1}{\alpha}} > 0.$$

The requirement $\text{Tr}(0) \neq 0, 1$ leads to

$$S \neq \left(\frac{2\alpha\Gamma(\beta)}{\mathbb{Q}\Gamma(\beta - \alpha + 1)} \right)^{1/\alpha}, \quad \left(\frac{\alpha\Gamma(\beta)}{\mathbb{Q}\Gamma(\beta - \alpha + 1)} \right)^{1/\alpha}, \quad (3.10)$$

which is equivalent to the condition $\lambda_{1,2}^i \neq 1$ for $i = 1, 2, 3, 4$.

To derive the normal form of system (3.8) at $\bar{S}' = 0$, we apply the transformation

$$\begin{pmatrix} N_{1,n} \\ N_{2,n} \end{pmatrix} = \begin{pmatrix} \wp_{12} & 0 \\ \tau - \wp_{11} & -\theta \end{pmatrix} \begin{pmatrix} \bar{P}_{1,n} \\ \bar{P}_{2,n} \end{pmatrix},$$

where

$$\tau = \Re(\lambda_1) = \frac{\text{Tr}(0)}{2}, \quad \theta = \Im(\lambda_2) = \frac{1}{2} \sqrt{4 \det(0) - \text{Tr}(0)^2}.$$

Under this transformation, system (3.8) becomes

$$\begin{cases} \bar{P}_{1,n+1} = \tau \bar{P}_{1,n} - \theta \bar{P}_{2,n} + \tilde{H}_1(\bar{P}_{1,n}, \bar{P}_{2,n}, S_0), \\ \bar{P}_{2,n+1} = \theta \bar{P}_{1,n} + \tau \bar{P}_{2,n} + \tilde{H}_2(\bar{P}_{1,n}, \bar{P}_{2,n}, S_0). \end{cases} \quad (3.11)$$

The nonlinear functions $\tilde{H}_1$ and $\tilde{H}_2$ consist of quadratic and cubic terms, whereas the higherorder contributions are represented by $\tilde{\mathcal{R}}_1$ and $\tilde{\mathcal{R}}_2$. In particular, they are expressed as

$$\begin{aligned} \tilde{H}_1(\bar{P}_{1,n}, \bar{P}_{2,n}, S_0) &= \frac{\wp_{13}}{\wp_{12}} N_{1,n}^2 + \frac{\wp_{14}}{\wp_{12}} N_{1,n} N_{2,n} + \frac{\wp_{15}}{\wp_{12}} N_{2,n}^2 + \frac{\wp_{16}}{\wp_{12}} N_{1,n}^3 + \frac{\wp_{17}}{\wp_{12}} N_{1,n}^2 N_{2,n} \\ &\quad + \frac{\wp_{18}}{\wp_{12}} N_{1,n} N_{2,n}^2 + \frac{\wp_{19}}{\wp_{12}} N_{2,n}^3 + \tilde{\mathcal{R}}_1(|N_{1,n}|, |N_{2,n}|)^4, \\ \tilde{H}_2(\bar{P}_{1,n}, \bar{P}_{2,n}, S_0) &= \left(\frac{\wp_{13}(\tau - \wp_{11})}{\theta \wp_{12}} - \frac{\wp_{23}}{\theta} \right) N_{1,n}^2 + \left(\frac{\wp_{14}(\tau - \wp_{11})}{\theta \wp_{12}} - \frac{\wp_{24}}{\theta} \right) N_{1,n} N_{2,n} \\ &\quad + \left(\frac{\wp_{15}(\tau - \wp_{11})}{\theta \wp_{12}} - \frac{\wp_{25}}{\theta} \right) N_{2,n}^2 + \left(\frac{\wp_{16}(\tau - \wp_{11})}{\theta \wp_{12}} - \frac{\wp_{26}}{\theta} \right) N_{1,n}^3 \\ &\quad + \left(\frac{\wp_{17}(\tau - \wp_{11})}{\theta \wp_{12}} - \frac{\wp_{27}}{\theta} \right) N_{1,n}^2 N_{2,n} + \left(\frac{\wp_{18}(\tau - \wp_{11})}{\theta \wp_{12}} - \frac{\wp_{28}}{\theta} \right) N_{1,n} N_{2,n}^2 \\ &\quad + \left(\frac{\wp_{19}(\tau - \wp_{11})}{\theta \wp_{12}} - \frac{\wp_{29}}{\theta} \right) N_{2,n}^3 + \tilde{\mathcal{R}}_2(|N_{1,n}|, |N_{2,n}|)^4, \end{aligned}$$

and

$$N_{1,n} = \wp_{12} \bar{P}_{1,n}, \quad N_{2,n} = (\tau - \wp_{11}) \bar{P}_{1,n} - \theta \bar{P}_{2,n}.$$

To determine the occurrence of a Neimark–Sacker bifurcation, we introduce the quantity

$$f_L = \Re(\lambda_2 i_{21}) - \Re\left(\frac{(1 - 2\lambda_1)\lambda_2^2}{1 - \lambda_1} i_{20} i_{11}\right) - \frac{1}{2}|i_{11}|^2 - |i_{02}|^2. \quad (3.12)$$

where

$$\begin{aligned}
 i_{20} &= \frac{1}{8} \left[\frac{\partial^2 \tilde{H}_1}{\partial \bar{P}_1^2} - \frac{\partial^2 \tilde{H}_1}{\partial \bar{P}_2^2} + 2 \frac{\partial^2 \tilde{H}_2}{\partial \bar{P}_1 \partial \bar{P}_2} + i \left(\frac{\partial^2 \tilde{H}_2}{\partial \bar{P}_1^2} - \frac{\partial^2 \tilde{H}_2}{\partial \bar{P}_2^2} - 2 \frac{\partial^2 \tilde{H}_1}{\partial \bar{P}_1 \partial \bar{P}_2} \right) \right] \Big|_{\bar{S}'=0}, \\
 i_{11} &= \frac{1}{4} \left[\frac{\partial^2 \tilde{H}_1}{\partial \bar{P}_1^2} + \frac{\partial^2 \tilde{H}_1}{\partial \bar{P}_2^2} + i \left(\frac{\partial^2 \tilde{H}_2}{\partial \bar{P}_1^2} + \frac{\partial^2 \tilde{H}_2}{\partial \bar{P}_2^2} \right) \right] \Big|_{\bar{S}'=0}, \\
 i_{02} &= \frac{1}{8} \left[\frac{\partial^2 \tilde{H}_1}{\partial \bar{P}_1^2} - \frac{\partial^2 \tilde{H}_1}{\partial \bar{P}_2^2} - 2 \frac{\partial^2 \tilde{H}_2}{\partial \bar{P}_1 \partial \bar{P}_2} + i \left(\frac{\partial^2 \tilde{H}_2}{\partial \bar{P}_1^2} - \frac{\partial^2 \tilde{H}_2}{\partial \bar{P}_2^2} + 2 \frac{\partial^2 \tilde{H}_1}{\partial \bar{P}_1 \partial \bar{P}_2} \right) \right] \Big|_{\bar{S}'=0}, \\
 i_{21} &= \frac{1}{16} \left[\frac{\partial^3 \tilde{H}_1}{\partial \bar{P}_1^3} + \frac{\partial^3 \tilde{H}_1}{\partial \bar{P}_1 \partial \bar{P}_2^2} + \frac{\partial^3 \tilde{H}_2}{\partial \bar{P}_1^2 \partial \bar{P}_2} + \frac{\partial^3 \tilde{H}_2}{\partial \bar{P}_2^3} + i \left(\frac{\partial^3 \tilde{H}_2}{\partial \bar{P}_1^3} + \frac{\partial^3 \tilde{H}_2}{\partial \bar{P}_1 \partial \bar{P}_2^2} - \frac{\partial^3 \tilde{H}_1}{\partial \bar{P}_1^2 \partial \bar{P}_2} - \frac{\partial^3 \tilde{H}_1}{\partial \bar{P}_2^3} \right) \right] \Big|_{\bar{S}'=0}.
 \end{aligned}$$

Theorem 3.5.1 (Neimark-Sacker bifurcation) Assume that condition (3.10) is satisfied and that $(R, \kappa, m, \epsilon, C, D, \beta, \alpha, S) \in \mathbb{N}_S$ with $f_L \neq 0$. Then system (3.8) undergoes a Neimark-Sacker bifurcation at the fixed point E when the bifurcation parameter S varies within a sufficiently small neighbourhood of

$$S_0 = \left(\frac{\alpha Q \Gamma(\beta)}{\mathbb{P} \Gamma(\beta - \alpha + 1)} \right)^{1/\alpha}.$$

Moreover, the sign of f_L determines the stability of the bifurcating invariant curve. If $f_L < 0$, a stable (attracting) invariant closed curve is created near the fixed point for $S > S_0$. On the other hand, if $f_L > 0$, the bifurcation produces an unstable (repelling) invariant closed curve for $S < S_0$.

3.6 Numerical simulations

In this section, we present bifurcation diagrams, phase portraits, and Lyapunov exponents for system (3.5) in order to support the preceding theoretical analysis. The emergence of complex dynamical behavior is demonstrated through numerical simulations and the 0–1 test.

Example 3.6.1 To validate the theoretical findings, system (3.5) is investigated using the parameter set $R = 1.2390$, $\kappa = 6.09$, $m = 1.2$, $\epsilon = 0.3$, $C = 1.5$, $D = 1.8$, $\alpha = 0.79$, and

$\beta = 1.2$. The bifurcation parameter S is varied over the interval $[0, 2]$, with initial conditions $(P_{1,0}, P_{2,0}) = (2.4, 3.14)$.

Taking S as the control parameter, numerical simulations show that at $S = 0.3675$, the positive equilibrium point $E = (4.1177, 2.2876)$ loses its stability. At this critical value, system (3.5) undergoes a flip bifurcation. The Jacobian matrix evaluated at E is given by

$$J(E) = \begin{pmatrix} 0.7633 & -0.4056 \\ 1.2739 & -1.2931 \end{pmatrix}.$$

Accordingly, the characteristic polynomial is

$$\Psi(\lambda) = \lambda^2 + 0.5297\lambda - 0.4703,$$

which yields the eigenvalues

$$\lambda_1 = -1.000, \quad \lambda_2 = 0.4703.$$

To further illustrate the dynamics, bifurcation diagrams and Lyapunov exponents are presented

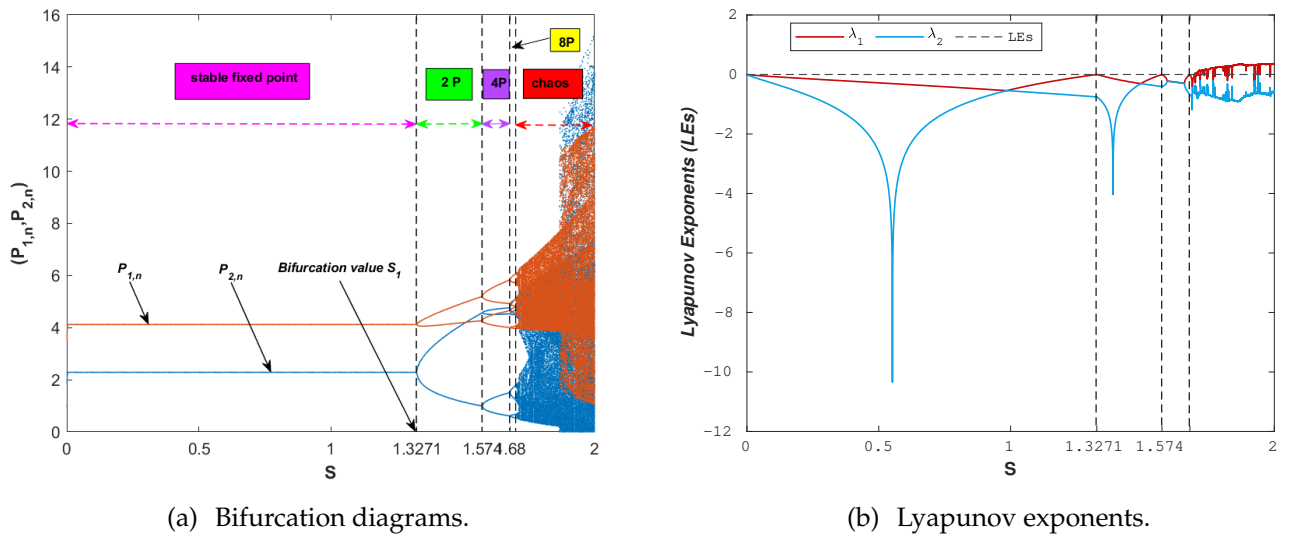


Figure 3.1: (a) Flip bifurcation diagram of system (3.5). (b) Corresponding Lyapunov exponents for the parameter values: $R = 1.2390$, $\kappa = 6.09$, $m = 1.2$, $\epsilon = 0.3$, $C = 1.5$, $D = 1.8$, $\beta = 1.2$, $\alpha = 0.79$ and $S \in [0, 2]$.

in Figure 3.1. Moreover, Figures 3.2 and 3.3 display the phase portraits and corresponding time series for selected values of the bifurcation parameter S . These results clearly indicate that variations in S significantly affect the qualitative behavior of the system, including stability changes, the emergence of periodic orbits, and transitions to more complex dynamical regimes.

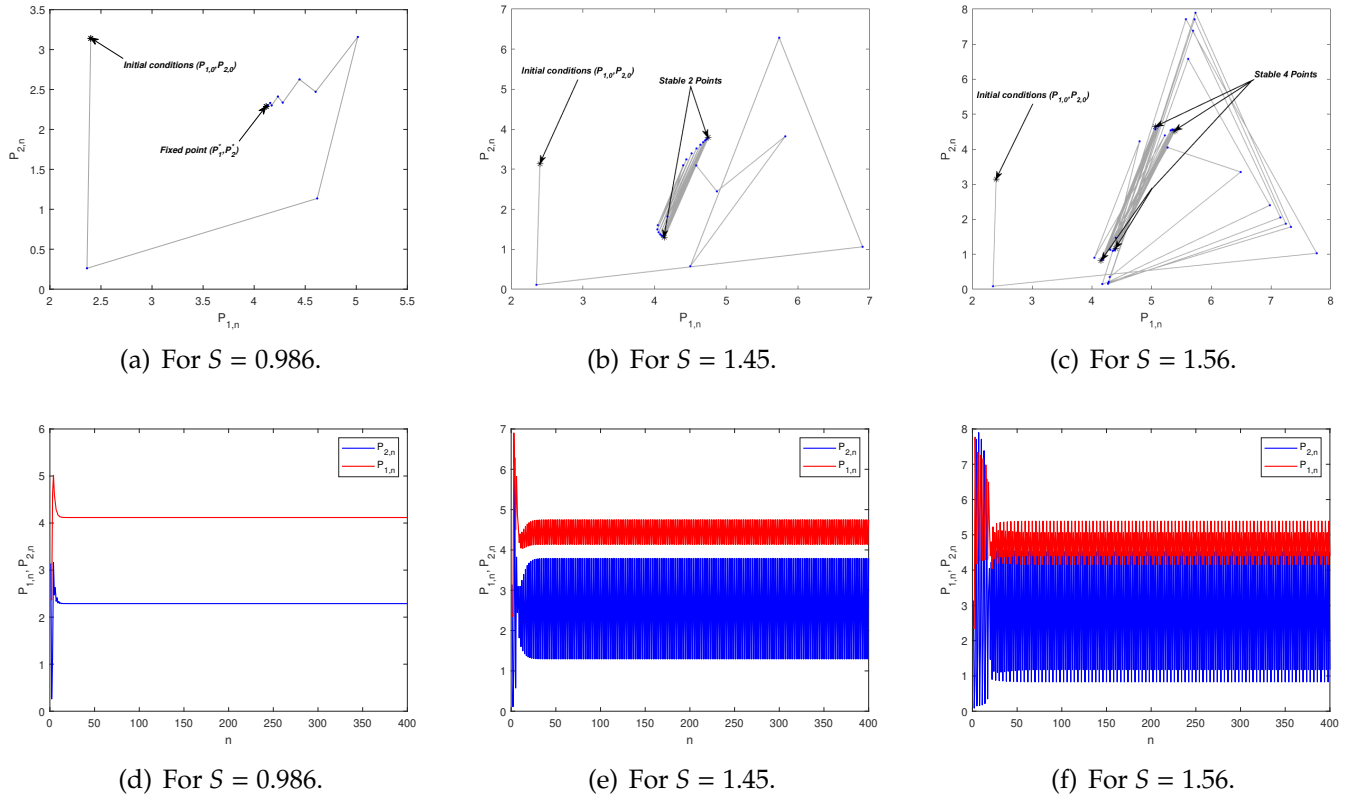


Figure 3.2: Phase portraits and corresponding time evolutions for different values of S : (a,d) $S = 0.986$, (b,e) $S = 1.45$, and (c,f) $S = 1.56$. The simulations are performed using the parameter values $R = 1.2390$, $\kappa = 6.09$, $m = 1.2$, $\epsilon = 0.3$, $C = 1.5$, $D = 1.8$, $\beta = 1.2$, $\alpha = 0.79$, with initial conditions $(P_{1,0}, P_{2,0}) = (2.4, 3.14)$.

Example 3.6.2 To validate the theoretical analysis, numerical experiments are performed on system (3.5) using the parameter set $R = 3.37$, $\kappa = 5.77$, $m = 0.1$, $\epsilon = 2.21$, $C = 1.18$, $D = 0.6$, $\beta = 0.6$, and $\alpha = 0.75$. The bifurcation parameter S is varied in the interval $[0, 1.28]$, and the initial conditions are chosen as $(P_{1,0}, P_{2,0}) = (2.5, 3.6)$.

Treating S as the bifurcation parameter, numerical simulations show that at $S = 1.01177$, the positive equilibrium point $E = (2.4424, 4.0707)$ loses its stability. At this critical value,

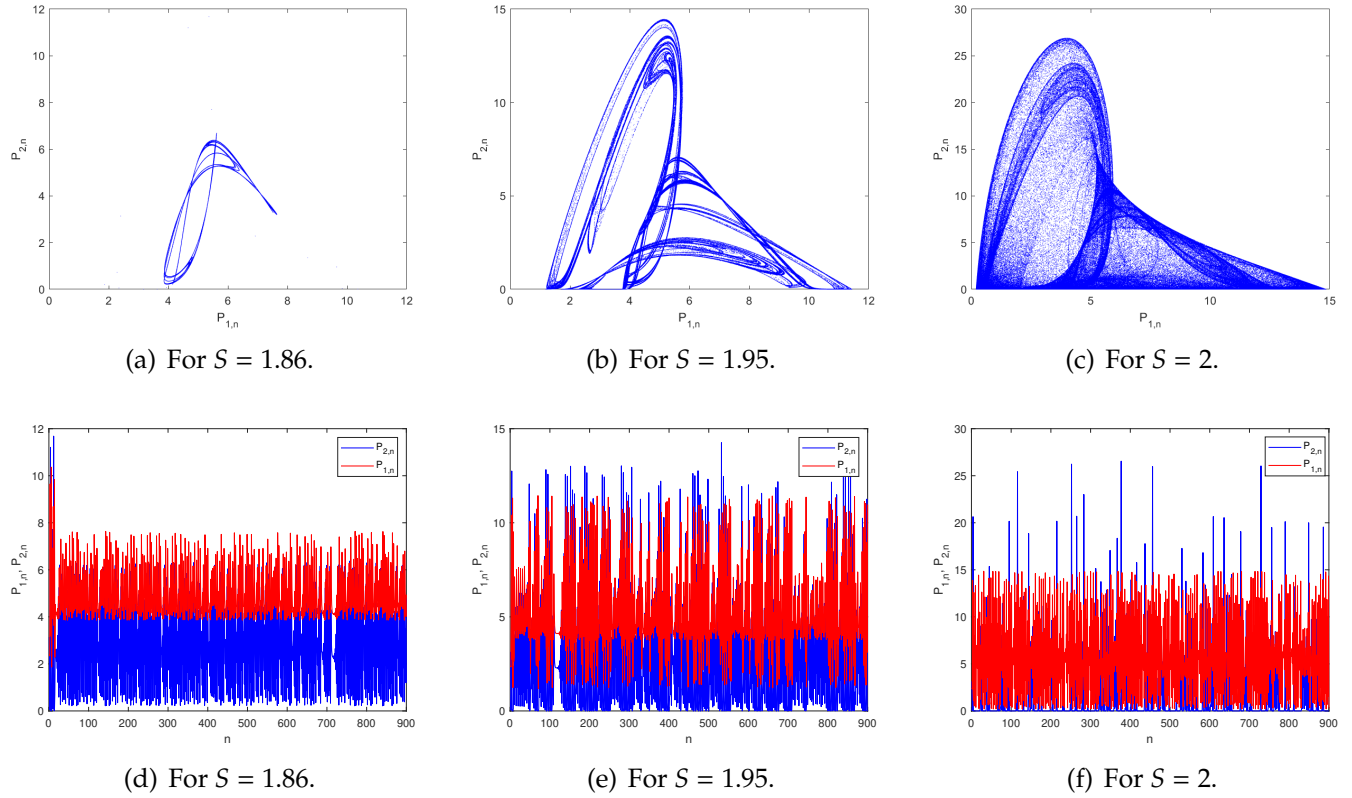


Figure 3.3: Phase portraits and corresponding time evolutions for different values of S : (a,d) $S = 1.86$, (b,e) $S = 1.95$, and (c,f) $S = 2$. The simulations are performed using the parameter values $R = 1.2390$, $\kappa = 6.09$, $m = 1.2$, $\epsilon = 0.3$, $C = 1.5$, $D = 1.8$, $\beta = 1.2$, $\alpha = 0.79$, with initial conditions $(P_{1,0}, P_{2,0}) = (2.4, 3.14)$.

system (3.5) undergoes a Neimark–Sacker bifurcation. The Jacobian matrix evaluated at E is

$$J(E) = \begin{pmatrix} 0.5018 & -0.5532 \\ 1.9762 & -0.1857 \end{pmatrix}.$$

The characteristic polynomial is given by

$$\Psi(\lambda) = \lambda^2 - 0.3161\lambda + 1,$$

with eigenvalues

$$\lambda_{1,2} = 0.1580 \pm 0.9875i, \quad |\lambda_{1,2}| = 1.$$

The existence of complex conjugate eigenvalues on the unit circle indicates that the equilibrium

E loses stability via a Neimark–Sacker bifurcation, leading to the emergence of quasi-periodic oscillations.

Figure 3.4 shows the Neimark–Sacker bifurcation diagram of system (3.5) with respect to S , together with the corresponding Lyapunov exponents. At the critical value $S = S_0 = 1.01177$, the stable fixed point loses stability and quasi-periodic dynamics emerge. This behavior is confirmed by the Lyapunov spectrum, where the largest Lyapunov exponent approaches zero, indicating neutral stability associated with invariant closed curves. As S increases further, the variation of Lyapunov exponents reflects transitions to more complex dynamics.

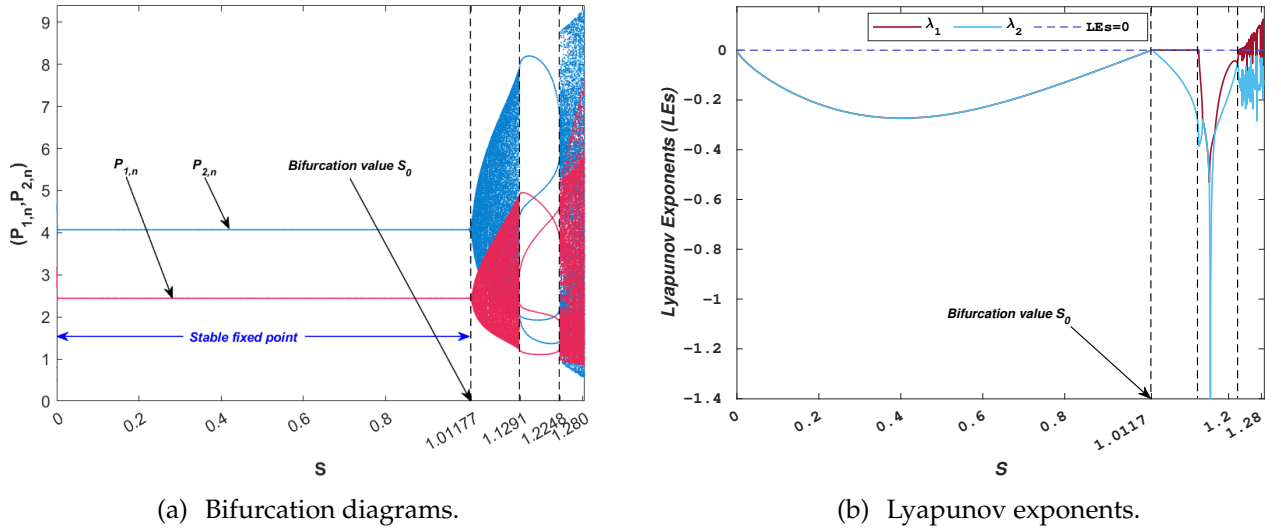


Figure 3.4: (a) Neimark-Sacker bifurcation diagram of system (3.5). (b) Corresponding Lyapunov exponents for the parameter values: $R = 3.37$, $\kappa = 5.77$, $m = 0.1$, $\epsilon = 2.21$, $C = 1.18$, $D = 0.6$, $\beta = 0.6$, $\alpha = 0.75$ and $S \in [0, 1.28]$.

The phase portraits and time evolution plots shown in Figure 3.5 further confirm that variations in S induce a Neimark–Sacker bifurcation. The fixed point loses stability and is replaced by invariant closed curves, corresponding to quasi-periodic oscillations in the phase space.

For the parameter set ($\kappa = 11.77$, $m = 2.1$, $\epsilon = 4.21$, $C = 1.18$, $D = 1.6$, $\beta = 0.6$, $\alpha = 0.75$, $S = 0.82$) with $R \in [0.7, 5.5]$, bifurcation diagrams and Lyapunov exponents (see Figure 3.6) illustrate how changes in the intrinsic growth rate of the prey affect the long-term dynamics.

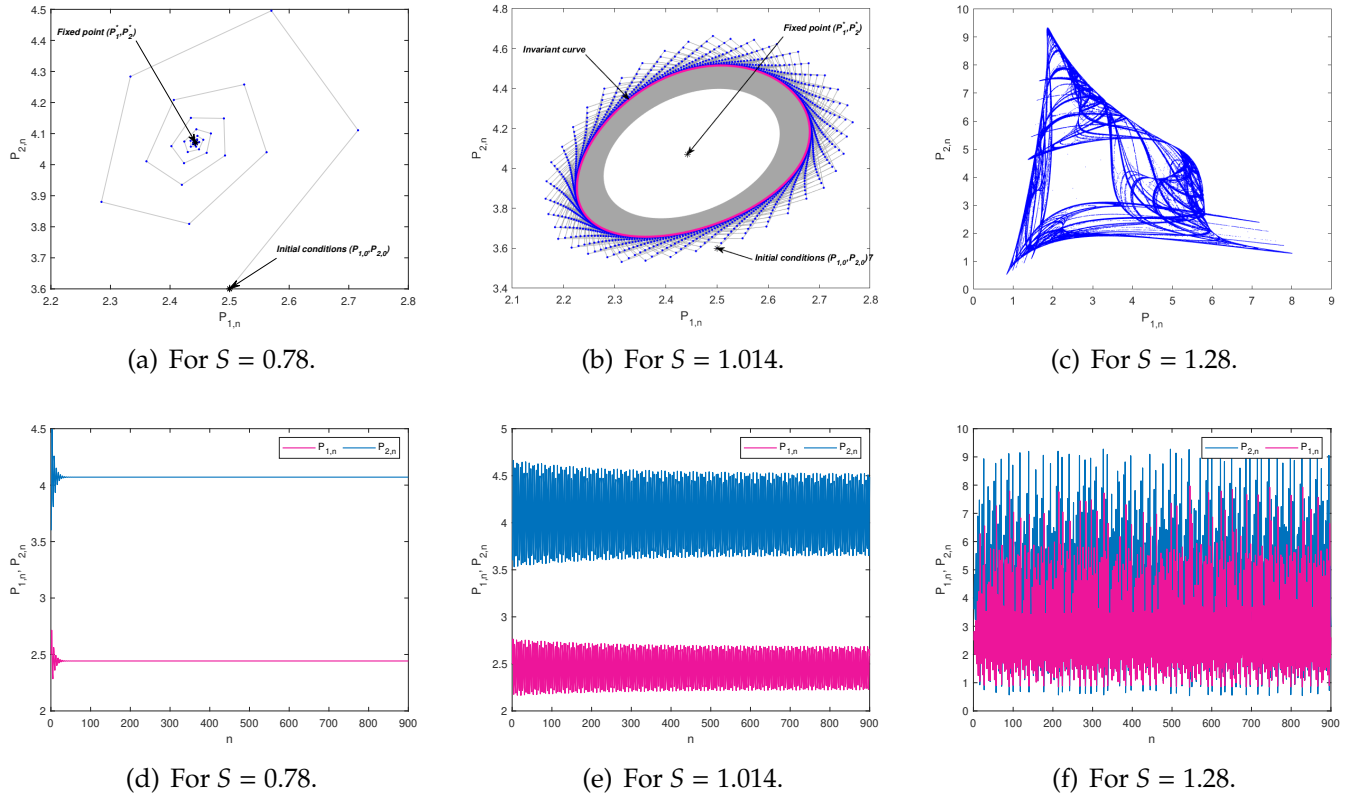


Figure 3.5: Phase portraits and corresponding time evolutions for different values of S : (a,d) $S = 1.86$, (b,e) $S = 1.95$, and (c,f) $S = 2$. The simulations are performed using the parameter values $R = 3.37$, $\kappa = 5.77$, $m = 0.1$, $\epsilon = 2.21$, $C = 1.18$, $D = 0.6$, $\beta = 0.6$, $\alpha = 0.75$, with initial conditions $(P_{1,0}, P_{2,0}) = (2.5, 3.6)$.

As R increases, the system first exhibits stable dynamics, where trajectories converge to the equilibrium E . This corresponds to a single branch in the bifurcation diagram and negative Lyapunov exponents, indicating asymptotic stability.

For intermediate values of R , the equilibrium loses stability and periodic oscillations emerge through a cascade of bifurcations. This is observed as branching in the bifurcation diagram, while one Lyapunov exponent approaches zero and the other remains negative, indicating periodic or quasi-periodic behavior.

As R increases further, the system exhibits complex and irregular dynamics. The bifurcation diagram becomes densely populated, indicating sensitive dependence on initial conditions and the appearance of chaotic attractors. This is confirmed when the largest Lyapunov exponent becomes positive over certain intervals, indicating chaos.

The results also reveal alternations between regular and irregular dynamics. In some parameter ranges, chaotic behavior disappears and periodic solutions reappear, as indicated by non-positive largest Lyapunov exponents.

From a biological perspective, the prey growth rate plays a crucial role in determining system stability. Moderate values of R support stable coexistence, whereas higher values may induce oscillatory or chaotic fluctuations, increasing the risk of population instability.

Overall, bifurcation diagrams and Lyapunov exponents provide complementary information: bifurcation diagrams describe qualitative transitions in dynamics, while Lyapunov exponents quantify stability and chaotic behavior.

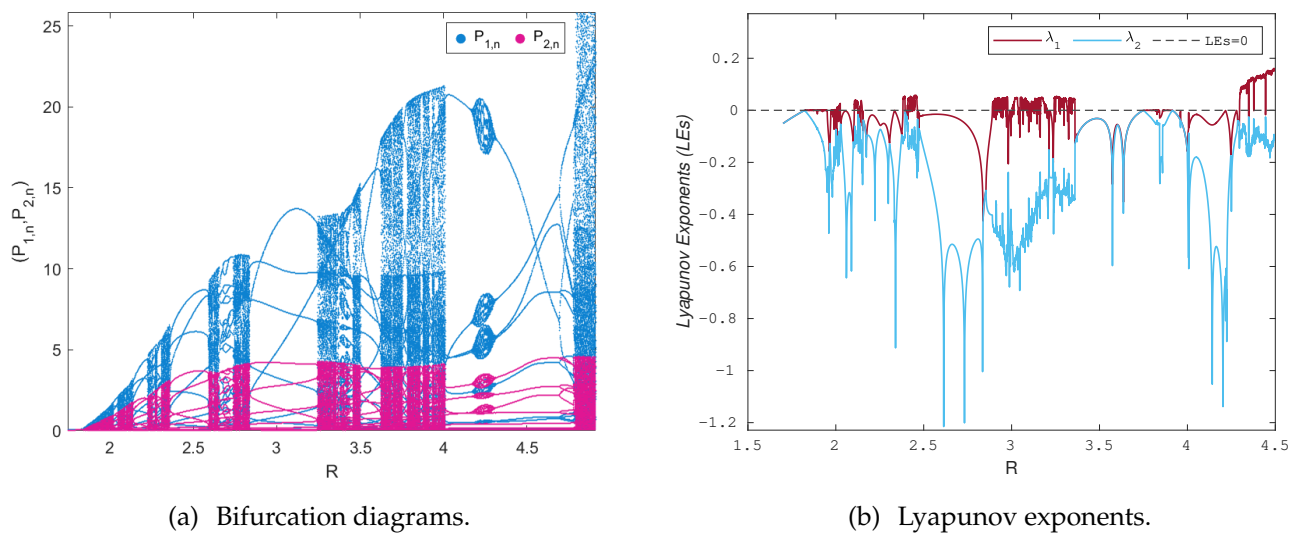


Figure 3.6: (a) Neimark-Sacker bifurcation diagram of system (3.5). (b) Corresponding Lyapunov exponents for the parameter values: $\kappa = 11.77$, $m = 2.1$, $\epsilon = 4.21$, $C = 1.18$, $D = 1.6$, $\beta = 0.6$, $\alpha = 0.75$, $S = 0.82$ and $R \in [0.7, 5.5]$.

Example 3.6.3 By applying the 0–1 test method for numerical simulations of flip bifurcation and Neimark-Sacker bifurcation, we obtain this Figures 3.7 and 3.8 display the (Π_c^1, Π_c^2) representations corresponding to both regular and chaotic regimes of the predator-prey system (3.5) for different values of the adjustment speed parameter S . When $S = 0.93$ and $S = 1.56$ (see Figures 3.7(a) and 3.7(b)), as well as for $S = 0.65$ and $S = 1.112$ (see Figures 3.8(a) and 3.8(b)), the trajectories remain confined within a bounded region, which is indicative of periodic

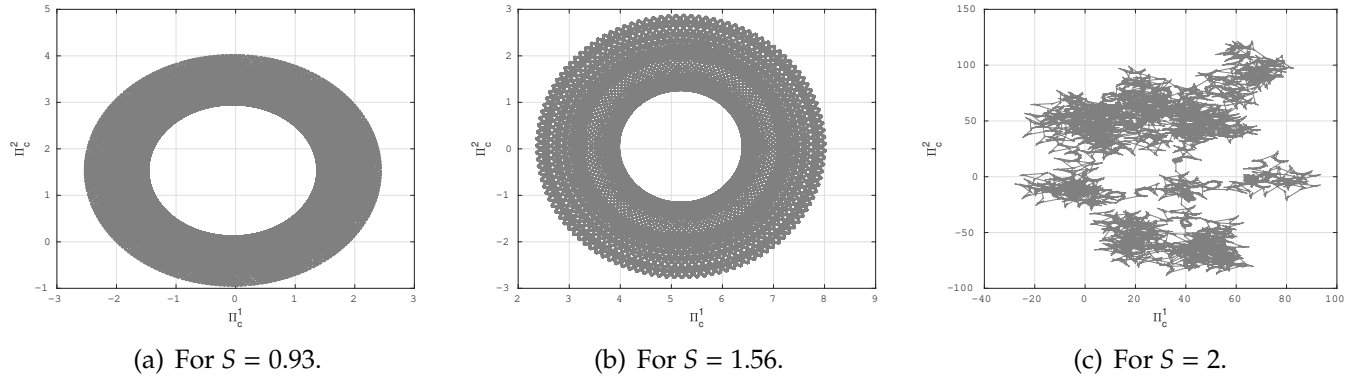


Figure 3.7: Results of the 0–1 test for the discrete system (3.5) corresponding to different values of S . The parameter set is given by $R = 1.2390$, $\kappa = 6.09$, $m = 1.2$, $\epsilon = 0.3$, $C = 1.5$, $D = 1.8$, $\beta = 1.2$, and $\alpha = 0.79$.

dynamics. In contrast, for $S = 2$ (Figure 3.7(c)) and $S = 1.28$ (Figure 3.8(c)), the trajectories exhibit diffusive, Brownian-like motion, providing clear evidence of chaotic behaviour. These observations are consistent with the outcomes of the 0–1 chaos test, the maximum Lyapunov exponent (MLE), and the bifurcation diagrams shown in Figures 3.1(b) and 3.4(b).

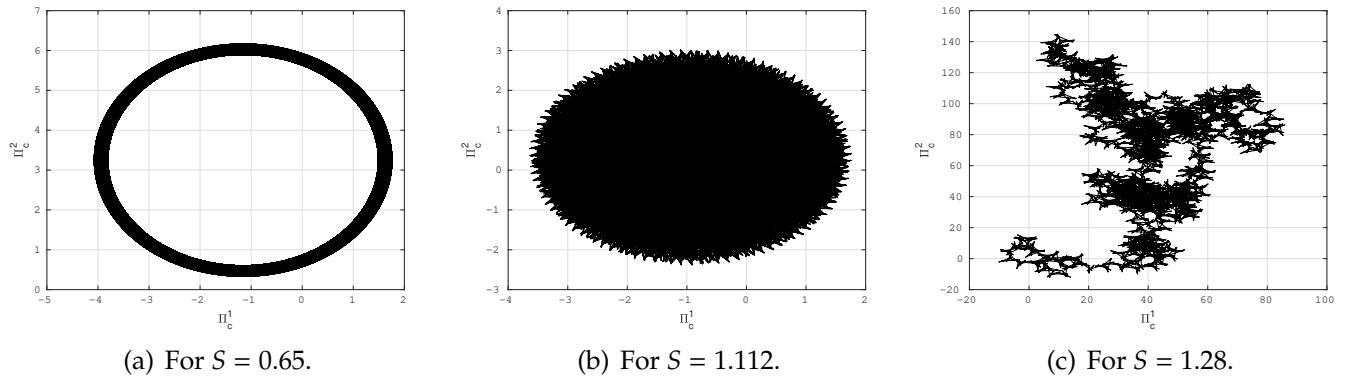


Figure 3.8: Results of the 0–1 test for the discrete system (3.5) corresponding to different values of S . The parameter set is given by $R = 3.37$, $\kappa = 5.77$, $m = 0.1$, $\epsilon = 2.21$, $C = 1.18$, $D = 0.6$, $\beta = 0.6$, and $\alpha = 0.75$.

3.7 Chaos control

The control of chaos in discrete dynamical systems has gained considerable attention due to the sensitive and unpredictable nature of chaotic behavior. Although chaotic systems may seem random, they are deterministic and contain numerous unstable periodic orbits within their attractors. Chaos control aims to stabilize these orbits using minimal interventions, converting chaotic motion into regular and predictable dynamics without altering the systems fundamental structure.

Common techniques include the Ott–Grebogi–Yorke (OGY) method [61], feedback control [62, 63], and hybrid control approaches [64]. The OGY method works by applying small, state-dependent parameter perturbations near the desired orbit. Feedback control uses corrective inputs based on deviations from a reference state, providing simplicity and robustness. Hybrid strategies combine multiple methods to enhance stability, performance, and robustness under uncertainty and noise.

In this section, we investigate chaos control for system (3.5) using a hybrid control approach. This method is straightforward to implement and is obtained by formulating the controlled version of system (3.5) as follows:

$$\begin{cases} P_{1,n+1} = \zeta P_{1,n} \exp \left(\left(R \left(\frac{1}{1 + mP_{2,n}} \right) \left(1 - \frac{P_{1,n}}{\kappa} \right) - \frac{\epsilon P_{2,n}}{P_{2,n} + P_{1,n}} \right) \frac{\Gamma(\beta - \alpha + 1)S^\alpha}{\alpha\Gamma(\beta)} \right) + (1 - \zeta) P_{1,n}, \\ P_{2,n+1} = \zeta P_{2,n} \exp \left(C \left(1 - \frac{DP_{2,n}}{P_{1,n}} \right) \frac{\Gamma(\beta - \alpha + 1)S^\alpha}{\alpha\Gamma(\beta)} \right) + (1 - \zeta) P_{2,n}, \end{cases} \quad (3.13)$$

where the parameter $\zeta \in (0, 1)$ determines the intensity of the control. This formulation represents a convex combination of the uncontrolled system and a stabilizing feedback term.

By appropriately selecting ζ , the qualitative behaviour of the system near E can be modified. In particular, the occurrence of bifurcations such as period-doubling and Neimark–Sacker can be delayed, advanced, or completely eliminated.

To analyse local stability, the Jacobian matrix of the controlled system is evaluated

at the fixed point $E = (P_1^*, P_2^*)$ and is given by

$$J^c(P_1^*, P_2^*) = \begin{pmatrix} 1 - \mathcal{A}\zeta \left(\frac{RDP_2^*}{\kappa(1 + mP_2^*)} - \frac{\epsilon D}{(D+1)^2} \right) - \mathcal{A}\epsilon \zeta \left(\frac{mDP_2^*}{(D+1)(1 + mP_2^*)} + \frac{D^2}{(D+1)^2} \right) \\ \frac{\mathcal{A}C\zeta}{D} & 1 - \mathcal{A}C\zeta \end{pmatrix}.$$

The corresponding characteristic equation is given by

$$\Psi^c(\lambda) = \lambda^2 - \text{Tr}(J^c)\lambda + \det(J^c), \quad (3.14)$$

where

$$\text{Tr}(J^c) = 2 - \frac{\zeta \Gamma(\beta - \alpha + 1) S^\alpha}{\alpha \Gamma(\beta)} \left(\frac{RDP_2^*}{\kappa(1 + mP_2^*)} - \frac{\epsilon D}{(D+1)^2} + C \right),$$

and

$$\begin{aligned} \det(J^c) = 1 + \left(\frac{\zeta \Gamma(\beta - \alpha + 1) S^\alpha}{\alpha \Gamma(\beta)} \right)^2 & \left(\frac{DCRP_2^*}{\kappa(1 + mP_2^*)} + \frac{m\epsilon CP_2^*}{(D+1)(1 + mP_2^*)} \right) - \left(\frac{\zeta \Gamma(\beta - \alpha + 1) S^\alpha}{\alpha \Gamma(\beta)} \right) \\ & \left(\frac{RDP_2^*}{\kappa(1 + mP_2^*)} - \frac{\epsilon D}{(D+1)^2} + C \right). \end{aligned}$$

The fixed point E is locally asymptotically stable if the eigenvalues of the Jacobian lie inside the unit circle. This requirement leads to the following conditions:

$$\Psi^c(1) = \left(\frac{\zeta \Gamma(\beta - \alpha + 1) S^\alpha}{\alpha \Gamma(\beta)} \right)^2 \left(\frac{DCRP_2^*}{\kappa(1 + mP_2^*)} + \frac{m\epsilon CP_2^*}{(D+1)(1 + mP_2^*)} \right) > 0, \quad (3.15)$$

$$\Psi^c(-1) = 4 - 2\text{Tr}(J^c) + \Psi^c(1) > 0, \quad (3.16)$$

$$\Psi^c(0) = \det(J^c) < 1. \quad (3.17)$$

These inequalities define the admissible region of the control parameter ζ for which the hybrid control strategy stabilizes the system.

Lemma 3.7.1 *The coexistence fixed point E of the controlled system (3.13) is locally asymptotically stable provided that conditions (3.15), (3.16), and (3.17) are satisfied.*

Example 3.7.1 Figure 3.9 illustrates the system dynamics before the implementation of the hybrid control strategy, while Figure 3.10 presents the corresponding behavior after control is applied. The simulations are performed using the parameter set $R = 3.37$, $\kappa = 5.77$, $m = 0.1$, $\epsilon = 2.21$, $C = 1.18$, $D = 0.6$, $\beta = 0.6$, $\alpha = 0.75$, and $S = 1.23$.

For the control parameter $\zeta \in (0, 0.859]$, the controlled system (3.13) stabilizes at the equilibrium point E . Figure 3.10(a) shows the bifurcation diagram of the controlled system, where all state variables converge to E throughout the considered range of ζ . This indicates that, after the introduction of the control mechanism, the system suppresses irregular oscillations and asymptotically approaches the fixed point.

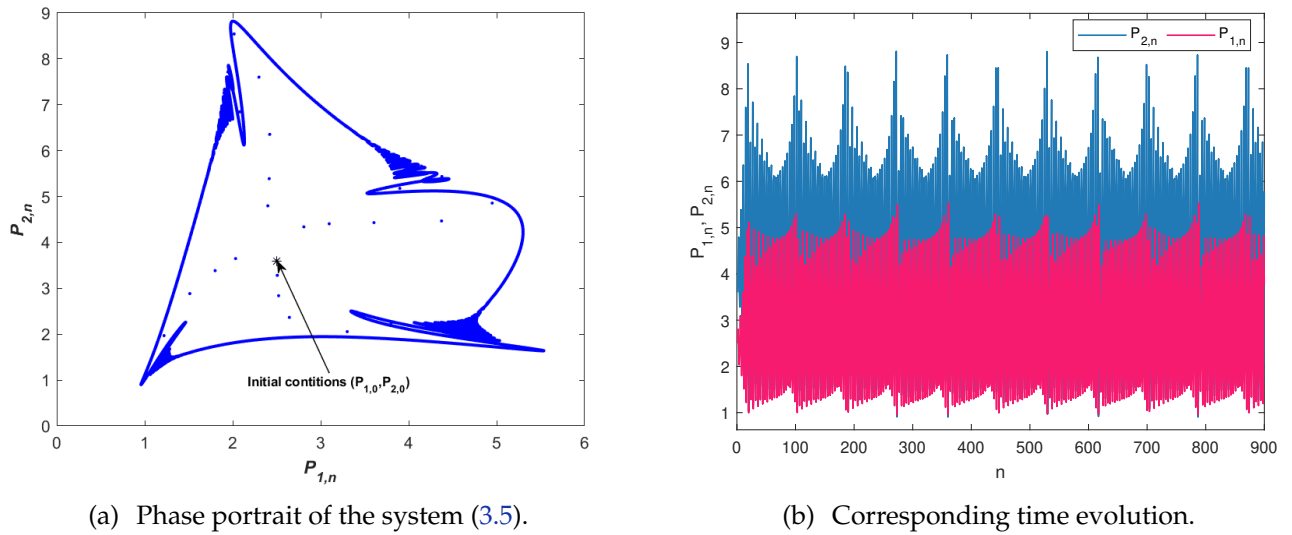
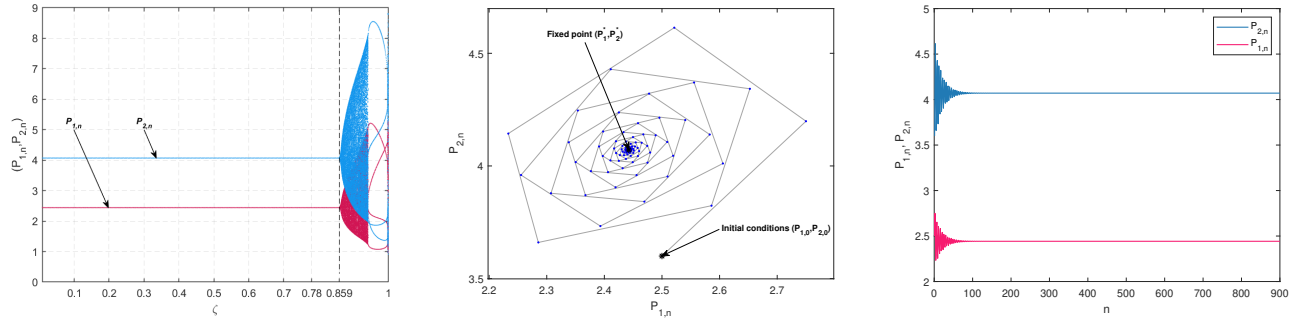


Figure 3.9: (a) Phase portrait of the system (3.5) before applying the hybrid control strategy. (b) Corresponding time evolution for the parameter values: $R = 3.37$, $\kappa = 5.77$, $m = 0.1$, $\epsilon = 2.21$, $C = 1.18$, $D = 0.6$, $\beta = 0.6$, $\alpha = 0.75$ and $S = 1.23$.

In contrast to the uncontrolled dynamics depicted in Figure 3.9, the post-control behavior exhibits a marked reduction in oscillatory and irregular patterns. This is further supported by the phase portraits and time series shown in Figures 3.10(b) and 3.10(c), which demonstrate that trajectories rapidly converge to the equilibrium without exhibiting complex fluctuations. These results confirm the effectiveness of the proposed hybrid control strategy in stabilizing the system at the desired fixed point E .



(a) Bifurcation diagrams of the system (3.13).

(b) Phase portrait of the system (3.13).

(c) Corresponding time evolution.

Figure 3.10: (a) Bifurcation diagrams of the system (3.13) in $\zeta \in (0, 1)$. (a) Phase portrait of the system (3.13). (c) Corresponding time evolution for the parameter values: $R = 3.37$, $\kappa = 5.77$, $m = 0.1$, $\epsilon = 2.21$, $C = 1.18$, $D = 0.6$, $\beta = 0.6$, $\alpha = 0.75$ and $S = 1.23$.

Conclusion

In this thesis, we investigated the dynamical behavior of a discrete fractional-order predator–prey model incorporating ratio-dependent interactions and the fear effect. The study focused on understanding the qualitative properties of the system and the influence of its parameters on the resulting dynamics.

The existence and stability of the fixed points were analyzed using fixed-point theory and the Jury stability criterion. Furthermore, the occurrence of flip and Neimark–Sacker bifurcations was examined, revealing the transition of the system from stable behavior to periodic and more complex dynamical regimes.

To characterize the chaotic dynamics of the model, Lyapunov exponents and the 0–1 test for chaos were employed. The obtained results confirmed the presence of chaos for certain parameter values and demonstrated the rich dynamical behavior generated by the discrete fractional-order system.

Finally, a hybrid chaos control strategy was implemented to suppress chaotic oscillations and stabilize the system. Numerical simulations showed that an appropriate choice of the control parameter successfully drives the system toward asymptotic stability, confirming the effectiveness of the proposed control method.

In addition, the results obtained in **Chapter 3** were organized and presented in a research paper, which has been submitted to a peer-reviewed scientific journal with a good impact factor.

Overall, the results of this work contribute to a deeper understanding of the dynamics of fractional-order predator–prey systems and highlight the important roles of memory effects, fear effects, and control mechanisms in shaping population dynamics.

Future Work

Several directions can be considered for future research. First, the model may be extended by incorporating additional ecological effects such as harvesting, time delays, or environmental noise, which may lead to richer dynamical behavior. Second, different definitions of fractional derivatives can be investigated and compared to further study the influence of memory effects on system dynamics. Third, the analysis may be extended to higher-dimensional systems, such as three-species food chain or food web models. Finally, more advanced and efficient control strategies could be developed to better regulate chaotic dynamics and improve the stability of complex ecological systems.

Qualitative analysis of a discretized generalized conformable fractional-order ratio-dependent Holling-Tanner predator-prey model incorporating the fear effect in the prey population

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Abstract — Predator-prey models are essential instruments for understanding population structure and ecosystem stability. This article analyses the dynamical behaviour of a discretized generalized conformable fractional-order ratio-dependent Holling-Tanner predator-prey model which contains a fear effect in the prey population. The concept of the generalized conformable fractional derivative is utilized to deal with the fractional derivative. Then, the piecewise constant argument approach is successfully invoked to find the discrete-time corresponding model. Moreover, the equilibrium points of the considered model are found. We then discuss the stability of these equilibrium points. The flip and Neimark–Sacker bifurcations are explored using the theory of the bifurcation. In order to determine whether the governed system exhibits regular or chaotic behaviour, the 0–1 test and chaos control are presented. Furthermore, we introduce some numerical confirmations for all parts to verify the used techniques. Finally, the used strategies show their validity to be used in other nonlinear systems.

Keywords: generalized conformable, prey-predator model, stability analysis, bifurcations, 0–1 test, chaos control.

MSC(2010): 37N25. 37N30. 34D35. 34D05. 34D08. 39A30. 39A05.

1. Introduction

The relationships between predators and their prey are an important topic in mathematical ecology, presenting an organized structure for understanding the mechanisms controlling population persistence, extinction, oscillations, and complicated dynamical behaviors. Continuous-time mathematical models, such as Lotka-Volterra and Holling-Tanner types, have been utilized for decades as foundational instruments for demonstrating significant ecological phenomena. Discrete-time schemes, nevertheless, are both theoretically advantageous and

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