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Monitoring and Analysis of a 4-D Memristor-Based Chaotic Circuit

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ملخص

تهدف هذه المذكرة لدراسة وتحليل وتثبيت دارة فوضوية قائمة على الميمريستور ذات بعد رباعي، والمشتقة من نظام تشوا الكلاسيكي. إن إدخال عنصر الميمريستور يمنح النظام ديناميكا لاختية قوية، قادرة على توليد سلوكيات معقدة جدًا مثل الفوضى والفوضى الفائقة.

من أجل قمع هذه الفوضى غير المرغوب فيها وضمان استقرار النظام المدروس، تم تقديم استراتيجية تحكم لاختية تعتمد على التحكم التكيفي. حيث يقوم المتحكم المقترح بضبط معاملات النظام بشكل ديناميكي اعتمادًا على متغيرات الحالة المقاسة، مما يضمن تقارب مسارات النظام نحو نقطة توازنه.

المحاكاة العددية تؤكد لنا فعالية قانون التحكم المقترح، من خلال إظهار قدرته على تثبيت المسارات الفوضوية وإعادة النظام إلى سلوك مستقر في حالة الاستقرار الدائم.

الكلمات المفتاحية : نظام فائق الفوضى، ميمريستور، دارة تشوا، الاستقرار، التحكم التكيفي.

Abstract

This work is devoted to the study, analysis, and stabilization of a four-dimensional memristor-based chaotic circuit derived from the classical Chua system. The introduction of the memristor element endows the system with strongly nonlinear dynamics, which may generate highly complex behaviors such as chaos and hyperchaos.

In order to suppress these undesirable oscillations and ensure the stability of the studied system, a nonlinear control strategy based on adaptive control is presented. The proposed controller dynamically adjusts the system parameters according to the measured state variables, thereby guaranteeing the asymptotic convergence of the trajectories toward a desired equilibrium point. Numerical simulations validate the effectiveness of the proposed control law, demonstrating its ability to stabilize chaotic trajectories and to restore stable steady-state behavior.

Keywords: Hyperchaotic system, Memristor, Chua's circuit, stability, adaptive control.

Résumé

Ce travail de mémoire est consacré à l'étude, l'analyse et la stabilisation d'un circuit chaotique à memristor en dimension quatre, issu du système classique de Chua. L'introduction de l'élément memristor confère au système une dynamique fortement non linéaire, susceptible de générer des comportements très complexes, tels que le chaos et l'hyperchaos.

Afin de supprimer ces oscillations indésirables et d'assurer la stabilité du système étudié, une stratégie de commande non linéaire basée sur le contrôle adaptatif est présentée. Le contrôleur proposé ajuste de manière dynamique les paramètres du système en fonction des variables d'état mesurées, assurant ainsi la convergence asymptotique des trajectoires vers un point d'équilibre désiré. Les simulations numériques valident l'efficacité de la loi de commande proposée, en montrant sa capacité à stabiliser les trajectoires chaotiques et à rétablir un comportement stable en régime permanent.

Mots clés : Système hyperchaotique, Memristor, circuit de Chua, stabilité, contrôle adaptatif.

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El Moatazz and Mohamed

Dedication

I dedicate this humble work to:

My father,

My mother,

My dear wife,

My dear daughter Maram,

My brothers and sisters,

Ferradji family,

My friends and all those who are dear to me

MOHAMED

*I dedicate this dissertation to my beloved mother and father and grandmother,
whose unwavering support and encouragement accompanied me
throughout my academic journey.*

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General Introduction

Nonlinear dynamical chaotic systems play an important role in many scientific and technological fields due to the richness of their behaviors and their potential applications in biological [1], financial [2] and engineering contexts [3]. Among these systems, chaotic circuits have attracted particular interest over the past decades, especially since the introduction of Chua's system [4], which is recognized as one of the simplest electronic models capable of generating complex chaotic behavior.

The emergence of the memristor as the fourth fundamental element of electrical circuits has opened new perspectives in the study of nonlinear systems. The integration of the memristor into classical chaotic circuits significantly enriches their dynamics by introducing memory effects and enhanced nonlinearity. These properties can lead to complex phenomena such as chaos and hyperchaos, characterized by strong sensitivity to initial conditions and a marked unpredictability of the system's time evolution.

In this context, this work focuses on the study, analysis, and stabilization of a modified four-dimensional memristor-based chaotic circuit derived from the classical Chua system [23]. The introduction of the memristor endows the studied model with strongly nonlinear dynamics capable of generating chaotic and hyperchaotic oscillations. Although such behaviors can be exploited in certain applications, they are generally undesirable in engineering systems that require stable and predictable operation.

In order to suppress these unwanted oscillations and ensure system stability, a nonlinear control strategy based on adaptive control is presented in this work. The presented controller dynamically adjusts the system parameters according to the measured

state variables, thereby guaranteeing the asymptotic convergence of the trajectories to a desired equilibrium point, even in the presence of parametric uncertainties.

The effectiveness of the presented control law is evaluated through extensive numerical simulations. The obtained results demonstrate the ability of the developed control strategy to stabilize the chaotic trajectories of the studied system and restore stable steady-state behavior, thus confirming the relevance of the adopted method.

This work is composed of two chapters: the first presents the fundamental concepts of chaotic dynamical systems, while the second is devoted to the study and stabilization of a four-dimensional memristor-based chaotic circuit derived from the modified Chua system.

An Overview of Key Concepts in Chaotic Dynamical Systems

This chapter introduces an overview of key concepts in chaotic dynamical systems. Unlike linear systems, nonlinear systems exhibit complex behaviors such as limit cycles, bifurcations, chaos, and strange attractors with fractal structure [5].

The discussion covers key topics including phase space representation, equilibrium points, stability analysis, and the mechanisms by which stability is preserved or lost. It then explores different types of attractors (regular and strange), followed by an overview of chaos theory.

1.1 Definition of a Dynamical System

A dynamical system is a triple (X, T, φ) , where T is a time set, X is a state space, and $f : X \rightarrow X$ is a family of evolution operators parameterized by $t \in T$, which allows us to define the state of the system at any time from a vector of initial conditions [6].

We distinguish two types of dynamical systems. Dynamical systems derived from iterative equations which are categorised as discrete time systems. Whereas, those governed by differential which are designated as continuous-time systems.

- If $T = \mathbb{R}$, the continuous-time dynamical system is represented by a system of

differential equations of the form:

$$\frac{dx}{dt} = f(x, t), \quad (1.1)$$

where $x \in \mathbb{R}^n$ denotes the state vector at time t , and f describes the system's evolution.

- If $T = \mathbb{Z}$, the discrete-time dynamical system is represented by the following equation:

$$x_{k+1} = f(x_k), \quad (1.2)$$

where $x_k \in \mathbb{R}^n$ denotes the state vector at the step k , and f a mapping that governs the system's evolution over time.

Example 1.1.1 Consider the simple pendulum described by the following system:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\sin(x_1). \end{cases} \quad (1.3)$$

where x_1 represents the angular displacement and x_2 the angular velocity.

This system is a continuous nonlinear dynamical system.

Example 1.1.2 Consider the Lotka–Volterra system which is given by:

$$\begin{cases} \dot{x} = x(a - by), \\ \dot{y} = y(-c + dx), \end{cases} \quad (1.4)$$

where x represents the prey population and y the predator population, and $a, b, c, d > 0$.

This system models the interaction between predator and prey populations.

In the rest of our work will focus on continuous dynamical system (1.1).

1.1.1 Autonomous and non autonomous systems

A dynamical system (1.1) is called autonomous if the vector field f does not depend explicitly on the time t . Conversely, if the term t appears explicitly in the system (1.1), the system is classified as non autonomous. Non-autonomous systems are characterised by a greater degree of complexity in their numerical handling. However, it is possible to transform such systems into autonomous ones by introducing appropriate variable changes, by increasing the dimension of the system by one. This transformation serves to simplify numerical solving by eliminating explicit dependencies.

1.1.2 Qualitative behaviour of dynamical systems

Here, we examine how the behaviour of dynamic systems can be understood through the geometric and qualitative analysis of their evolution over time. We will explore how the system's state traces out trajectories in state space or phase space, and how these trajectories form orbits that reveal patterns in the system's dynamics. The notion of flows will also be introduced to describe the continuous motion of states and to comprehend the system's structure and its evolution over time.

Phase space and state space

The phase space constitutes a multidimensional framework that facilitates the geometric interpretation of the temporal behaviour of a dynamic system. The state space denotes a set of coordinates or variables required to characterise the system fully

Flow

The flow of a dynamical system (1.1) is the expression of the trajectories or the bundle of its trajectories in phase space [6].

Let $x(t, x_0)$, $x_0 \in U$, be a solution of (1.1) with initial condition $x(0) = x_0$. The map $\varphi_t : U \rightarrow \mathbb{R}^n$ defined by

$$\varphi_t(x_0) = x(t, x_0) \tag{1.5}$$

is called the flow of (1.1). It possesses the following properties:

- (i) φ_t is of class $C^{+\infty}$.
- (ii) $\varphi_0(x_0) = x_0$.
- (iii) $\varphi_{t+s}(x_0) = \varphi_t(\varphi_s(x_0))$.

Equilibrium Points

Equilibrium points constitute a fundamental concept to understand and analyse a system's local behaviour. These points also called fixed, critical points play a crucial role in dynamical systems.

Definition 1.1.1 *The equilibrium point of the system (1.1) is a state x^* in the phase space satisfying the following equation*

$$f(x^*) = 0. \quad (1.6)$$

1.1.3 Limit sets

Definition 1.1.2 *A point a is said to be an ω -limit point of a trajectory $x(x_0, t)$ of (1.1), if there exists a sequence $t_n \rightarrow +\infty$ such that :*

$$\lim_{n \rightarrow +\infty} \varphi_{t_n}(x_0) = a.$$

Definition 1.1.3 *A point b is said to be an α -limit point of a trajectory $x(x_0, t)$ of (1.1) if there exists a sequence $t_n \rightarrow -\infty$ such that:*

$$\lim_{n \rightarrow +\infty} \varphi_{t_n}(x_0) = b.$$

Remark 1.1.1 *The set of α -limit points (respectively ω -limit points) of $x(x_0, t)$ is denoted by $\alpha(x_0)$ (respectively $\omega(x_0)$). The set $\alpha(x_0) \cup \omega(x_0)$ is called the limit set of $x(x_0, t)$.*

Example 1.1.3 *Consider the system:*

$$\dot{x} = -x.$$

The solution is:

$$x(t) = x_0 e^{-t}.$$

Since $\lim_{t \rightarrow \infty} x(t) = 0$, the ω -limit set is $\{0\}$.

Limit cycles

A closed trajectory C in the phase space defines a limit cycle, it represents an isolated periodic orbit of a dynamical system. The autonomous system (1.1) may admit a limit cycle if for every point $x_0 \in C$, we have

$$\phi_{t+T}(x_0) = \phi_t(x_0), \forall t \in \mathbb{R}, \quad (1.7)$$

where T is the period of the cycle C .

Remark 1.1.2 A limit cycle C of the system (1.1) is said to be:

- *Stable*, if all trajectories initiated sufficiently close to C asymptotically approach C as $t \rightarrow \infty$.
- *Unstable*, if nearby trajectories diverge from C as $t \rightarrow \infty$.
- *Semi-stable*, if some trajectories that initiate near C approach it as $t \rightarrow \infty$, whereas others diverge from it.

1.2 Stability analysis of dynamical systems

The notion of stability refers to behaviour that persists over time, allowing us to formalise the concept of whether the trajectory becomes the solution at a point close to an equilibrium point. This is an important consideration because, in practice, initial conditions are uncertain and it is desirable for two neighbouring initial conditions to lead to neighbouring trajectories at all times, including infinitely long times. A natural approach to studying this behaviour would be to solve the differential equation and analyse the solutions. In general, however, differential equations are not solvable in closed form.

1.2.1 Stability in the Sense of Lyapunov

The notion of stability refers to behaviour that persists over time, allowing us to formalize the concept of whether the trajectory remains close to an equilibrium point. Initial conditions are uncertain in practice, so it is desirable for two neighbouring initial conditions to lead to neighbouring trajectories at all times.

$$\begin{cases} \dot{x} = f(x(t), t) \\ x(t_0) = 0 \end{cases} \quad (1.8)$$

Definition 1.2.1 (Stability) *The solution x^* is a stable point of the system (1.1), if there exists a positive real $\delta(\epsilon)$ such that:*

$$\forall \epsilon > 0, \exists \delta(\epsilon) > 0 : \|x_0 - x^*\| < \delta \Rightarrow \|x(t) - x^*\| < \epsilon.$$

Definition 1.2.2 (Attractiveness) *The equilibrium x^* is an attractive point of the system (1.1), if there exists $\rho > 0$ such that:*

$$\|x_0 - x^*\| < \rho \Rightarrow \lim_{t \rightarrow \infty} \|x(t) - x^*\| = 0.$$

Definition 1.2.3 (Asymptotic stability) *The solution x^* is asymptotically stable if it is both stable and attractive:*

$$\|x_0 - x^*\| < \delta \Rightarrow \lim_{t \rightarrow \infty} \|x(t, x_0) - x^*\| = 0.$$

Definition 1.2.4 (Exponential stability) *The equilibrium x^* is exponentially stable if there exist $a, b > 0$ such that:*

$$\|x(t_0) - x^*\| < \delta \implies \|x(t) - x^*\| \leq a\|x(t_0) - x^*\|e^{-b(t-t_0)}, \quad \forall t \geq t_0.$$

Remark 1.2.1 *Asymptotic stability requires that the trajectories converge to the equilibrium point. In contrast, stability only implies that the trajectories remain in a neighbourhood of equilibrium point without necessarily approaching it.*

So, the difference between a stable and an asymptotically stable equilibrium remains in the

response of the system to small perturbations of the initial state. For a stable equilibria, such perturbations may lead to small but persistent oscillations. In contrast, around an asymptotically stable equilibrium, the oscillations diminish over time and the system reverts to equilibrium.

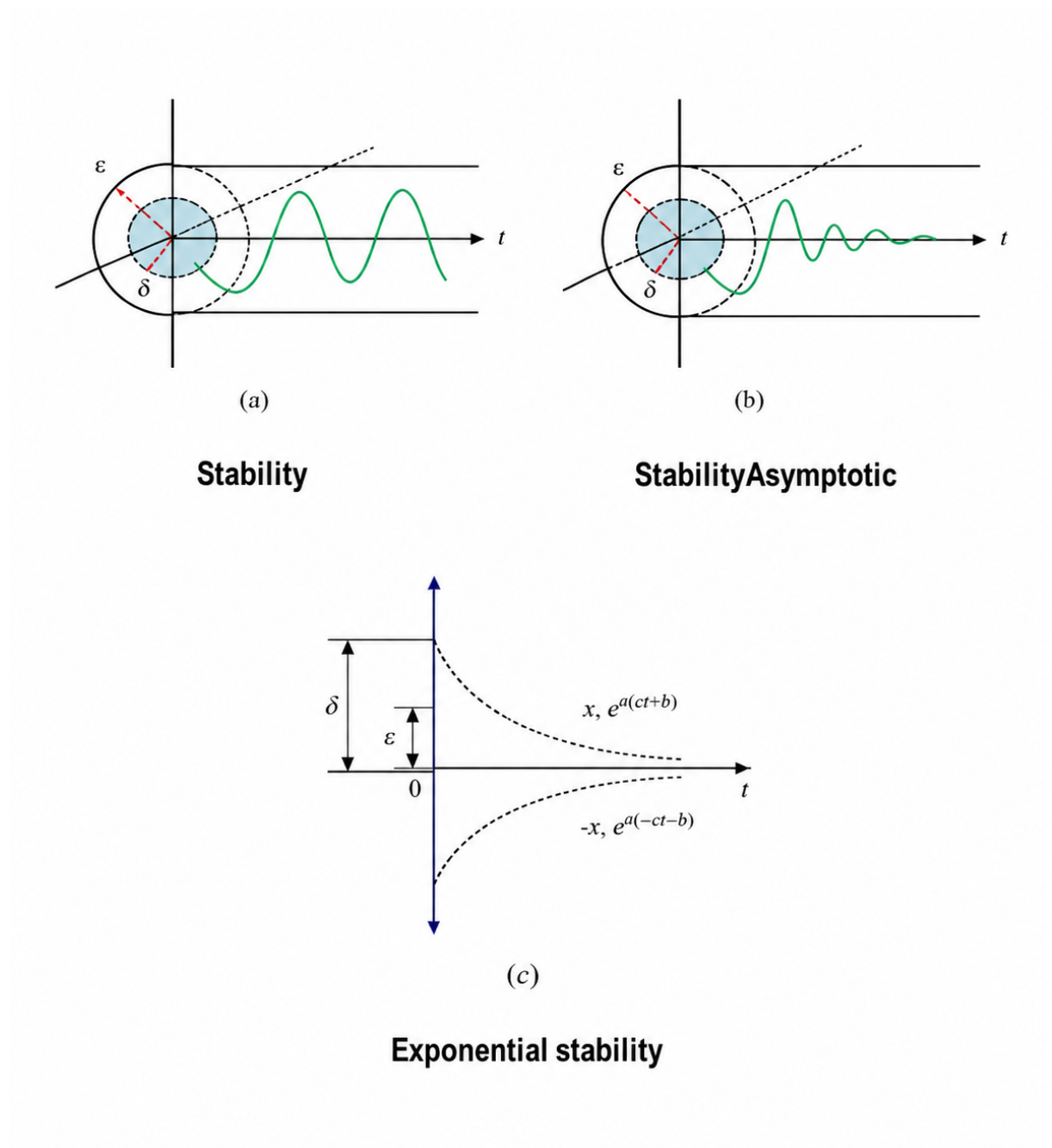


Figure 1.1: Different types of Lyapunov stability for dynamical systems.

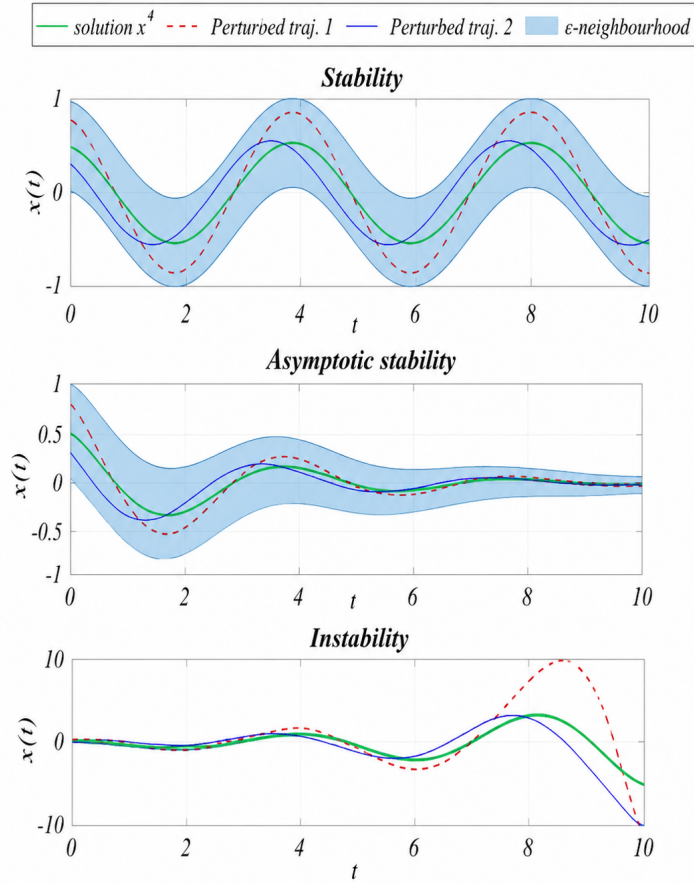


Figure 1.2: Geometric interpretation of stability concepts in phase space.

1.2.2 Stability of a Nonlinear System

To study the stability of a nonlinear system, we present two methods:

- Direct method (Lyapunov function)
- Indirect method (linearization)

Lyapunov's direct Method

The Lyapunov's direct method is fundamental to analyse the stability of nonlinear systems (1.1), without needing their analytical solutions. It is based construction of a

scalar function, known as a Lyapunov function, whose properties reflect the stability of the system.

Let $\bar{x}$ be an equilibrium point of (1.1). Let $V : W \rightarrow \mathbb{R}$ be a differentiable function defined on a neighborhood W of $\bar{x}$ such that $V(\bar{x}) = 0$ and $V(x) > 0$ if $x \neq \bar{x}$.

Let:

$$\dot{V} = \sum_{j=1}^n \frac{\partial V}{\partial x_j} \dot{x}_j = \sum_{j=1}^n \frac{\partial V}{\partial x_j} f_j(x).$$

We have the following theorem .

Theorem 1.2.2 (Lyapunov stability theorem [7])

- (i) If $\dot{V} \leq 0$ in $W - \{\bar{x}\}$, then $\bar{x}$ is stable.
- (ii) If $\dot{V} < 0$ in $W - \{\bar{x}\}$, then $\bar{x}$ is asymptotically stable.

Example 1.2.1 Consider the following system:

$$\begin{cases} \dot{x}_1 = -x_1, \\ \dot{x}_2 = -x_2. \end{cases} \quad (1.9)$$

Define:

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2).$$

Then:

$$\dot{V}(x) = -x_1^2 - x_2^2 \leq 0.$$

Thus, the equilibrium $(0,0)$ is stable.

Indirect Method (Linearization)

The aim of indirect method is to study the behavior of the nonlinear system in the neighborhood of their equilibrium points. For an equilibrium point taken as the origin, we can write the Taylor expansion of f as:

$$f(x) = Df(0)x + \frac{1}{2}D^2f(0)(x, x) + \dots \quad (1.10)$$

The matrix

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}(x) \quad (1.11)$$

is called the Jacobian matrix of f (its determinant is the Jacobian). For small x , the system (1.10) shows that the behavior of the system near 0 is that of the linearized system:

$$\dot{x} = Df(0)x, \quad (1.12)$$

where:

$$Df(0) = A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}.$$

If A has n distinct eigenvalues $\lambda_i, i = 1, \dots, n$, the solution of (1.12) is given by:

$$x = \sum_{i=1}^n c_i \alpha^{(i)} e^{\lambda_i t},$$

where $\alpha^{(i)}$ is the eigenvector corresponding to the eigenvalue λ_i and $c_i, i = 1, 2, \dots, n$ are constants (determined by the initial conditions). We deduce that:

- (a) If all eigenvalues λ_i have negative real parts, the equilibrium point is asymptotically stable.
- (b) If one of the eigenvalues has a positive real part, the fixed point is unstable.
- (c) If $Df(0)$ has no zero or purely imaginary eigenvalue, the equilibrium point is a hyperbolic point; otherwise, it is non-hyperbolic.
- (d) If there exist i and j such that $\operatorname{Re}\lambda_i < 0$ and $\operatorname{Re}\lambda_j > 0$, the fixed point is a saddle point.
- (e) If all eigenvalues of $Df(0)$ are real and of the same sign, the equilibrium point is a node. (A stable node is a sink. An unstable node is a source).

Example 1.2.2 Consider the linear system [7]:

$$\dot{x} = Ax, \quad \text{with } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

The eigenvalues of matrix A are solutions of the equation:

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0, \quad \text{with } \text{tr}(A) = a + d, \quad \det(A) = ad - bc.$$

Assuming matrix A is diagonalizable, the solutions of (1.7) are then of the form:

$$x(t) = c_1 \alpha^{(1)} e^{\lambda_1 t} + c_2 \alpha^{(2)} e^{\lambda_2 t}, \quad \lambda_{1,2} = (\text{tr}(A) \pm \sqrt{\Delta})/2, \quad \text{if } \Delta = \text{tr}(A)^2 - 4 \det(A) \neq 0.$$

$$x(t) = (c_1 \alpha^{(1)} + t c_2 \alpha^{(2)}) e^{tr(A)t/2}, \quad \text{if } \Delta = \text{tr}(A)^2 - 4 \det(A) = 0.$$

where $\alpha^{(1)}, \alpha^{(2)}$ are eigenvectors of A , c_1, c_2 are integration constants. We deduce the classification of fixed points:

- (a) If $\Delta = \text{tr}(A)^2 - 4 \det(A) < 0$, $\text{tr}(A) \neq 0$: the equilibrium point is asymptotically stable if $\text{tr}(A) < 0$, unstable if $\text{tr}(A) > 0$. Trajectories near 0 are spirals. The fixed point is called a focus.
- (b) If $\Delta = \text{tr}(A)^2 - 4 \det(A) < 0$, $\text{tr}(A) = 0$: the fixed point is a center or elliptic point. Trajectories near 0 are ellipses. The origin is a stable equilibrium point but not asymptotically stable.
- (c) If $\Delta = \text{tr}(A)^2 - 4 \det(A) = 0$, $\text{tr}(A) \neq 0$: the equilibrium point is a node, asymptotically stable if $\text{tr}(A) < 0$, unstable if $\text{tr}(A) > 0$.
- (d) If $\Delta = \text{tr}(A)^2 - 4 \det(A) > 0$, $\det(A) > 0$: the eigenvalues are real and of the same sign, the fixed point is an improper node, asymptotically stable if $\text{tr}(A) < 0$, unstable if $\text{tr}(A) > 0$.
- (e) If $\Delta = \text{tr}(A)^2 - 4 \det(A) > 0$, $\det(A) < 0$: the eigenvalues are real and of opposite signs, the fixed point is a saddle point.

The different equilibrium points in the phase space $\mathbb{R}^2$ are indicated in Figure 1.2.2, where we also distinguish nodes (equal real eigenvalues) and improper nodes (real eigenvalues of the same sign but unequal). The type and stability of equilibrium points of a linear system in $\mathbb{R}^2$ is given in Table 1.1.

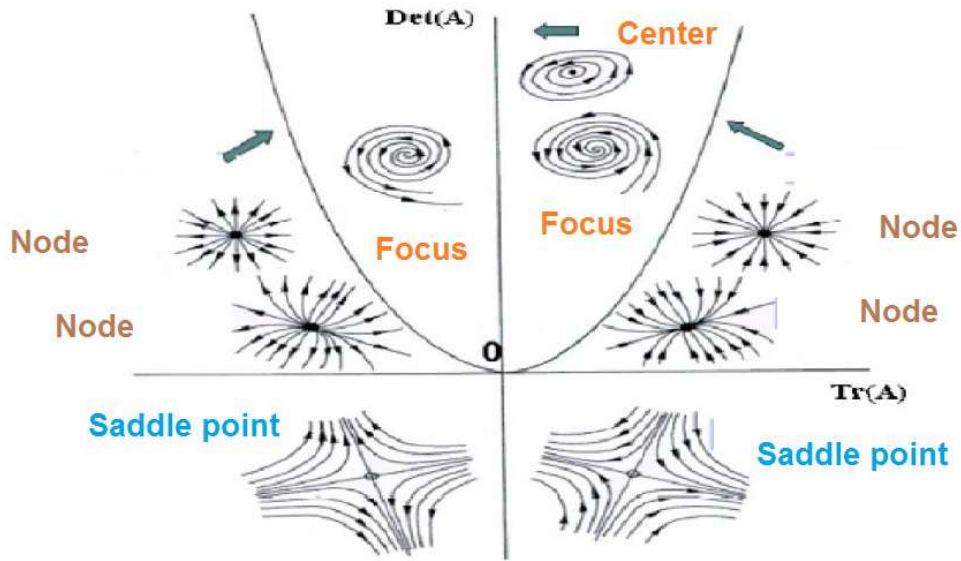


Figure 1.3: Topological classification of equilibrium points in two-dimensional systems

Eigenvalues	Equilibrium point type	Stability
$\lambda_1 > \lambda_2 > 0$	Node	Unstable
$\lambda_1 < \lambda_2 < 0$	Node	Asymptotically stable
$\lambda_2 < 0 < \lambda_1$	Saddle point	Unstable
$\lambda_1 = \lambda_2 > 0$	Proper or improper node	Unstable
$\lambda_1 = \lambda_2 < 0$	Proper or improper node	Asymptotically stable
$\lambda_1, \lambda_2 = a \pm bi, a > 0$	Focus	Unstable
$\lambda_1, \lambda_2 = a \pm bi, a < 0$	Focus	Asymptotically stable
$\lambda_1 = bi, \lambda_2 = -bi$	Center	Stable

Table 1.1: Type and stability of equilibrium points of a linear system in $\mathbb{R}^2$

1.2.3 Hartman-Grobman Theorem

In mathematics, when studying dynamical systems, the Hartman-Grobman theorem or linearization theorem is an important theorem concerning the local behavior of dynamical systems in the neighborhood of a hyperbolic equilibrium point.

Definition 1.2.5 *Two flows φ_t and ψ_t are said to be topologically equivalent in a neighborhood of an equilibrium point if there exists a homeomorphism h that maps the equilibrium point of the first flow to the equilibrium point of the second flow and conjugates the flows, i.e., $h \circ \varphi_t = \psi_t \circ h$.*

Theorem 1.2.3 (Hartman-Grobman Theorem [8]) *Consider a system (1.1) with flow φ_t . If x^* is a hyperbolic equilibrium point, then there exists a neighborhood V of x^* on which the flow φ_t is topologically equivalent to the flow of the linearized system at x^* .*

1.2.4 Routh-Hurwitz Criterion

In order to prove that an equilibrium point is asymptotically stable [9], it is necessary to calculate the n eigenvalues λ_i of the matrix A and verify that their real parts are strictly negative, i.e., $\forall i, \text{Re}(\lambda_i) < 0$. An algebraic method was developed by Routh-Hurwitz. It relies on the calculation of particular determinants called Routh-Hurwitz determinants.

Consider the following linear system of dimension n :

$$\dot{x} = Ax,$$

where $A = [a_{ij}]$ is a square matrix of dimension n with constant coefficients. The eigenvalues of A are solutions of the characteristic equation:

$$P(\lambda) = \det(A - \lambda I) = 0 \iff \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_{n-1}\lambda + a_n = 0.$$

Consider the following n determinants:

$$H_1 = |a_1|,$$

$$H_2 = \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix},$$

$$H_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix},$$

$$H_k = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots \\ 1 & a_2 & a_4 & \cdots \\ 0 & a_1 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_k \end{vmatrix}.$$

Theorem 1.2.4 (Routh-Hurwitz Criterion) *Let $p(\lambda)$ be a characteristic polynomial such that $a_0 > 0$. For the fixed point to be uniformly asymptotically stable (u.a.s), it is necessary that the principal determinants of the Hurwitz matrices are strictly positive.*

Routh-Hurwitz Criterion in $\mathbb{R}^3$

Consider the system:

$$\dot{x} = Ax,$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

The characteristic equation is:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0.$$

The Routh-Hurwitz determinants are:

$$\begin{aligned}
 H_1 &= |a_1| = a_1, \\
 H_2 &= \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix} = a_1 a_2 - a_3, \\
 H_3 &= \begin{vmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{vmatrix} = a_3 H_2,
 \end{aligned}$$

If $H_1 > 0$, $H_2 > 0$, and $H_3 > 0$, then the origin is asymptotically stable.

1.3 Attractors

In the theory of dynamical systems, an *attractor* is a set in the phase space toward which trajectories of the system evolve after a sufficiently long time. Attractors represent the long-term behavior of a dynamical system and provide important information about the stability and qualitative properties of the system.

More formally, an attractor is an invariant set A such that trajectories starting from a neighborhood of A converge to it as time tends to infinity. An attractor can take different forms depending on the nature of the dynamical system.

The most common types of attractors include:

- **Fixed-point attractor:** also called an equilibrium point, where the trajectory converges to a constant state.
- **Limit cycle:** a closed periodic orbit that attracts nearby trajectories.
- **Quasi-periodic attractor:** associated with motion on invariant tori produced by the interaction of several independent frequencies.
- **Chaotic attractor:** also known as a strange attractor, characterized by sensitive dependence on initial conditions and a fractal geometric structure.

Chaotic attractors arise in many nonlinear systems and exhibit complex geometrical structures that are neither purely periodic nor completely random.

1.3.1 Basin of Attraction

The *basin of attraction* of an attractor is defined as the set of all initial conditions in the phase space that lead trajectories to converge to that attractor as time evolves.

Mathematically, if A is an attractor of a dynamical system, the basin of attraction $B(A)$ is defined as

$$B(A) = \{x_0 \in \mathbb{R}^n : \lim_{t \rightarrow \infty} \phi(t, x_0) \in A\},$$

where $\phi(t, x_0)$ denotes the trajectory of the system starting from the initial condition x_0 . Basins of attraction are important because many nonlinear systems possess multiple attractors. In such cases, the long-term behavior of the system strongly depends on the initial conditions. The boundaries separating different basins can exhibit highly complex and even fractal structures, particularly in chaotic systems [7].

Understanding the basin of attraction allows researchers to determine which stable behavior a system will eventually approach for a given initial condition.

1.3.2 Poincaré Map

The *Poincaré map* is a powerful tool used to analyze continuous dynamical systems by reducing them to discrete mappings. The method was introduced by Henri Poincaré to simplify the qualitative study of differential equations[7].

Instead of studying the entire trajectory of the system in phase space, the Poincaré map observes the intersections of trajectories with a lower-dimensional surface called the *Poincaré section*.

Consider a continuous dynamical system defined by

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n.$$

Let Σ be a hypersurface that is transversal to the trajectories of the system. The Poincaré map is defined as

$$P : \Sigma \rightarrow \Sigma$$

which maps a point on the surface Σ to the next intersection of the trajectory with the

same surface.

This construction transforms the continuous-time system into a discrete dynamical system. Fixed points of the Poincaré map correspond to periodic orbits of the original system. For chaotic systems, the Poincaré map often reveals complex geometric structures that reflect the presence of strange attractors.

1.4 Bifurcation Theory

Bifurcation theory studies qualitative changes in the behavior of a dynamical system when a control parameter is varied. These changes correspond to the creation, disappearance, or modification of equilibria, periodic orbits, or more complex invariant sets. Bifurcations constitute a fundamental mechanism through which simple dynamical regimes evolve into complex and chaotic behaviors.

Consider a parameter-dependent dynamical system of the form

$$\dot{x} = f(x, \mu), \tag{1.13}$$

where $x \in \mathbb{R}^n$ is the state variable and $\mu \in \mathbb{R}$ is a control parameter. A bifurcation occurs at a critical value $\mu = \mu_c$ when the qualitative structure of the phase portrait changes.

Bifurcations are commonly classified into local and global bifurcations. Local bifurcations can be detected using linearization techniques, whereas global bifurcations involve large-scale changes in invariant sets.

1.4.1 Local Bifurcations in One Dimension

Local bifurcations in one-dimensional dynamical systems arise when the stability or number of equilibrium points changes as a parameter varies. Consider the autonomous system

$$\dot{x} = f(x, \mu), \tag{1.14}$$

with $x \in \mathbb{R}$. An equilibrium point x_0 satisfies

$$f(x_0, \mu) = 0. \quad (1.15)$$

A local bifurcation occurs when the Jacobian

$$\frac{\partial f}{\partial x}(x_0, \mu) \quad (1.16)$$

vanishes, leading to a loss of hyperbolicity.

Saddle-Node Bifurcation

In a saddle-node (fold) bifurcation, two equilibrium points one stable and one unstable collide and annihilate each other as the control parameter crosses a critical value.

Transcritical Bifurcation

A transcritical bifurcation involves an exchange of stability between two intersecting equilibrium branches.

Pitchfork Bifurcation

Pitchfork bifurcations occur in symmetric systems. A single equilibrium loses stability and gives rise to two new symmetric equilibria in the supercritical case, while unstable equilibria merge into one in the subcritical case.

1.4.2 Bifurcations in Dimension Greater than One

In higher-dimensional systems ($n \geq 2$), bifurcation phenomena are richer due to the presence of oscillatory modes and invariant manifolds. Such systems are described by

$$\dot{x} = \mathbf{f}(x, \mu), \quad x \in \mathbb{R}^n. \quad (1.17)$$

A fundamental bifurcation in higher dimensions is the Hopf bifurcation, which occurs when a pair of complex conjugate eigenvalues of the Jacobian matrix crosses the imaginary axis. This bifurcation leads to the birth or destruction of a limit cycle and plays a

crucial role in the onset of oscillatory and chaotic dynamics.

Example 1.4.1 *Consider the system:*

$$\begin{cases} \dot{x} = \mu x - y - x(x^2 + y^2), \\ \dot{y} = x + \mu y - y(x^2 + y^2), \end{cases} \quad (1.18)$$

where μ is a real parameter.

The origin $(0, 0)$ is an equilibrium point.

When $\mu < 0$, the equilibrium is stable. When $\mu > 0$, a limit cycle appears around the origin.

Thus, the system undergoes a Hopf bifurcation at $\mu = 0$.

1.5 Chaotic Systems

Chaos refers to a class of behaviors observed in deterministic nonlinear dynamical systems that exhibit apparent randomness and long-term unpredictability despite being governed by precise evolution laws.

Chaotic dynamics are encountered in numerous physical, biological, and engineering systems and represent a cornerstone of nonlinear science.

1.5.1 Definitions of Chaos

Several definitions of chaos exist in the literature. A widely accepted mathematical definition is due to Devaney. According to this definition, a dynamical system is chaotic if it satisfies the following properties [10].

- Sensitivity to initial conditions,
- Topological transitivity,
- Density of periodic orbits.

1.5.2 Characteristics of Chaotic Systems

Chaotic systems exhibit several characteristic features

- Extreme sensitivity to initial conditions,
- Aperiodic long-term behavior,
- Bounded trajectories,
- Existence of strange attractors with fractal structure.

1.5.3 Routes to Chaos (Scenarios of Transition to Chaos)

Chaos typically emerges through a sequence of bifurcations as a system parameter is varied. The most common routes to chaos include:

- Period-doubling (Feigenbaum) cascade,
- Quasi-periodic route via torus breakdown,
- Intermittency,
- Crisis-induced chaos.

1.5.4 Quantification of Chaos

Quantitative tools are required to characterize chaotic dynamics. Common measures include Lyapunov exponents, fractal dimensions, entropy measures, and statistical indicators .

Lyapunov Exponents

The Lyapunov exponents measure the average exponential rate at which nearby trajectories. The Lyapunov exponent λ is defined by

$$\lambda = \lim_{t \rightarrow +\infty} \frac{1}{t - t_0} \log_2 \frac{d(t)}{d_0}, \quad (1.19)$$

where, d_0 define the initial infinitesimal separation between two nearby trajectories, $d(t)$ is the separation at time t , and t_0 is the initial time.

A Positive Lyapunov exponent is a hallmark of chaotic behaviour, indicating extreme sensitivity to initial conditions. For an n dimensional system, there exists a spectrum of n Lyapunov exponents, denoted $\lambda_1 > \lambda_2 > \dots > \lambda_n$.

The following classification of attractors is obtained based on the number of positive exponents :

- If $\lambda_i < 0$ for all i , the system exhibits an asymptotically stable equilibrium point.
- If $\lambda_1 = 0$ and $\lambda_i < 0$ for all $i \geq 2$, the system admits a stable limit cycle.
- If at least one Lyapunov exponent satisfies $\lambda_i > 0$, the system displays chaotic dynamics.
- If at least two Lyapunov exponents satisfy $\lambda_i > 0$ and the system dimension satisfies $n \geq 4$, the dynamics are hyperchaotic.

Fractal dimension

Chaotic attractors often exhibit complex geometric structure and self-similarity quantified by fractal dimensions. A Fractal dimension is a quantitative measure that captures the inherent geometrical complexity of an attractor, often taking non-integer values that reflect how the structure of an attractor scales across different spatial resolutions. Several notions of fractal dimension have been introduced in the literature, such as Hausdorff Dimension, Box-Counting Dimension, Correlation Dimension and Kaplan-Yorke Dimension.

Kaplan-Yorke Dimension

It combines the information from the Lyapunov exponents into a single measure of attractor complexity. Let the Lyapunov exponents $\lambda_1 > \lambda_2 > \dots > \lambda_n$ satisfying:

$$\sum_{i=1}^j \lambda_i \geq 0 \text{ and } \sum_{i=1}^{j+1} \lambda_i \leq 0. \quad (1.20)$$

Hence, the Kaplan-Yorke dimension is given by:

$$D_{KY} = j + \frac{\sum_{i=1}^j \lambda_i}{|\lambda_{j+1}|} \quad (1.21)$$

Modified Memristor-Based Chua's Circuit

In 1971, Leon Chua theoretically introduced the memristor as the fourth fundamental circuit element alongside the resistor, capacitor, and inductor [4]. Later, in 2008, researchers at Hewlett-Packard (HP) laboratories reported the successful fabrication of a physical implementation of the memristor [13]. This prototype uses thin TiO_2 films with doped and undoped regions located between two nanoscale metal electrodes.

A memristor is a nonlinear resistor with memory effect that may be either flux-controlled or charge-controlled [14]. It is a two-terminal device defined by the relationship

$$f(\varphi, q) = 0.$$

The equations (2.1) and (2.2) describe flux-controlled and charge-controlled memristor respectively [14, 15]

$$W(\varphi) = \frac{dq(\varphi)}{d\varphi}, i = W(\varphi)v, \quad (2.1)$$

$$M(q) = \frac{d\varphi(q)}{dq}, v = M(q)i, \quad (2.2)$$

where φ and q denote magnetic flux and charge, $W(\varphi)$ and $M(q)$ are called the memductance and memristance, respectively.

Several memristor-based models have been recently studied [16, 17, 18, 23, 19, 20, 21, 22].

Abdelouhab et al. [23] proposed an alternative memristor-based Chua's circuit by using a negative conductance and a passive flux-controlled memristor described by Eq.(2.1) in parallel, instead of the classical nonlinear diode in the original circuit's and by shifting the position of the inductance which is moved between the two capacitances as shown in Figure 2.1.

The flux-controlled memristor considered in this study is described by the piecewise quadratic function $q(\varphi)$.

$$q(\varphi) = -a\varphi + \frac{b}{2}\varphi|\varphi|.$$

Where a and b are both positive parameters and its memductance function is given by

$$W(\varphi) = \frac{dq(\varphi)}{d\varphi} = -a + b|\varphi|.$$

2.1 Mathematical Analysis of Modified Memristor-Based Chua's Circuit

Using Kirchhoff Laws, the dynamics of the modified memristor-based Chua's circuit [23] can be described by the following set of differential equations:

$$\begin{cases} \dot{x} = \alpha[z + \xi x - (-a + b|w|)x], \\ \dot{y} = y - z, \\ \dot{z} = -\beta(x - y) - \gamma z, \\ \dot{w} = x, \end{cases} \quad (2.3)$$

where x , y , z and w are the states and α , β , γ , ξ , a and b assumed to be positive constant parameters.

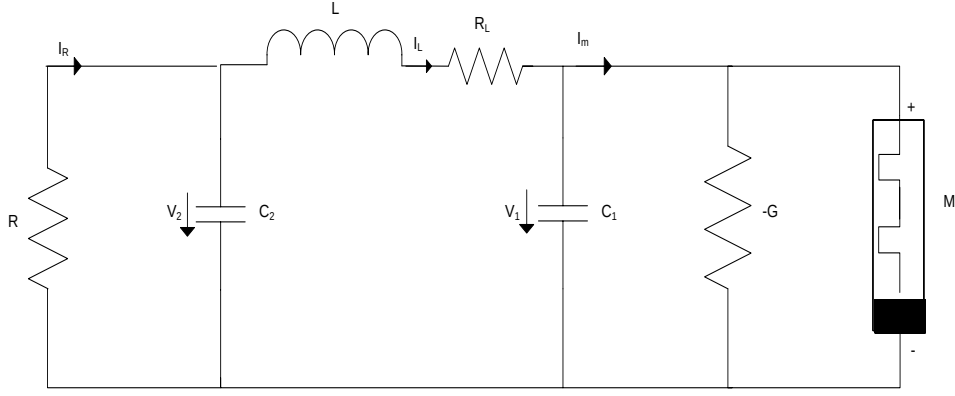


Figure 2.1: Modified memristor-based Chua's circuit

2.1.1 Dissipation

The system (2.3) can be written in its vector form as:

$$\frac{dX}{dt} = f(X) = \begin{bmatrix} f_1(x, y, z, w) \\ f_2(x, y, z, w) \\ f_3(x, y, z, w) \\ f_4(x, y, z, w) \end{bmatrix}, \quad (2.4)$$

where

$$\begin{cases} f_1(x, y, z, w) = \alpha[z(t) + \xi x(t) - (-a + b|w(t)|)x(t)], \\ f_2(x, y, z, w) = y(t) - z(t), \\ f_3(x, y, z, w) = -\beta(x(t) - y(t)) - \gamma z(t), \\ f_4(x, y, z, w) = x(t). \end{cases} \quad (2.5)$$

So

$$\text{div} f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} + \frac{\partial f_4}{\partial w} = \alpha(\xi + a - b|w|) + 1 - \gamma. \quad (2.6)$$

When $|w| > \frac{\xi + a}{b} - \frac{\gamma - 1}{\alpha b}$, the system (2.3) is dissipative.

2.1.2 Equilibrium Points

The equilibrium points are obtained by solving the following equations:

$$\begin{cases} \alpha[z(t) + \xi x(t) - (-a + b|w|)x(t)] = 0, \\ y(t) - z(t) = 0, \\ -\beta(x(t) - y(t)) - \gamma z(t) = 0, \\ x(t) = 0. \end{cases} \quad (2.7)$$

So, the equilibrium points are as the form:

$$X_e = \{(x, y, z, w); x = 0, y = 0, z = 0 \text{ and } w = w_e\}. \quad (2.8)$$

We conclude from X_e that each point on the w – axis is an equilibrium point of (2.3).

The Jacobian matrix at each equilibrium point X_e is

$$J(X_e) = \begin{bmatrix} \alpha(\xi - W(w_e)) & 0 & \alpha & 0 \\ 0 & 1 & -1 & 0 \\ -\beta & \beta & -\gamma & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad (2.9)$$

where

$$W(w_e) = -a + b|w_e|.$$

Its characteristic polynomial is given by:

$$\begin{aligned} P(\lambda) = \lambda [\lambda^3 + [\gamma - 1 - \alpha(\xi - W(w_e))]\lambda^2 + [(-\gamma + 1)\alpha(\xi - W(w_e)) + (1 + \alpha)\beta - \gamma]\lambda \\ + \alpha[(\gamma - \beta)(\xi - W(w_e)) - \beta]] \end{aligned} \quad (2.10)$$

For the system parameter values

$$\alpha = 5, \beta = 5, \gamma = 0.1, \xi = 3, a = 1.5, \text{ and } W(w_e) = -a + b|w_e|, \quad (2.11)$$

the characteristic equation becomes

$$\lambda \left[\lambda^3 + (5b |w_e| - 23.4) \lambda^2 + (50.15 - 4.5b |w_e|) \lambda + 24.5b |w_e| - 135.25 \right] = 0. \quad (2.12)$$

Clearly one eigenvalue vanishes for all values of b and w_e . In order to determine the range of b and w_e where system (2.3) have a three-dimensional stable manifold, one considers the projection of system (2.3) to the eigenspace corresponding to the three other eigenvalues (regardless of the influence of zero eigenvalue), then the Routh-Hurwitz stability criterion is applied to Eq.(2.13).

$$\lambda^3 + (5b |w_e| - 23.4) \lambda^2 + (50.15 - 4.5b |w_e|) \lambda + 24.5b |w_e| - 135.25 = 0. \quad (2.13)$$

Thus the inequalities (2.14) must be satisfied in order to obtain all the roots of the characteristic equation (2.13) with negative real parts.

$$\begin{cases} 5b |w_e| - 23.4 > 0, \\ 24.5b |w_e| - 135.25 > 0, \\ -22.5 |bw_e|^2 + 331.55b |w_e| - 1038.3 > 0, \end{cases} \quad (2.14)$$

hence

$$5.5204 < b |w_e| < 10.221,$$

i.e;

$$\frac{5.5204}{|w_e|} < b < \frac{10.221}{|w_e|}.$$

In contrast, chaos has a greater possibility of occurrence if Eq.(2.12) has one or more roots with positive real parts, that is

$$b |w_e| < 5.5204, \text{ or } b |w_e| > 10.221. \quad (2.15)$$

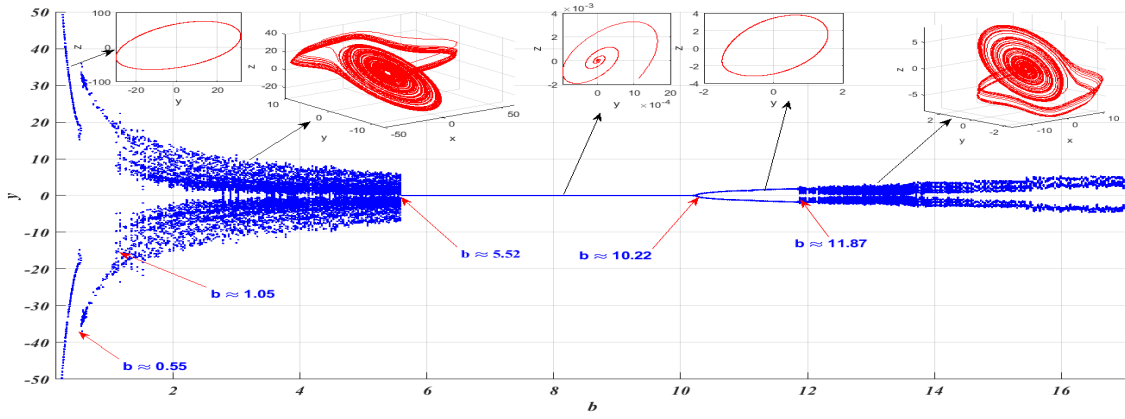


Figure 2.2: Bifurcation diagram of y versus b , with $w_e = -1$

2.1.3 Bifurcation and Lyapunov Exponents

Bifurcation Diagram Versus b and Corresponding Lyapunov Exponents

In this section, one considers the parameter values $\alpha = 5, \beta = 5, \gamma = 0.1, a = 1.5, \xi = 3$ and letting b vary over a certain interval to discuss the complex dynamics of system (2.3) with the initial condition $(x, y, z, w_e) = (-0.5, 0.1, 0.01, -1)$. The bifurcation diagram of y and the corresponding Lyapunov exponents spectrum for b varying from 0.2 to 17, with a step size $h = 0.001$ are obtained as illustrated in Figure 2.2 and Figure 2.3, respectively, which are in good agreement.

From these figures it is clear that system (2.3) displays period 1 orbit for $b \in]0.2, 1.05[\cup]10.22, 11.87[$ and a period 2 orbit for $b \in]10.22, 11.87[$ followed by a period 4 orbit when $b \approx 11.87$ then rapidly it displays hyperchaotic Attractor. For $b \in]1.05, 5.22[\cup]11.88, 17[$ system (2.3) exhibits chaotic and hyperchaotic behavior.

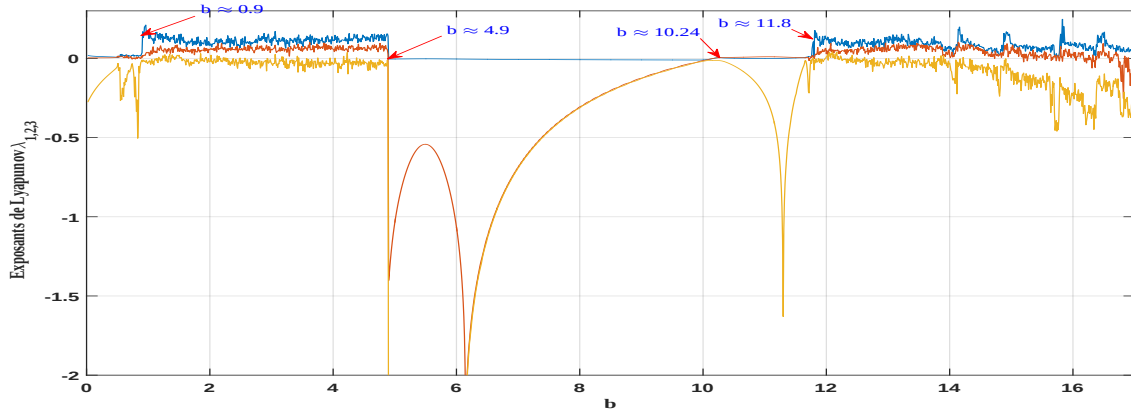


Figure 2.3: The three largest Lyapunov exponents of system (2.3) versus b for $w_e = -1$

In particular, for $b = 12$ the Lyapunov exponent are

$$L_1 \approx 0.1625, L_2 \approx 0.0508, L_3 \approx -0.0124, L_4 \approx -33.1206. \quad (2.16)$$

Since $L_1 + L_2 + L_3 + L_4 = -32.9197 < 0$, $L_1 > 0$, $L_2 > 0$, then the system (2.3) is dissipative. The Kaplan-Yorke dimension of its attractor is

$$D_{KY} = 3 + \frac{L_1 + L_2 + L_3}{|L_4|} = 3.006, \quad (2.17)$$

which is a fractal dimension.

Bifurcation Diagram Versus w_0 and Corresponding Lyapunov Exponents

In order to examine the impact of initial condition values on dynamical behavior of the system (2.3), for the set of parameter values (6), different diagrams are presented to identify chaos.

Considering the initial condition $(x, y, z, w_e) = (-0.5, 0.1, 0.01, w_0)$, the Lyapunov exponents spectrum and the corresponding bifurcation diagram of y , for w_e varying from -15 to 15 with step 0.01 are obtained as shown in Figure 2.4 and Figure 2.5, respectively. From these diagrams one observes that, when the value of initial state w_0 belongs to the following four intervals: $[-15, -11.91]$, $[-5.52, -0.9]$, $[0.9, 5.52]$, $[11.91, 15]$, then system

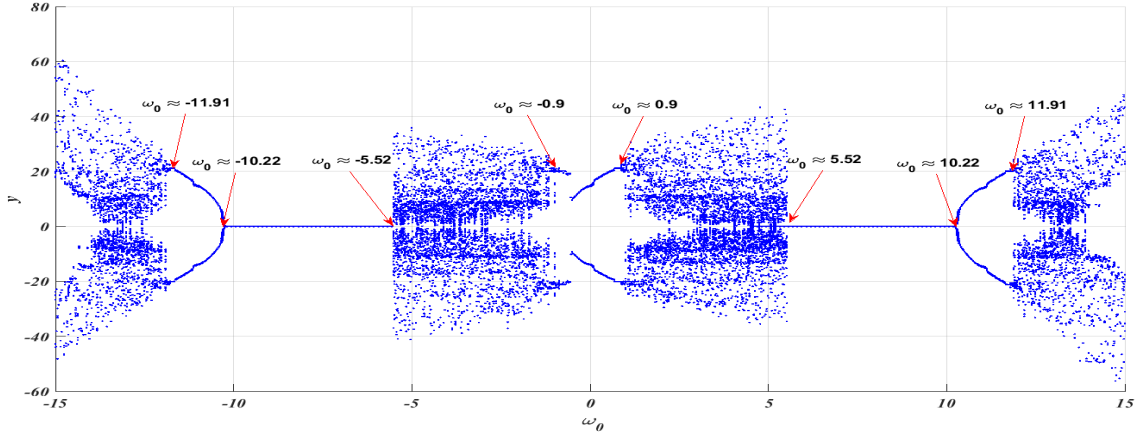


Figure 2.4: Bifurcation diagram of y versus w_e , when $b = 1$

(2.3) exhibits chaos. Furthermore the two diagrams indicate a symmetry versus $w_0 = 0$. Particularly, for $w_e = -3$ the Lyapunov exponents are calculated using Wolf algorithm [12] as

$$L_1 = 0.0978, L_2 = 0.0659, L_3 = -0.0338, L_4 = -38.0524. \quad (2.18)$$

Since $L_1 + L_2 + L_3 + L_4 = -37.9226 < 0$, $L_1 > 0, L_2 > 0$, then the system (2.3) is dissipative and hyperchaotic. The Kaplan-Yorke dimension of its attractor is

$$D_{KY} = 3 + \frac{L_1 + L_2 + L_3}{|L_4|} = 3.0034,$$

which is a fractal dimension.

Some phase portraits are shown in Figure 2.6 for different values w_e . In particular, a period-1 orbits are shown in 2.6(b), 2.6(e), and 2.6(h). Moreover, 2.6(c), 2.6(g) represents a stable equilibrium point, and 2.6(a), 2.6(d), 2.6(f) and 2.6(i) displays chaotic attractors. Figure 2.7 highlight the coexistence phenomena in system (2.3) where different type of attractors (fixed points, limit cycles, chaotic and hyperchaotic attractors) are depicted for different initial conditions: $(-0.5, 0.1, 0.01, -14), (-0.5, 0.1, 0.01, -11), (-0.5, 0.1, 0.01, 0.6), (-0.5, 0.1, 0.01, 3), (-0.5, 0.1, 0.01, 7.5), (-0.5, 0.1, 0.01, 11), (-0.5, 0.1, 0.01, 14)$ with fixed parameter value $b = 1$.

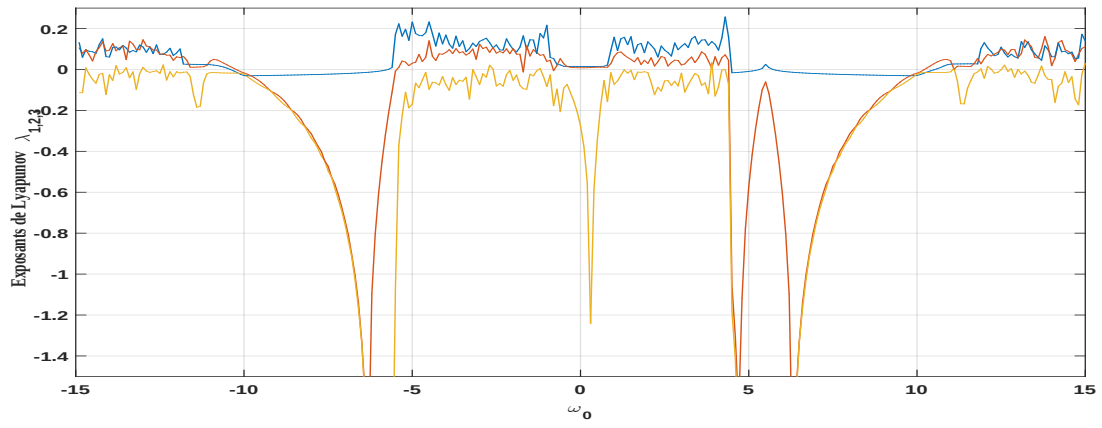


Figure 2.5: The three largest Lyapunov exponents of system (2.3) versus w_e , when $b = 1$

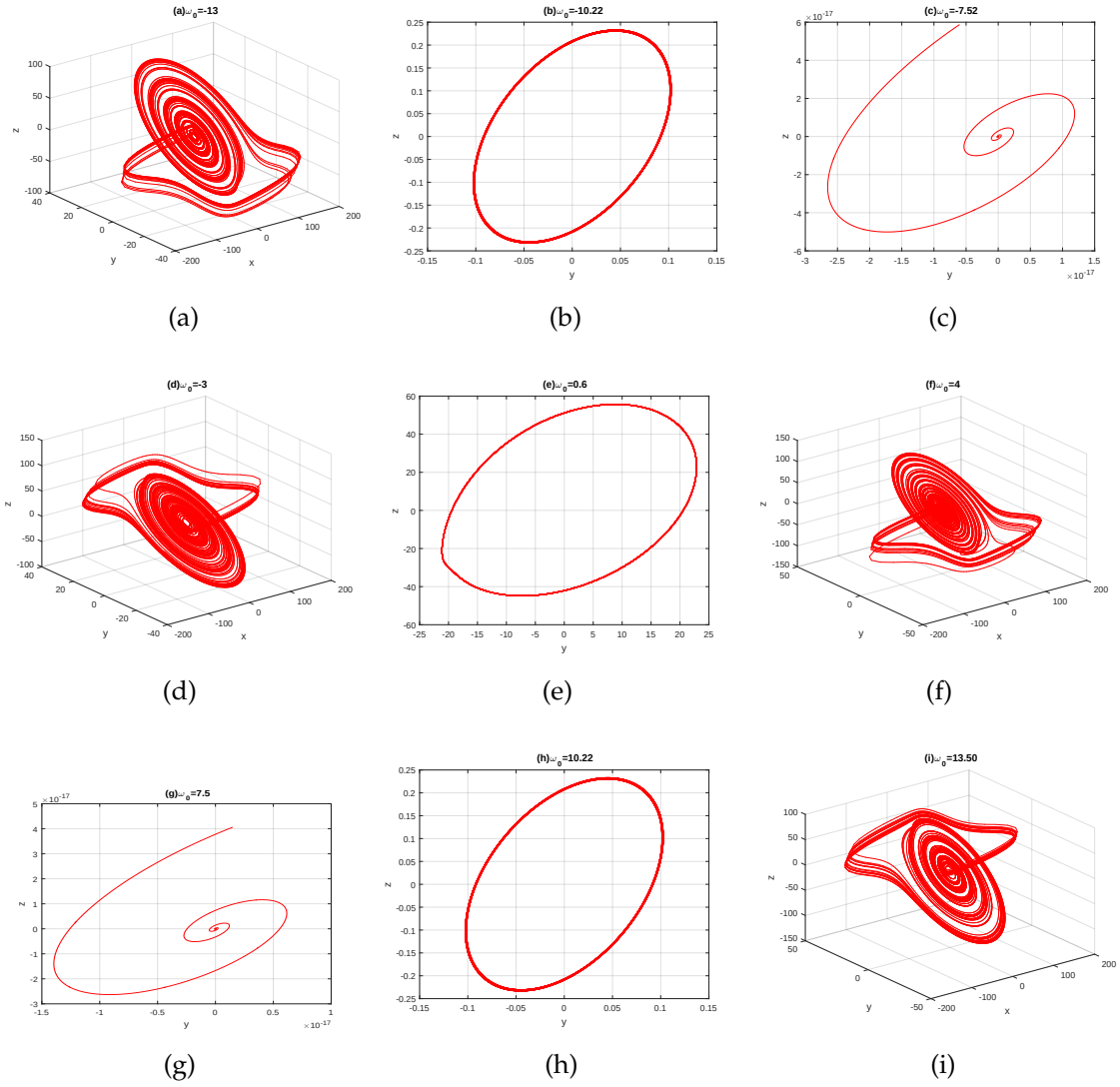


Figure 2.6: Some attractors for different values of initial condition w_e : (a) $w_e = -13$, (b) $w_e = -10.22$, (c) $w_e = -7.52$, (d) $w_e = -3$, (e) $w_e = 0.6$, (f) $w_e = 4$, (g) $w_e = 7.5$, (h) $w_e = 10.22$, (i) $w_e = 13.5$.

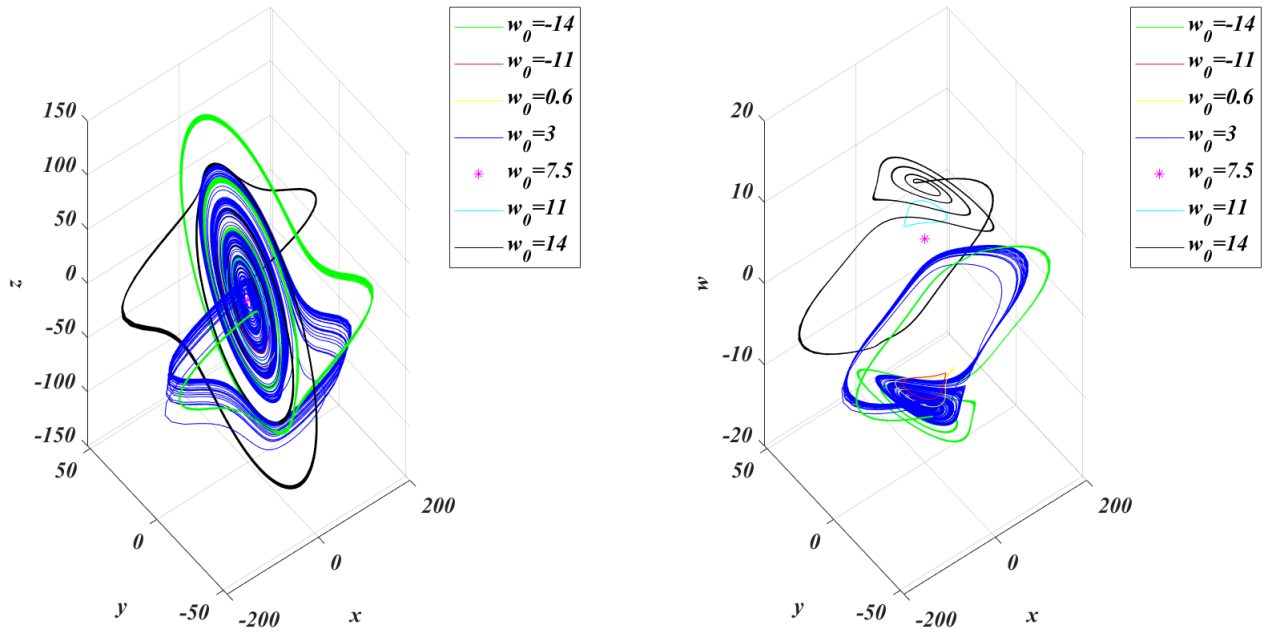


Figure 2.7: Phase portrait of system (2.3) showing various coexisting behaviors for different values of w_e : $w_e = -14, w_e = -11, w_e = 0.6, w_e = 3, w_e = 7.5, w_e = 11, w_e = 14$ with $b = 1$.

2.2 Stability Analysis

Here, we design an adaptive control law to stabilize the modified memristor system, when system parameters are unknown. The stability of the system under adaptive controller is verified using the Lyapunov stability theorem.

2.2.1 Theoretical Study

We consider the controlled memristor-based Chua's system:

$$\begin{cases} \dot{x} = \alpha[z + \xi x - (-a + b|w|)x] + u_1 \\ \dot{y} = y - z + u_2 \\ \dot{z} = -\beta(x - y) - \gamma z + u_3 \\ \dot{w} = x + u_4 \end{cases} \quad (2.19)$$

where α, β, γ, a and ξ are unknown constant parameters.

Assume that the adaptive control laws u_i are selected as:

$$\begin{cases} u_1 = -\hat{\alpha}[z + \xi x - (-a + b|w|)x] - k_1 x \\ u_2 = -y + z - k_2 y \\ u_3 = \hat{\beta}(x - y) + \hat{\gamma} z - k_3 z \\ u_4 = -x - k_4(w - w_e), \end{cases} \quad (2.20)$$

where $\hat{\alpha}, \hat{\beta}$ and $\hat{\gamma}$ are estimates of the parameters α, β and γ , respectively, and $k_i > 0$, $i = 1, 2, 3, 4$ are positive gain constants.

Substituting (2.20) into (2.19), we get:

$$\begin{cases} \dot{x} = e_\alpha[z + \xi x - (-a + b|w|)x] - k_1x \\ \dot{y} = -k_2y \\ \dot{z} = -e_\beta(x - y) - e_\gamma z - k_3z \\ \dot{w} = -k_4(w - w_e), \end{cases} \quad (2.21)$$

where e_α , e_β and e_γ are parameter estimation errors defined as

$$\begin{cases} e_\alpha = \alpha - \hat{\alpha} \\ e_\beta = \beta - \hat{\beta} \\ e_\gamma = \gamma - \hat{\gamma} \end{cases} \quad (2.22)$$

Taking the derivative of (2.24) with respect to time t , we get:

$$\begin{cases} \dot{e}_\alpha = -\dot{\hat{\alpha}}, \\ \dot{e}_\beta = -\dot{\hat{\beta}}, \\ \dot{e}_\gamma = -\dot{\hat{\gamma}}. \end{cases} \quad (2.23)$$

To ensure the convergence of the estimation errors, we choose the Lyapunov candidate function as:

$$V = \frac{1}{2} (x^2 + y^2 + z^2 + (w - w_e)^2 + e_\alpha^2 + e_\beta^2 + e_\gamma^2) \quad (2.24)$$

It is obvious that the function V is a positive definite on $\mathbb{R}^7$.

By differentiating V , we obtain:

$$\begin{aligned} \dot{V} &= -k_1x^2 - k_2y^2 - k_3z^2 - k_4(w - w_e)^2 \\ &\quad + e_\alpha (zx + (\xi + a - b|w|)x^2 - \dot{\hat{\alpha}}) \\ &\quad + e_\beta (-(x - y)z - \dot{\hat{\beta}}) + e_\gamma (-z^2 - \dot{\hat{\gamma}}). \end{aligned} \quad (2.25)$$

In view of (2.25), we take the parameter update law as:

$$\begin{cases} \dot{\hat{\alpha}} = zx + (\xi + a - b|w|)x^2, \\ \dot{\hat{\beta}} = -(x - y)z, \\ \dot{\hat{\gamma}} = -z^2. \end{cases} \quad (2.26)$$

We have the following theorem.

Theorem 2.2.1 *The controlled system (2.19) is asymptotically stable under the adaptive control (2.20), and the parameter update (2.26).*

Proof. Substituting (2.26) into (2.25), we get:

$$\dot{V} = -k_1x^2 - k_2y^2 - k_3z^2 - k_4(w - w_e)^2 < 0,$$

for all $(x, y, z, w) \neq (0, 0, 0, w_e)$

So, by Lyapunov stability theorem, the equilibrium point $(0, 0, 0, w_e)$ is asymptotically stable. ■

2.2.2 Numerical Simulations

The numerical solutions of the system is computed applying MATLAB software program based on Rung- Kutta method. Here, we implement the proposed adaptive control law (2.20) to stabilize equilibrium point $X_e(0, 0, 0, -3)$.

The system's parameters are taken as

$$\alpha = 5, \beta = 5, \gamma = 0.1, \xi = 3, a = 1.5, \text{ and } b = 1$$

The initial conditions are taken as

$$x(0) = 42.4467, y(0) = 7.3891, z(0) = -0.0251 \text{ and } w(0) = 26.5562.$$

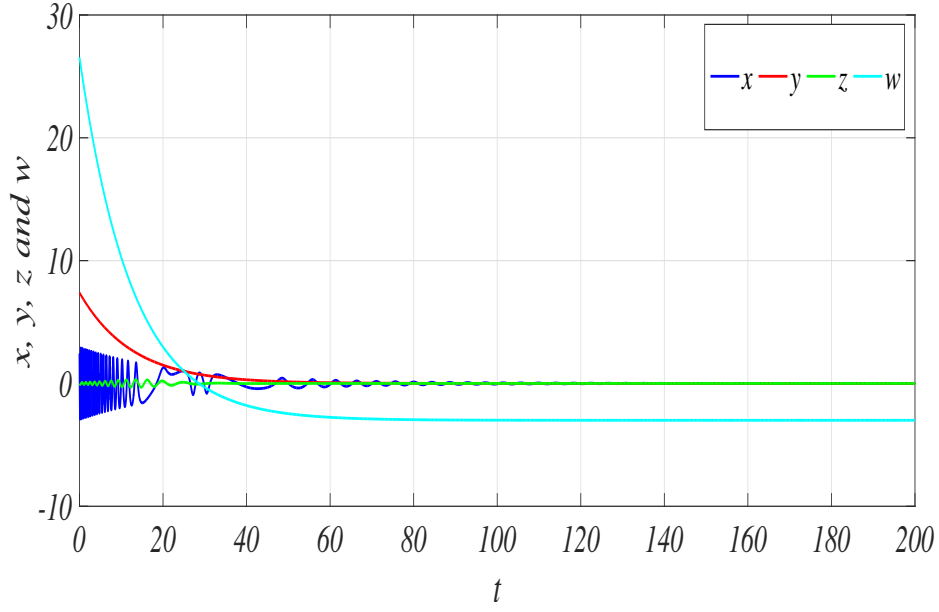


Figure 2.8: Stabilization of the equilibrium point $X_e(0, 0, 0, -3)$

The initial conditions for the parameter estimates, are selected as

$$\hat{\alpha}(0) = 3.3501, \hat{\beta}(0) = 4.9317, \text{ and } \hat{\gamma}(0) = 0.7$$

and the gain constants are taken as

$$k_1 = 0.08, k_2 = 0.08, k_3 = 0.2 \text{ and } k_4 = 0.08.$$

Figure 2.8 illustrates the stabilization of the equilibrium point $X_e(0, 0, 0, -3)$, under adaptive control law (2.20). Temporal evolution of the estimated parameters $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ are shown in Figure 2.9.

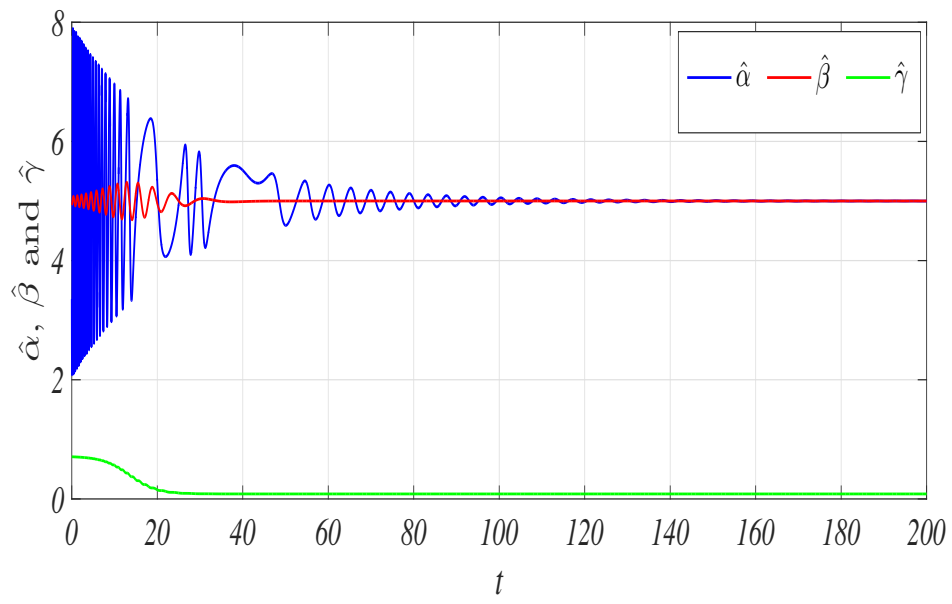


Figure 2.9: Temporal evolution of the estimated parameters $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$

General Conclusion

This work focused on the study, analysis, and stabilization of a modified memristor-based chaotic circuit derived from the classical Chua system. The introduction of the memristor enriched the system's nonlinear dynamics and revealed complex behavior such as chaos and hyperchaos. A dynamical analysis of the model was carried out in order to characterize its essential properties, particularly the stability of equilibrium points and the chaotic nature of the trajectories. To control these behaviors and ensure stable operation of the system, a nonlinear control strategy based on adaptive control was presented. This approach allows the system parameters to be dynamically adjusted according to the state variables, thereby guaranteeing asymptotic convergence to a desired equilibrium point, even in the presence of parametric uncertainties. Numerical simulations confirmed the effectiveness of the proposed control law, demonstrating a rapid stabilization of the chaotic trajectories and the restoration of steady-state behavior.

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