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Dedication

First and foremost, I dedicate this work to **Allah** Almighty, whose guidance and care have never left me, and who granted me the strength, determination, and courage to move forward step by step along the path He has chosen for me.

To my dear mother **Hamida** and my dear father **Ahsan**, this work is the fruit of your labor, and today it has come to fruition. You didn't just encourage me, you saw me at the top. Thank you for believing in me when I doubted myself.

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Dedication

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Noussaiba

Dedication

First and foremost, I dedicate this work to **Allah** Almighty, whose guidance and care have never left me, and who granted me the strength, determination, and courage to move forward step by step along the path He has chosen for me.

To my dear father **Nordine**, this work is the fruit of your labor, and today it has come to fruition. You didn't just encourage me, you saw me at the top. Thank you for believing in me when I doubted myself.

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Lina

Abstract

This master thesis is devoted to the study of three systems of nonlinear difference equations related to Narayana numbers, with emphasis on deriving explicit forms of the solutions and analyzing their dynamical behavior.

In the first chapter, the Narayana sequence and some related sequences are presented, highlighting their algebraic properties as well as the fundamental results used later in the study.

The second chapter deals with the first system of difference equations, which is transformed into a simpler form allowing the solutions to be expressed in terms of Narayana numbers. An analysis of equilibrium points, stability, and convergence behavior is also carried out. This part is supported by numerical examples illustrating the validity of the results and the effectiveness of the adopted approach.

In the third chapter, two additional nonlinear difference equation systems are studied, for which explicit solutions are obtained using properties of Narayana numbers. The dynamics of these solutions are then analyzed through the study of stability, convergence, and equilibrium points. The final results show a close connection between these systems and Narayana numbers, highlighting their importance in the analysis of discrete dynamical systems.

Keywords : Narayana numbers, systems of difference equations, equilibrium points, local and global stability, asymptotic behavior.

Résumé

Ce mémoire est consacrée à l'étude de trois systèmes d'équations aux différences non linéaires liés aux nombres de Narayana, en mettant l'accent sur l'extraction de formes explicites des solutions ainsi que sur l'analyse de leur comportement dynamique.

Dans le premier chapitre, on présente la suite de Narayana et certaines suites associées, en mettant en évidence leurs propriétés algébriques ainsi que les résultats fondamentaux utilisés par la suite.

Le deuxième chapitre traite du premier système d'équations aux différences, qui est transformé en une forme plus simple permettant d'exprimer ses solutions en fonction des nombres de Narayana. Une étude des points d'équilibre, de la stabilité et de l'analyse de convergence y est également effectuée. Cette partie est appuyée par des exemples numériques illustrant la validité des résultats et l'efficacité de l'approche adoptée.

Dans le troisième chapitre, deux autres systèmes d'équations aux différences non linéaires sont étudiés, pour lesquels des solutions explicites sont obtenues en s'appuyant sur les propriétés des nombres de Narayana. La dynamique de ces solutions est ensuite analysée à travers l'étude de la stabilité, de la convergence et des points d'équilibre. Les résultats finaux montrent un lien étroit entre ces systèmes et les nombres de Narayana mettant en évidence leur importance dans l'analyse des systèmes dynamiques discrets.

Mots-clés : nombres de Narayana, systèmes d'équations aux différences, points d'équilibres, stabilité locale et globale, comportement asymptotique.

ملخص

تُكرّس هذه المذكرة لدراسة ثلاث جمل من معادلات الفروق غير الخطية المرتبطة بأعداد نارايانا، مع التركيز على استخراج الصيغ الصريحة للحلول وتحليل سلوكها الديناميكي.

في الفصل الأول، يتم تقديم متتالية نارايانا وبعض المتتاليات المرتبطة بها مع إبراز خصائصها الجبرية والنتائج الأساسية المستعملة لاحقًا.

أما الفصل الثاني فيتناول الجملة الأولى من معادلات الفروق، حيث تُحوّل إلى صيغة أبسط تسمح باستخراج حلولها بدلالة أعداد نارايانا، مع دراسة نقاط التوازن والاستقرار والتحليل التقاربي. كما يُدعم هذا الجزء بأمثلة عددية توضح صحة النتائج وفعالية المنهج المتبع.

وفي الفصل الثالث، تُدرس جملتان إضافيتان من معادلات الفروق غير الخطية، حيث تُستخرج الحلول الصريحة اعتمادًا على خصائص أعداد نارايانا. كما يتم تحليل ديناميكيات هذه الحلول من خلال دراسة الاستقرار والتقارب ونقاط التوازن. وتُظهر النتائج النهائية ارتباطًا وثيقًا بين هذه الأنظمة وأعداد نارايانا، مما يبرز أهميتها في تحليل الأنظمة الديناميكية المتقطعة.

الكلمات الأساسية: أعداد نارايانا، أنظمة المعادلات الفرقية، نقاط التوازن، الاستقرار المحلي و العام، السلوك التقاربي.

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Introduction

General Introduction

The study of difference equations constitutes an important branch of discrete mathematics and dynamical systems. Difference equations arise naturally in the modeling of phenomena evolving over discrete time intervals and have numerous applications in biology, economics, population dynamics, probability theory, numerical analysis and other scientific fields [4]. Their importance stems from their ability to describe the evolution of sequences and discrete processes through recurrence relations.

Historically, one of the earliest and most celebrated examples of a recurrence relation is the Fibonacci sequence, introduced by Leonardo Fibonacci in his book *Liber Abbaci* in 1202 while studying a rabbit population growth problem [3]. Since then, the theory of difference equations has experienced remarkable development through the contributions of many mathematicians and has become an active area of research in modern mathematics.

In recent years, considerable attention has been devoted to obtaining explicit solutions of nonlinear systems of difference equations and investigating their qualitative behavior. Several authors have established connections between such systems and well-known integer sequences, including Fibonacci, Lucas, Padovan, and Balancing sequences [13, 7, 8, 5]. These connections have provided powerful tools for deriv-

ing closed-form solutions and studying the long-term behavior of nonlinear discrete dynamical systems.

Among the integer sequences that have attracted increasing interest is the Narayana sequence, which possesses several remarkable algebraic and combinatorial properties. Various generalizations and applications of Narayana numbers have been investigated in recent years [9, 12, 11, 10]. Motivated by these developments, this dissertation focuses on exploring new connections between Narayana numbers and nonlinear systems of difference equations.

The principal objective of this dissertation is to derive explicit formulas for the solutions of certain nonlinear systems of difference equations in terms of Narayana numbers and to investigate the asymptotic behavior and stability properties of their solutions. In particular, we establish new relationships between nonlinear discrete dynamical systems and the Narayana sequence, extending the existing literature on sequence-based solution methods for difference equations [1, 2, 6].

This dissertation is organized into three chapters.

In **Chapter 1**, we present the mathematical background necessary for the remainder of the work. Special attention is devoted to the Narayana sequence and its fundamental properties. We also review several classical sequences and recurrence relations that play an important role in the study of difference equations.

In **Chapter 2**, we investigate the nonlinear system

$$x_{n+1} = \frac{y_n x_{n-1} + 1}{y_n x_{n-1}}, \quad y_{n+1} = \frac{x_n y_{n-1} + 1}{x_n y_{n-1}}, \quad n \in \mathbb{N}_0,$$

with real initial conditions x_{-1}, x_0, y_{-1}, y_0 . We derive explicit solution formulas in terms of Narayana numbers, study the asymptotic behavior of solutions, determine the unique equilibrium point, and analyze its local stability. This chapter establishes a first connection between Narayana numbers and a system of rational difference equations [6].

In **Chapter 3**, we consider two classes of generalized nonlinear systems of difference equations,

$$x_{n+1} = f^{-1}(g(y_n) + g(y_{n-2})), \quad y_{n+1} = g^{-1}(f(x_n) + f(x_{n-2})),$$

and

$$x_{n+1} = f^{-1}\left(1 + \frac{1}{g(y_n)f(x_{n-1})}\right), \quad y_{n+1} = g^{-1}\left(1 + \frac{1}{f(x_n)g(y_{n-1})}\right),$$

subject to suitable real initial conditions. Using transformations that reduce these nonlinear systems to linear recurrence relations associated with the Narayana sequence, we derive explicit formulas for their solutions, identify their equilibrium points, and investigate their stability properties.

Through this work, we demonstrate how the properties of integer sequences, particularly the Narayana sequence, can be effectively employed to analyze nonlinear systems of difference equations. The obtained results further highlight the important role played by special sequences in understanding the qualitative behavior of discrete dynamical systems and contribute to the growing body of research connecting recurrence sequences with nonlinear difference equations.

An introduction to the Narayana sequence

1.1 Introduction

In this chapter, we present the basic notions related to the Narayana sequence. We first recall some classical recurrence sequences, especially the Fibonacci sequence, then we introduce the Narayana sequence, its explicit formula, characteristic equation, and some of its main properties. We also present certain generalizations such as the k -Narayana sequence and the Narayana-Lucas sequence.

1.2 Classical linear recurrence sequences

Linear recurrence sequences are fundamental objects in discrete mathematics and difference equations. They are defined by relations in which each term depends linearly on preceding terms. Among the most well-known examples is the Fibonacci sequence which has many applications in mathematics and science. These sequences provide the basis for studying more general sequences such as the Narayana sequence.

1.2.1 The Fibonacci sequence

In 1202, Fibonacci studied the problem of the growth of a rabbit population under ideal conditions. The problem is as follows: We start with one newly born pair of rabbits. A rabbit that is one month old is capable of reproduction. Each pair of rabbits (of reproductive age) gives birth to another pair of rabbits every month. Rabbits do not die.

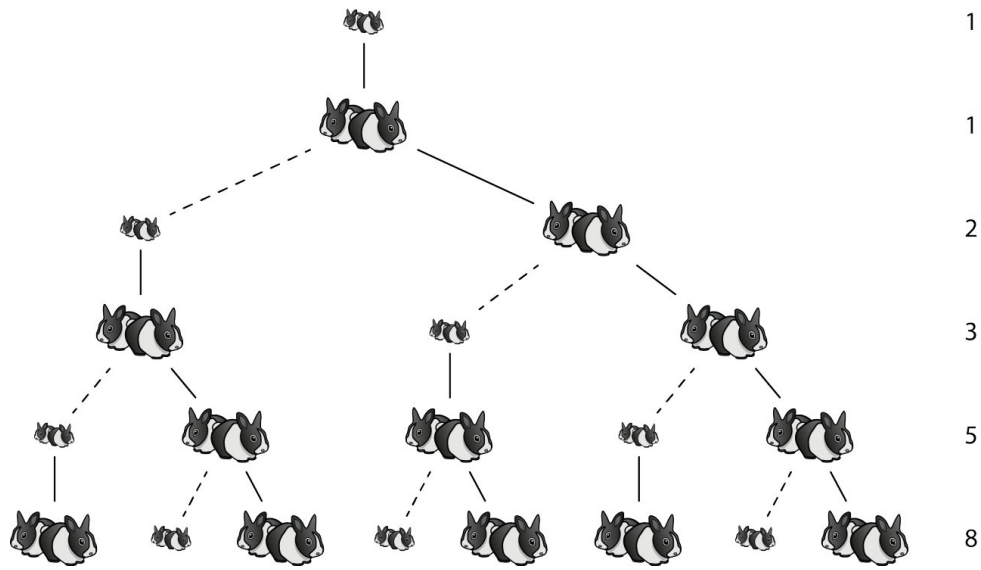


Figure 1.1: Growth of a Rabbit Population According to the Fibonacci Sequence

The Fibonacci sequence $\{F_n\}_{n \geq 0}$ is defined by the following second-order linear recurrence relation[4]:

$$F_{n+2} = F_{n+1} + F_n, \quad \text{for } n \geq 0, \quad (1.1)$$

with the initial conditions

$$F_0 = 0, \quad F_1 = 1. \quad (1.2)$$

The first few terms of Fibonacci sequence are:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \dots \quad (1.3)$$

The characteristic equation associated with this recurrence relation is given by:

$$\lambda^2 - \lambda - 1 = 0. \quad (1.4)$$

Solving for the roots, we obtain:

$$\alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}. \quad (1.5)$$

Thus, the general solution for F_n can be expressed as:

$$F_n = c_1 \alpha^n + c_2 \beta^n. \quad (1.6)$$

By using the initial conditions, we solve for the constants c_1 and c_2 :

$$\begin{aligned} c_1 + c_2 &= 0, \\ c_1 \alpha + c_2 \beta &= 1. \end{aligned} \quad (1.7)$$

Solving this system, we obtain:

$$c_1 = \frac{1}{\alpha - \beta}, \quad c_2 = \frac{-1}{\alpha - \beta}. \quad (1.8)$$

Thus, the explicit formula for $\{F_n\}$ is given by:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}. \quad (1.9)$$

Which is called the Binet formula of Fibonacci numbers.

1.2.2 Fundamental properties of Fibonacci sequence

The Fibonacci sequence $\{F_n\}_{n \geq 0}$ possesses several properties, that can be derived using its recurrence relation and characteristic equation:

(i) **Cassini's identity:** For $n > 0$, we have

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n. \quad (1.10)$$

(ii) **d'Ocagne's identity:** For $n, r \in \mathbb{N}$, we have

$$F_{n+r}F_{n+1} - F_{n+r+1}F_n = (-1)^n F_r. \quad (1.11)$$

(iii) **Johnson's identity:** For $k, l, m, n, r \in \mathbb{N}$ such that $k + l = m + n$, we have

$$F_k F_l - F_m F_n = (-1)^r (F_{k-r} F_{l-r} - F_{m-r} F_{n-r}). \quad (1.12)$$

limit of the ratio of two successive terms

It converges that the ratio of two consecutive terms of the Fibonacci sequence tends, in the limit, to the largest root of the characteristic equation.

$$\lim_{n \rightarrow +\infty} \frac{F_{n+1}}{F_n} = \alpha. \quad (1.13)$$

Remark 1.2.1 *The limit in (1.13) shows that the long-term behavior of the Fibonacci sequence is governed by the dominant real root*

$$\alpha = \frac{1 + \sqrt{5}}{2} \approx 1.618034,$$

which is known as the golden ratio. This remarkable constant appears in many areas of mathematics, geometry, nature, and art. In the context of the Fibonacci sequence, the golden ratio determines the asymptotic growth rate of the sequence, since consecutive Fibonacci numbers become approximately proportional to powers of α . Consequently, for large values of (n) , the Fibonacci numbers satisfy

$$F_n \approx \frac{\alpha^n}{\sqrt{5}},$$

highlighting the fundamental role of the golden ratio in the behavior of the sequence.

1.2.3 Generalizations of the Fibonacci sequence

Generalizations of the Fibonacci sequence naturally extend the classical recurrence by modifying its defining relation while preserving key structural properties. This extension has led to a wide family of related sequences and has developed into an independent research area within number theory and discrete mathematics. The importance of this field is reflected in the journal **Fibonacci Quarterly**, which is dedicated to research on the Fibonacci sequence, recurrence relations, and their generalizations. In this section, we present some important generalizations of the Fibonacci sequence relevant to our study.

1.2.4 Tribonacci sequence

The Tribonacci sequence $\{W_n\}_{n \geq 0}$ is defined by the third-order recurrence relation:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \quad \text{for } n \geq 3, \quad (1.14)$$

Where r, s and t are real numbers with $t \neq 0$.

with the initial conditions:

$$W_0 = a, \quad W_1 = b, \quad W_2 = c. \quad (1.15)$$

We consider the case $r=1, s=0, t=1$. A generalized Narayana sequence $\{W_n\}_{n \geq 0}$ is defined by the third-order recurrence relations[12]:

$$W_n = W_{n-1} + W_{n-3}, \quad \text{for } n \geq 3, \quad (1.16)$$

With the initial conditions:

$$W_0, W_1, W_2. \quad (1.17)$$

The characteristic equation associated with this recurrence relation is given by:

$$\lambda^3 - \lambda^2 - 1 = 0. \quad (1.18)$$

Solving for the roots, we obtain:

$$\begin{aligned} \alpha &= \frac{1}{3} + \left(\frac{29}{54} + \sqrt{\frac{31}{108}} \right)^{1/3} + \left(\frac{29}{54} - \sqrt{\frac{31}{108}} \right)^{1/3}, \\ \beta &= \frac{1}{3} + \omega \left(\frac{29}{54} + \sqrt{\frac{31}{108}} \right)^{1/3} + \omega^2 \left(\frac{29}{54} - \sqrt{\frac{31}{108}} \right)^{1/3}, \\ \gamma &= \frac{1}{3} + \omega^2 \left(\frac{29}{54} + \sqrt{\frac{31}{108}} \right)^{1/3} + \omega \left(\frac{29}{54} - \sqrt{\frac{31}{108}} \right)^{1/3}. \end{aligned} \quad (1.19)$$

With:

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp\left(\frac{2\pi i}{3}\right). \quad (1.20)$$

There are the following relations between the roots of characteristic equation:

$$\begin{aligned} \alpha + \beta + \gamma &= 1, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= 0, \\ \alpha\beta\gamma &= 1. \end{aligned} \quad (1.21)$$

The explicit formula for the Narayana sequence is given by:

$$\begin{aligned} W_n &= \frac{p_1}{(\alpha - \beta)(\alpha - \gamma)} \alpha^n + \frac{p_2}{(\beta - \alpha)(\beta - \gamma)} \beta^n + \frac{p_3}{(\gamma - \alpha)(\gamma - \beta)} \gamma^n \\ &= A_1 \alpha^n + B_1 \beta^n + C_1 \gamma^n, \end{aligned} \quad (1.22)$$

where

$$\begin{aligned} p_1 &= W_2 - (\beta + \gamma)W_1 + \beta\gamma W_0, \\ p_2 &= W_2 - (\alpha + \gamma)W_1 + \alpha\gamma W_0, \\ p_3 &= W_2 - (\alpha + \beta)W_1 + \alpha\beta W_0. \end{aligned} \quad (1.23)$$

and

$$\begin{aligned} A_1 &= \frac{p_1}{(\alpha - \beta)(\alpha - \gamma)} = \frac{W_2 - (\beta + \gamma)W_1 + \beta\gamma W_0}{(\alpha - \beta)(\alpha - \gamma)}, \\ B_1 &= \frac{p_2}{(\beta - \alpha)(\beta - \gamma)} = \frac{W_2 - (\alpha + \gamma)W_1 + \alpha\gamma W_0}{(\beta - \alpha)(\beta - \gamma)}, \\ C_1 &= \frac{p_3}{(\gamma - \alpha)(\gamma - \beta)} = \frac{W_2 - (\alpha + \beta)W_1 + \alpha\beta W_0}{(\gamma - \alpha)(\gamma - \beta)}. \end{aligned} \tag{1.24}$$

1.3 The Narayana Sequence

The Narayana numbers were introduced by the Indian mathematician Narayana in the 14th century while studying a population problem involving cows and calves under idealized conditions. The problem is analogous to Fibonacci's rabbit problem, since it starts with a single newborn cow, A cow becomes capable of reproduction after three years. From that point onward, each adult cow gives birth to one calf per year. It is assumed that cows do not die.



Figure 1.2: Narayana Pandita (1340-1400)

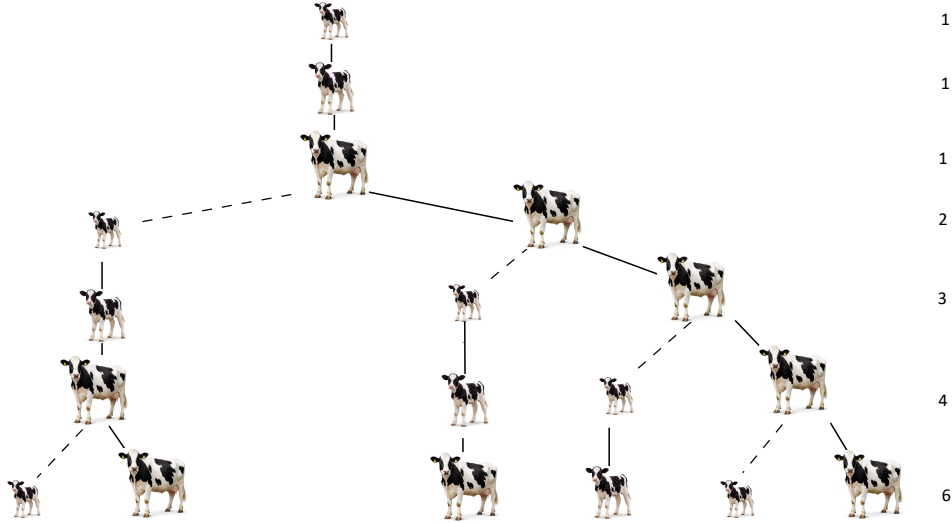


Figure 1.3: Growth of a Cow Population According to the Narayana Sequence

The Narayana sequence $\{N_n\}_{n \geq 0}$ is defined by the recurrence relation[11]:

$$N_{n+3} = N_{n+2} + N_n, \quad \text{for } n \geq 0, \quad (1.25)$$

where the initial conditions are given by:

$$N_0 = 0, \quad N_1 = 1, \quad N_2 = 1. \quad (1.26)$$

The first few terms of the Narayana sequence are:

$$0, 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, \dots \quad (1.27)$$

The characteristic equation associated with this recurrence relation is given by:

$$\lambda^3 - \lambda^2 - 1 = 0. \quad (1.28)$$

We note that this is the same characteristic equation as in (1.18), hence it has the same

roots (1.19).

Thus, the general solution for N_n can be expressed as:

$$N_n = A_2\alpha^n + B_2\beta^n + C_2\gamma^n. \quad (1.29)$$

By using the initial conditions, we solve for the constants A_2 , B_2 and C_2 .

$$\begin{aligned} A_2 + B_2 + C_2 &= 0, \\ A_2\alpha + B_2\beta + C_2\gamma &= 1, \\ A_2\alpha^2 + B_2\beta^2 + C_2\gamma^2 &= 1. \end{aligned} \quad (1.30)$$

Solving this system, we obtain:

$$\begin{aligned} A_2 &= \frac{\alpha}{(\alpha - \beta)(\alpha - \gamma)}, \\ B_2 &= \frac{\beta}{(\beta - \alpha)(\beta - \gamma)}, \\ C_2 &= \frac{\gamma}{(\gamma - \alpha)(\gamma - \beta)}. \end{aligned} \quad (1.31)$$

Thus, the explicit formula for N_n is:

$$N_n = \frac{\alpha^{n+1}}{(\alpha - \beta)(\alpha - \gamma)} + \frac{\beta^{n+1}}{(\beta - \alpha)(\beta - \gamma)} + \frac{\gamma^{n+1}}{(\gamma - \alpha)(\gamma - \beta)}. \quad (1.32)$$

1.3.1 Properties of the Narayana Sequence

The Narayana sequence $\{N_n\}_{n \geq 0}$ possesses several properties, that can be derived using its recurrence relation and characteristic equation[10]:

Summation Identities

The Narayana sequence satisfies several summation identities:

For $n, k \in \mathbb{N}$.

(i) Sum of the first n terms:

$$\sum_{k=1}^n N_k = N_{n+3} - N_3. \quad (1.33)$$

Proof. From Binet's formula, we get

$$\begin{aligned} \sum_{k=1}^n N_k &= A_2(\alpha^1 + \alpha^2 + \cdots + \alpha^n) + B_2(\beta^1 + \beta^2 + \cdots + \beta^n) + C_2(\gamma^1 + \gamma^2 + \cdots + \gamma^n) \\ &= A_2\left(\frac{\alpha(\alpha^n - 1)}{\alpha - 1}\right) + B_2\left(\frac{\beta(\beta^n - 1)}{\beta - 1}\right) + C_2\left(\frac{\gamma(\gamma^n - 1)}{\gamma - 1}\right) \\ &= A_2(\alpha^3(\alpha^n - 1)) + B_2(\beta^3(\beta^n - 1)) + C_2(\gamma^3(\gamma^n - 1)) \\ &= A_2\alpha^{n+3} + B_2\beta^{n+3} + C_2\gamma^{n+3} - (A_2\alpha^3 + B_2\beta^3 + C_2\gamma^3) \\ &= N_{n+3} - N_3. \end{aligned} \quad (1.34)$$

(ii) sum of terms with even indices:

$$\sum_{k=0}^n N_{2k} = \frac{1}{3} (N_{2n+2} + N_{2n+1} + 2N_{2n} - 2). \quad (1.35)$$

Proof. We prove this identity by mathematical induction: For $n = 0$,

$$\sum_{k=0}^0 N_{2k} = N_0 = 0.$$

The right-hand side gives

$$\frac{1}{3}(N_2 + N_1 + 2N_0 - 2) = \frac{1}{3}(1 + 1 + 0 - 2) = 0.$$

Thus, the formula holds for $n = 0$.

Assume the formula holds for n , and prove that it holds for $n + 1$:

$$\sum_{k=0}^n N_{2k} = \frac{1}{3} (N_{2n+2} + N_{2n+1} + 2N_{2n} - 2).$$

Then

$$\begin{aligned} \sum_{k=0}^{n+1} N_{2k} &= \sum_{k=0}^n N_{2k} + N_{2n+2} \\ &= \frac{1}{3} (N_{2n+2} + N_{2n+1} + 2N_{2n} - 2) + N_{2n+2}. \end{aligned}$$

Combining terms

$$= \frac{1}{3} (4N_{2n+2} + N_{2n+1} + 2N_{2n} - 2).$$

Using the recurrence relation

$$N_n = N_{n-1} + N_{n-3}.$$

We obtain

$$N_{2n+4} = N_{2n+2} + N_{2n+1} + N_{2n}.$$

A direct computation shows that

$$4N_{2n+2} + N_{2n+1} + 2N_{2n} = N_{2n+4} + N_{2n+3} + 2N_{2n+2}.$$

Hence,

$$\sum_{k=0}^{n+1} N_{2k} = \frac{1}{3} (N_{2n+4} + N_{2n+3} + 2N_{2n+2} - 2).$$

Thus, by mathematical induction, the property is true for all $n \in \mathbb{N}$. ■

(iii) The sum of terms with odd indices :

$$\sum_{k=0}^n N_{2k+1} = \frac{1}{3} (2N_{2n+2} + 2N_{2n+1} + N_{2n} - 4). \quad (1.36)$$

proof. We prove this identity by mathematical induction: For $n = 0$,

$$\sum_{k=0}^0 N_{2k+1} = N_1 = 1.$$

The right-hand side gives

$$\frac{1}{3}(2N_2 + 2N_1 + N_0 - 1) = \frac{1}{3}(2 + 2 + 0 - 1) = 1.$$

Assume the formula holds for n , and prove that it holds for $n + 1$:

$$\sum_{k=0}^n N_{2k+1} = \frac{1}{3} (2N_{2n+2} + 2N_{2n+1} + N_{2n} - 1).$$

Then

$$\begin{aligned} \sum_{k=0}^{n+1} N_{2k+1} &= \sum_{k=0}^n N_{2k+1} + N_{2n+3} \\ &= \frac{1}{3} (2N_{2n+2} + 2N_{2n+1} + N_{2n} - 1) + N_{2n+3}. \end{aligned}$$

Using the recurrence relation

$$N_{2n+3} = N_{2n+2} + N_{2n},$$

and simplifying, we obtain

$$\sum_{k=0}^{n+1} N_{2k+1} = \frac{1}{3} (2N_{2n+4} + 2N_{2n+3} + N_{2n+2} - 1).$$

Thus, by mathematical induction, the property is true for all $n \in \mathbb{N}$.

(iv) of terms with step 3 (starting at 3):

$$\sum_{k=1}^n N_{3k} = N_{3n+1} - N_1. \quad (1.37)$$

Proof. From Binet's formula, we get

$$\begin{aligned} \sum_{k=1}^n N_{3k} &= A_2(\alpha^3 + \alpha^6 + \cdots + \alpha^{3n}) + B_2(\beta^3 + \beta^6 + \cdots + \beta^{3n}) + C_2(\gamma^3 + \gamma^6 + \cdots + \gamma^{3n}) \\ &= A_2 \frac{\alpha^3(\alpha^{3n} - 1)}{\alpha^3 - 1} + B_2 \frac{\beta^3(\beta^{3n} - 1)}{\beta^3 - 1} + C_2 \frac{\gamma^3(\gamma^{3n} - 1)}{\gamma^3 - 1} \\ &= A_2\alpha(\alpha^{3n} - 1) + B_2\beta(\beta^{3n} - 1) + C_2\gamma(\gamma^{3n} - 1) \\ &= A_2\alpha^{3n+1} + B_2\beta^{3n+1} + C_2\gamma^{3n+1} - (A_2\alpha + B_2\beta + C_2\gamma) \\ &= N_{3n+1} - N_1. \end{aligned} \quad (1.38)$$

(v) Sum of terms with step 3 (starting at 4):

$$\sum_{k=1}^n N_{3k+1} = N_{3n+2} - N_2. \quad (1.39)$$

Proof. From Binet's formula, we get

$$\begin{aligned} \sum_{k=1}^n N_{3k+1} &= A_2(\alpha^4 + \alpha^7 + \cdots + \alpha^{3n+1}) + B_2(\beta^4 + \beta^7 + \cdots + \beta^{3n+1}) + C_2(\gamma^4 + \gamma^7 + \cdots + \gamma^{3n+1}) \\ &= A_2\alpha(\alpha^3 + \alpha^6 + \cdots + \alpha^{3n}) + B_2\beta(\beta^3 + \beta^6 + \cdots + \beta^{3n}) + C_2\gamma(\gamma^3 + \gamma^6 + \cdots + \gamma^{3n}) \\ &= A_2\alpha \frac{\alpha^3(\alpha^{3n} - 1)}{\alpha^3 - 1} + B_2\beta \frac{\beta^3(\beta^{3n} - 1)}{\beta^3 - 1} + C_2\gamma \frac{\gamma^3(\gamma^{3n} - 1)}{\gamma^3 - 1} \\ &= A_2\alpha^2(\alpha^{3n} - 1) + B_2\beta^2(\beta^{3n} - 1) + C_2\gamma^2(\gamma^{3n} - 1) \\ &= A_2\alpha^{3n+2} + B_2\beta^{3n+2} + C_2\gamma^{3n+2} - (A_2\alpha^2 + B_2\beta^2 + C_2\gamma^2) \\ &= N_{3n+2} - N_2. \end{aligned} \quad (1.40)$$

limit of the ratio of two successive terms

It converges that the ratio of two consecutive terms of the Narayana sequence tends, in the limit, to the largest root of the characteristic equation.

$$\lim_{n \rightarrow +\infty} \frac{N_{n+1}}{N_n} = \alpha. \quad (1.41)$$

Remark 1.3.1 *The limit in (1.41) indicates that the long-term behavior of the Narayana sequence is determined by the dominant real root α of the characteristic equation*

$$x^3 - x^2 - 1 = 0.$$

This root, numerically given by $\alpha \approx 1.465571$, can be interpreted as the Narayana constant, since it governs the asymptotic exponential growth rate of the sequence.

In this context, α plays a role similar to other well-known algebraic constants arising from linear recurrence relations. For instance, the golden ratio

$$\varphi = \frac{1 + \sqrt{5}}{2} \approx 1.618034$$

describes the growth of the Fibonacci sequence, while the plastic number

$$\rho \approx 1.324718,$$

which is the real root of $x^3 - x - 1 = 0$, governs the asymptotic behavior of the Padovan sequence. Similarly, the silver ratio

$$\delta = 1 + \sqrt{2} \approx 2.414214$$

is associated with Pell-type sequences and their growth properties.

A comparison between these fundamental constants shows the ordering

$$\rho < \alpha < \varphi < \delta,$$

that is,

$$1.324718 < 1.465571 < 1.618034 < 2.414214.$$

Hence, the Narayana sequence exhibits an intermediate growth rate: it increases asymptotically faster than the Padovan sequence, yet more slowly than both the Fibonacci and Pell-type sequences. This places it naturally between these classical families of linear recurrence sequences in terms of exponential growth behavior.

Relation to the Narayana-Lucas sequence

The Narayana sequence and the Narayana-Lucas sequence are related by:

$$L_n = 3N_{n+1} - 2N_n. \quad (1.42)$$

1.4 Generalizations of the Narayana sequence

1.4.1 k -Narayana sequence

The k -Narayana sequence is defined by the recurrence relation[9]:

$$N_{k,n} = kN_{k,n-1} + N_{k,n-3}, \quad n \geq 0, \quad k \geq 3. \quad (1.43)$$

With the initial conditions

$$b_{k,0} = 0, \quad b_{k,1} = 1, \quad b_{k,2} = k. \quad (1.44)$$

The first few terms of the K -Narayana sequence are:

$$0, 1, k, k^2, k^3 + 1, k^4 + 2k, k^5 + 3k^2, k^6 + 4k^3 + 1, \dots \quad (1.45)$$

The characteristic equation associated with this recurrence relation is given by:

$$\lambda^3 - k\lambda^2 - 1 = 0. \quad (1.46)$$

Solving for the roots, we obtain:

$$\begin{aligned} \alpha_k &= \frac{1}{3} \left(k + k^2 \sqrt[3]{\frac{2}{27 + 2k^3 + 3\sqrt{81 + 12k^3}}} + \sqrt[3]{\frac{27 + 2k^3 + 3\sqrt{81 + 12k^3}}{2}} \right), \\ \beta_k &= \frac{1}{3} \left(k - \omega k^2 \sqrt[3]{\frac{2}{27 + 2k^3 + 3\sqrt{81 + 12k^3}}} + \omega^2 \sqrt[3]{\frac{27 + 2k^3 + 3\sqrt{81 + 12k^3}}{2}} \right), \\ \gamma_k &= \frac{1}{3} \left(k + \omega^2 k^2 \sqrt[3]{\frac{2}{27 + 2k^3 + 3\sqrt{81 + 12k^3}}} + \omega \sqrt[3]{\frac{27 + 2k^3 + 3\sqrt{81 + 12k^3}}{2}} \right). \end{aligned} \quad (1.47)$$

With

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp\left(\frac{2\pi i}{3}\right). \quad (1.48)$$

The k -Narayana numbers are given by Binet's formula.

$$N_{k,n} = \frac{\alpha_k^{n+1}}{(\alpha_k - \beta_k)(\alpha_k - \gamma_k)} + \frac{\beta_k^{n+1}}{(\beta_k - \alpha_k)(\beta_k - \gamma_k)} + \frac{\gamma_k^{n+1}}{(\gamma_k - \beta_k)(\gamma_k - \alpha_k)}. \quad (1.49)$$

Hence, as $n \rightarrow +\infty$, it follows that:

$$\lim_{n \rightarrow +\infty} \frac{N_{k,n+1}}{N_{k,n}} = \alpha_k. \quad (1.50)$$

Finally, it can be observed that for $k=1$, the sequence reduces to the classical Narayana sequence.

1.4.2 The Narayana-Lucas sequence

The **Narayana – Lucas sequence** $\{L_n\}_{n \geq 0}$ is defined by the recurrence relation[12] :

$$L_{n+3} = L_{n+2} + L_n, \quad n \geq 0, \quad (1.51)$$

With the initial conditions:

$$L_0 = 3, \quad L_1 = L_2 = 1. \quad (1.52)$$

The characteristic equation associated with this recurrence relation is given by:

$$\lambda^3 - \lambda^2 - 1 = 0. \quad (1.53)$$

Solving for the roots, we obtain:

$$\begin{aligned} \alpha &= \frac{1}{3} + \left(\frac{29}{54} + \sqrt{\frac{31}{108}} \right)^{1/3} + \left(\frac{29}{54} - \sqrt{\frac{31}{108}} \right)^{1/3}, \\ \beta &= \frac{1}{3} + \omega \left(\frac{29}{54} + \sqrt{\frac{31}{108}} \right)^{1/3} + \omega^2 \left(\frac{29}{54} - \sqrt{\frac{31}{108}} \right)^{1/3}, \\ \gamma &= \frac{1}{3} + \omega^2 \left(\frac{29}{54} + \sqrt{\frac{31}{108}} \right)^{1/3} + \omega \left(\frac{29}{54} - \sqrt{\frac{31}{108}} \right)^{1/3}. \end{aligned} \quad (1.54)$$

With:

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp\left(\frac{2\pi i}{3}\right). \quad (1.55)$$

Thus, the explicit formula for $\{L_n\}$ is given by:

$$L_n = \alpha^n + \beta^n + \gamma^n. \quad (1.56)$$

Which is called Binet formula of Narayana-Lucas numbers.

1.4.3 Properties of the Narayana-Lucas sequence

The Narayana-Lucas sequence $\{L_n\}_{n \geq 0}$ possesses several properties that can be derived using its recurrence relation and characteristic equation[11]

Summation identities

The Narayana-Lucas sequence satisfies several summation identities:

(i) The sum of the first n terms:

$$\sum_{k=0}^n L_k = L_{n+3} - 1. \quad (1.57)$$

(ii) The sum of terms with even indices:

$$\sum_{k=0}^n L_{2k} = \frac{1}{3} (L_{2n+2} + L_{2n+1} + 2L_{2n} + 1). \quad (1.58)$$

(iii) The sum of terms with odd indices :

$$\sum_{k=0}^n L_{2k+1} = \frac{1}{3} (2L_{2n+2} + 2L_{2n+1} + L_{2n} - 4). \quad (1.59)$$

limit of the ratio of two successive terms

It converges that the ratio of two consecutive terms of the Narayana-Lucas sequence tends, in the limit, to the largest root of the characteristic equation.

$$\lim_{n \rightarrow +\infty} \frac{L_{n+1}}{L_n} = \alpha. \quad (1.60)$$

Relation to the Narayana sequence

The Narayana-Lucas sequence and the Narayana sequence are related by:

$$L_n = 3N_{n+1} - 2Nn. \tag{1.61}$$

1.5 Conclusion

In this chapter, we studied some well-known and widely used sequences, particularly the Fibonacci sequence. We also introduced the Narayana sequence and some related sequences, and presented several of their algebraic properties, which will be useful in the subsequent chapters.

System of difference equations linked to Narayana sequence

2.1 Introduction

In this chapter, we study the nonlinear system of second-order rational difference equations

$$x_{n+1} = \frac{y_n x_{n-1} + 1}{y_n x_{n-1}}, \quad y_{n+1} = \frac{x_n y_{n-1} + 1}{x_n y_{n-1}}, \quad n \in \mathbb{N}_0, \quad (2.1)$$

with initial conditions $x_{-1}, x_0, y_{-1}, y_0 \in \mathbb{R}$.

We derive explicit formulas for the solutions of the system in terms of the Narayana sequence and investigate its equilibrium point and stability properties.

The study of nonlinear difference equations in connection with well-known recursive sequences has attracted significant attention in recent years. Many researchers have successfully derived explicit solutions for various difference equations and systems by relating them to special sequences, leading to a deeper understanding of their dynamical behavior and long-term properties.

In [5], the authors obtained explicit solution formulas for the following system of

difference equations in terms of the Balancing numbers

$$x_{n+1} = \frac{1}{6 - y_{n-k}}, \quad y_{n+1} = \frac{1}{6 - x_{n-k}}.$$

Furthermore, [7] derived explicit solutions for the following system in terms of the Fibonacci numbers.

$$x_{n+1} = \frac{1}{\pm 1 \pm y_{n-k}}, \quad y_{n+1} = \frac{1}{\pm 1 \pm x_{n-k}}.$$

In [8], the authors derived the solutions of a system of difference equations in terms of the Lucas numbers.

$$x_{n+1} = \frac{5y_{n-k} - 5}{y_{n-k}}, \quad y_{n+1} = \frac{5x_{n-k} - 5}{x_{n-k}}.$$

Also, in [13], Yazlik obtained the form of the solutions of the following system of difference equations.

$$x_{n+1} = \frac{x_{n-1} \pm 1}{y_n x_{n-1}}, \quad y_{n+1} = \frac{y_{n-1} \pm 1}{x_n y_{n-1}}.$$

All these works express the solutions of systems of difference equations in terms of well-known sequences such as the Fibonacci and Padovan numbers. We are interested in studying the system of difference equations (2.1), whose solutions are related to the Narayana numbers.

This work is considered the **first** to be presented in terms of these numbers. To the best of our knowledge, no previous study has expressed the solutions of a difference equation system in terms of Narayana numbers.

The main results obtained in this chapter form the basis of a research article [6] that has been submitted for publication in an international scientific journal.

2.2 Form of the solution of the system (2.1)

Determining the form of the solutions of a system of difference equations is important for understanding its behavior and studying its properties. In this section, we derive the form of the solutions of system (2.1).

Before presenting the main result, we introduce the following lemma, which plays a fundamental role in determining the form of the solutions of system (2.1). The next lemma provides useful relations that will be used in the subsequent proofs.

Lemma 2.2.1 *Consider the two homogeneous third order linear autonomous differences equations:*

$$y_{n+2} - y_{n+1} - y_{n-1} = 0, \quad (2.2)$$

$$z_{n+2} + z_{n+1} + z_{n-1} = 0. \quad (2.3)$$

With the initial conditions $y_{-1}, y_0, y_1, z_{-1}, z_0$ and $z_1 \in \mathbb{R}$. Hence, for all $n \in N_0$ we have

$$y_n = y_{-1}N_n + y_0N_{n-1} + y_1N_{n+1}, \quad (2.4)$$

$$z_n = (-1)^{n+1}(z_{-1}N_n - z_0N_{n-1} + z_1N_{n+1}). \quad (2.5)$$

Proof.

The characteristic equation of (2.2) equation is given by:

$$\lambda^3 - \lambda^2 - 1 = 0. \quad (2.6)$$

We have three roots α, β , and γ .

The general solution to the recurrence relation (2.2) is given by

$$y_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n. \quad (2.7)$$

To find c_1 , c_2 and c_3 , we use the initial conditions y_{-1} , y_0 and y_1 that is:

$$\begin{aligned}\frac{c_1}{\alpha} + \frac{c_2}{\beta} + \frac{c_3}{\gamma} &= y_{-1}, \\ c_1 + c_2 + c_3 &= y_0, \\ c_1\alpha + c_2\beta + c_3\gamma &= y_1.\end{aligned}$$

We write the system in matrix form:

$$\begin{pmatrix} \frac{1}{\alpha} & \frac{1}{\beta} & \frac{1}{\gamma} \\ 1 & 1 & 1 \\ \alpha & \beta & \gamma \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} y_{-1} \\ y_0 \\ y_1 \end{pmatrix}.$$

Using Cramer's method, we obtain

$$\begin{aligned}c_1 &= \frac{y_{-1} + \alpha(\alpha - 1)y_0 + \alpha y_1}{(\alpha - \beta)(\alpha - \gamma)}, \\ c_2 &= \frac{y_{-1} + \beta(\beta - 1)y_0 + \beta y_1}{(\beta - \alpha)(\beta - \gamma)}, \\ c_3 &= \frac{y_{-1} + \gamma(\gamma - 1)y_0 + \gamma y_1}{(\gamma - \alpha)(\gamma - \beta)}.\end{aligned}$$

Substituting c_1, c_2 and c_3 into (2.7), we obtain

$$\begin{aligned}y_n &= \alpha^n \left(\frac{y_{-1} + \alpha(\alpha - 1)y_0 + \alpha y_1}{(\alpha - \beta)(\alpha - \gamma)} \right) \\ &+ \beta^n \left(\frac{y_{-1} + \beta(\beta - 1)y_0 + \beta y_1}{(\beta - \alpha)(\beta - \gamma)} \right) \\ &+ \gamma^n \left(\frac{y_{-1} + \gamma(\gamma - 1)y_0 + \gamma y_1}{(\gamma - \alpha)(\gamma - \beta)} \right) \\ &= y_{-1} \left(\frac{\alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{\beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{\gamma^n}{(\gamma - \alpha)(\gamma - \beta)} \right) \\ &+ y_0 \left(\frac{\alpha^{n+1}(\alpha - 1)}{(\alpha - \beta)(\alpha - \gamma)} + \frac{\beta^{n+1}(\beta - 1)}{(\beta - \alpha)(\beta - \gamma)} + \frac{\gamma^{n+1}(\gamma - 1)}{(\gamma - \alpha)(\gamma - \beta)} \right) \\ &+ y_1 \left(\frac{\alpha^{n+1}}{(\alpha - \beta)(\alpha - \gamma)} + \frac{\beta^{n+1}}{(\beta - \alpha)(\beta - \gamma)} + \frac{\gamma^{n+1}}{(\gamma - \alpha)(\gamma - \beta)} \right).\end{aligned}$$

According to formula (1.32), we obtain:

$$\begin{aligned} y_n &= y_{-1}N_n + y_0(N_{n+2} - N_{n+1}) + N_{n+1}y_1 \\ &= y_{-1}N_n + y_0N_{n-1} + y_1N_{n+1}. \end{aligned}$$

The characteristic equation of (2.3) is given by:

$$\lambda^3 + \lambda^2 + 1 = 0.$$

The roots of the characteristic equation are $-\alpha$, $-\beta$ and $-\gamma$.

Thus, the general solution of (2.3) is given by:

$$z_n = c_1(-\alpha)^n + c_2(-\beta)^n + c_3(-\gamma)^n. \quad (2.8)$$

To find c_1 , c_2 and c_3 , we use the initial conditions z_{-1} , z_0 and z_1 that is:

$$\begin{aligned} \frac{c_1}{-\alpha} + \frac{c_2}{-\beta} + \frac{c_3}{-\gamma} &= z_{-1}, \\ c_1 + c_2 + c_3 &= z_0, \\ c_1(-\alpha) + c_2(-\beta) + c_3(-\gamma) &= z_1. \end{aligned}$$

Using Cramer's method, we obtain

$$\begin{aligned} c_1 &= \frac{-z_{-1} + \alpha(\alpha - 1)z_0 - \alpha z_1}{(\alpha - \beta)(\alpha - \gamma)}, \\ c_2 &= \frac{-z_{-1} + \beta(\beta - 1)z_0 - \beta z_1}{(\beta - \alpha)(\beta - \gamma)}, \\ c_3 &= \frac{-z_{-1} + \gamma(\gamma - 1)z_0 - \gamma z_1}{(\gamma - \alpha)(\gamma - \beta)}. \end{aligned}$$

We replace c_1 , c_2 and c_3 in (2.8), we obtain:

$$z_n = (-1)^{n+1}(z_{-1}N_n - z_0N_{n-1} + z_1N_{n+1}).$$

■

Before presenting Lemma 2.2.2, we introduce a transformation that decouples the system into simpler third-order linear difference equations related to the Narayana sequence. Using the results of Lemma 2.2.1, we obtain explicit formulas for the even and odd terms of the solutions.

Lemma 2.2.2 *Consider the linear system of third order linear autonomous differences equations:*

$$\begin{aligned} u_{n+2} - v_{n+1} - v_{n-1} &= 0, \\ v_{n+2} - u_{n+1} - u_{n-1} &= 0. \end{aligned} \tag{2.9}$$

With the initial conditions $u_{-1}, u_0, u_1, v_{-1}, v_0$ and $v_1 \in \mathbb{R}$.

Then

$$\begin{cases} u_{2n+1} = u_{-1}N_{2n+1} + v_0N_{2n} + u_1N_{2n+2}, \\ u_{2n+2} = v_{-1}N_{2n+3} + u_0N_{2n+1} + v_1N_{2n+3}, \end{cases} \tag{2.10}$$

$$\begin{cases} v_{2n+1} = v_{-1}N_{2n+1} + u_0N_{2n+2} + v_1N_{2n+4}, \\ v_{2n+2} = u_{-1}N_{2n+2} + v_0N_{2n+1} + u_1N_{2n+3}. \end{cases} \tag{2.11}$$

Proof. Consider the linear system of third order linear autonomous differences equations:

$$u_{n+2} - v_{n+1} - v_{n-1} = 0, \quad v_{n+2} - u_{n+1} - u_{n-1} = 0, \quad n \in N_0. \tag{2.12}$$

Through the combination of addition and subtraction of equations, we get

$$u_{n+2} + v_{n+2} = v_{n+1} + u_{n+1} + v_{n-1} + u_{n-1},$$

$$u_{n+2} - v_{n+2} = v_{n+1} - u_{n+1} + v_{n-1} - u_{n-1}.$$

So

$$\begin{cases} u_{n+2} + v_{n+2} = (v_{n+1} + u_{n+1}) + (v_{n-1} + u_{n-1}), \\ u_{n+2} - v_{n+2} = -(u_{n+1} - v_{n+1}) - (u_{n-1} - v_{n-1}). \end{cases}$$

By posing the following change of variables

$$R_n = u_n + v_n, \quad S_n = u_n - v_n. \quad (2.13)$$

We obtain two homogeneous linear difference equations of third order:

$$R_{n+2} = R_{n+1} + R_{n-1}. \quad (2.14)$$

$$S_{n+2} = -S_{n+1} - S_{n-1}. \quad (2.15)$$

It is clear that equations (2.14) and (2.2) are the same. Therefore, by Lemma 2.2.1, we have

$$R_n = R_{-1}N_n + R_0N_{n-1} + R_1N_{n+1}.$$

Similarly, equations (2.15) and (2.3) are the same. Therefore, by Lemma 2.2.1, we have

$$S_n = (-1)^{n+1}(S_{-1}N_n - S_0N_{n-1} + S_1N_{n+1}).$$

We have from (2.13) that

$$\begin{aligned} u_n &= \frac{1}{2}(R_n + S_n), & v_n &= \frac{1}{2}(R_n - S_n), \\ u_{2n} &= \frac{1}{2}(R_{2n} + S_{2n}), & v_{2n} &= \frac{1}{2}(R_{2n} - S_{2n}), \\ u_{2n+1} &= \frac{1}{2}(R_{2n+1} + S_{2n+1}), & v_{2n+1} &= \frac{1}{2}(R_{2n+1} - S_{2n+1}). \end{aligned}$$

We obtain

$$\begin{cases} u_{2n+1} = \frac{1}{2} [(R_{-1} + S_{-1})N_{2n+1} + (R_0 - S_0)N_{2n} + (R_1 + S_1)N_{2n+2}], \\ u_{2n+2} = \frac{1}{2} [(R_{-1} - S_{-1})N_{2n+3} + (R_0 + S_0)N_{2n+1} + (R_1 - S_1)N_{2n+3}], \end{cases} \quad (2.16)$$

$$\begin{cases} v_{2n+1} = \frac{1}{2} [(R_{-1} - S_{-1})N_{2n+1} + (R_0 + S_0)N_{2n+2} + (R_1 - S_1)N_{2n+4}], \\ v_{2n+2} = \frac{1}{2} [(R_{-1} + S_{-1})N_{2n+2} + (R_0 - S_0)N_{2n+1} + (R_1 + S_1)N_{2n+3}]. \end{cases} \quad (2.17)$$

Then

$$\begin{cases} u_{2n+1} = u_{-1}N_{2n+1} + v_0N_{2n} + u_1N_{2n+2}, \\ u_{2n+2} = v_{-1}N_{2n+2} + u_0N_{2n+1} + v_1N_{2n+3}, \end{cases} \quad (2.18)$$

$$\begin{cases} v_{2n+1} = v_{-1}N_{2n+1} + u_0N_{2n} + v_1N_{2n+2}, \\ v_{2n+2} = u_{-1}N_{2n+2} + v_0N_{2n+1} + u_1N_{2n+3}. \end{cases} \quad (2.19)$$

■

The following theorem gives the explicit form of the solutions of system (2.1). This result is essential for analyzing the behavior of the system and studying its main properties.

Theorem 2.2.1 *Let $\{x_n, y_n\}_{n \geq -1}$ be a well defined solution of (2.1). Then, for $n = 0, 1, 2, \dots$ we have*

$$\begin{cases} x_{2n+1} = \frac{N_{2n+2} + x_{-1}N_{2n+1} + y_0x_{-1}N_{2n+3}}{N_{2n+1} + x_{-1}N_{2n} + y_0x_{-1}N_{2n+2}}, \\ x_{2n+2} = \frac{N_{2n+3} + y_{-1}N_{2n+2} + x_0y_{-1}N_{2n+4}}{N_{2n+2} + y_{-1}N_{2n+1} + x_0y_{-1}N_{2n+3}}, \\ y_{2n+1} = \frac{N_{2n+2} + y_{-1}N_{2n+1} + x_0y_{-1}N_{2n+3}}{N_{2n+1} + y_{-1}N_{2n} + x_0y_{-1}N_{2n+2}}, \\ y_{2n+2} = \frac{N_{2n+3} + x_{-1}N_{2n+2} + y_0x_{-1}N_{2n+4}}{N_{2n+2} + x_{-1}N_{2n+1} + y_0x_{-1}N_{2n+3}}. \end{cases} \quad (2.20)$$

Proof. Use a change of variables:

$$x_n = \frac{u_{n+1}}{v_n}, \quad y_n = \frac{v_{n+1}}{u_n}. \quad (2.21)$$

In (2.1) we obtain

$$x_{n+1} = \frac{u_{n+2}}{v_{n+1}}, \quad y_{n+1} = \frac{v_{n+2}}{u_{n+1}}.$$

So

$$\begin{aligned} \frac{u_{n+2}}{v_{n+1}} &= 1 + \frac{1}{\left(\frac{v_{n+1}}{u_n}\right)\left(\frac{u_n}{v_{n-1}}\right)}, \\ &= \frac{v_{n+1} + v_{n-1}}{v_{n+1}}. \end{aligned}$$

and

$$\begin{aligned}\frac{v_{n+2}}{u_{n+1}} &= 1 + \frac{1}{\left(\frac{u_{n+1}}{v_n}\right)\left(\frac{v_n}{u_{n-1}}\right)}, \\ &= \frac{u_{n+1} + u_{n-1}}{u_{n+1}}.\end{aligned}$$

Therefore, system (2.1) becomes

$$\begin{aligned}u_{n+2} - v_{n+1} - v_{n-1} &= 0, \\ v_{n+2} - u_{n+1} - u_{n-1} &= 0.\end{aligned}\tag{2.22}$$

Then it follows from Lemma 2.2.2 that

$$\begin{cases} x_{2n+1} = \frac{(R_{-1} - S_{-1})N_{2n+2} + (R_0 + S_0)N_{2n+1} + (R_1 - S_1)N_{2n+3}}{(R_{-1} - S_{-1})N_{2n+1} + (R_0 + S_0)N_{2n} + (R_1 - S_1)N_{2n+2}}, \\ x_{2n+2} = \frac{(R_{-1} + S_{-1})N_{2n+3} + (R_0 - S_0)N_{2n+2} + (R_1 + S_1)N_{2n+4}}{(R_{-1} + S_{-1})N_{2n+2} + (R_0 - S_0)N_{2n+1} + (R_1 + S_1)N_{2n+3}}, \end{cases}\tag{2.23}$$

$$\begin{cases} y_{2n+1} = \frac{(R_{-1} + S_{-1})N_{2n+2} + (R_0 - S_0)N_{2n+1} + (R_1 + S_1)N_{2n+3}}{(R_{-1} + S_{-1})N_{2n+1} + (R_0 - S_0)N_{2n} + (R_1 + S_1)N_{2n+2}}, \\ y_{2n+2} = \frac{(R_{-1} - S_{-1})N_{2n+3} + (R_0 + S_0)N_{2n+2} + (R_1 - S_1)N_{2n+4}}{(R_{-1} - S_{-1})N_{2n+2} + (R_0 + S_0)N_{2n+1} + (R_1 - S_1)N_{2n+3}}. \end{cases}\tag{2.24}$$

Then

$$\begin{cases} x_{2n+1} = \frac{N_{2n+2} + \frac{R_0+S_0}{R_{-1}-S_{-1}}N_{2n+1} + \frac{R_1-S_1}{R_{-1}-S_{-1}}N_{2n+3}}{N_{2n+1} + \frac{R_0+S_0}{R_{-1}-S_{-1}}N_{2n} + \frac{R_1-S_1}{R_{-1}-S_{-1}}N_{2n+2}}, \\ x_{2n+2} = \frac{N_{2n+3} + \frac{R_0-S_0}{R_{-1}+S_{-1}}N_{2n+2} + \frac{R_1+S_1}{R_{-1}+S_{-1}}N_{2n+4}}{N_{2n+2} + \frac{R_0-S_0}{R_{-1}+S_{-1}}N_{2n+1} + \frac{R_1+S_1}{R_{-1}+S_{-1}}N_{2n+3}}, \end{cases}\tag{2.25}$$

$$\begin{cases} y_{2n+1} = \frac{N_{2n+2} + \frac{R_0-S_0}{R_{-1}+S_{-1}}N_{2n+1} + \frac{R_1+S_1}{R_{-1}+S_{-1}}N_{2n+3}}{N_{2n+1} + \frac{R_0-S_0}{R_{-1}+S_{-1}}N_{2n} + \frac{R_1+S_1}{R_{-1}+S_{-1}}N_{2n+2}}, \\ y_{2n+2} = \frac{N_{2n+3} + \frac{R_0+S_0}{R_{-1}-S_{-1}}N_{2n+2} + \frac{R_1-S_1}{R_{-1}-S_{-1}}N_{2n+4}}{N_{2n+2} + \frac{R_0+S_0}{R_{-1}-S_{-1}}N_{2n+1} + \frac{R_1-S_1}{R_{-1}-S_{-1}}N_{2n+3}}. \end{cases}\tag{2.26}$$

We have

$$x_{-1} = \frac{u_0}{v_{-1}} = \frac{R_0 + S_0}{R_{-1} - S_{-1}}, \quad x_0 = \frac{u_1}{v_0} = \frac{R_1 + S_1}{R_0 - S_0}.\tag{2.27}$$

$$y_{-1} = \frac{v_0}{u_{-1}} = \frac{R_0 - S_0}{R_{-1} + S_{-1}}, \quad y_0 = \frac{v_1}{u_0} = \frac{R_1 - S_1}{R_0 + S_0}.\tag{2.28}$$

From (2.27) and (2.28) it follows that

$$\begin{aligned} \frac{R_1 - S_1}{R_{-1} - S_{-1}} &= \frac{R_1 - S_1}{R_0 + S_0} \frac{R_0 + S_0}{R_{-1} - S_{-1}} = y_0 x_{-1}. \\ \frac{R_1 + S_1}{R_{-1} + S_{-1}} &= \frac{R_1 + S_1}{R_0 - S_0} \frac{R_0 - S_0}{R_{-1} + S_{-1}} = x_0 y_{-1}. \end{aligned} \quad (2.29)$$

From (2.25), (2.26), (2.27), (2.28), and (2.29) we obtain

$$\begin{cases} x_{2n+1} = \frac{N_{2n+2} + x_{-1}N_{2n+1} + y_0x_{-1}N_{2n+3}}{N_{2n+1} + x_{-1}N_{2n} + y_0x_{-1}N_{2n+2}}, \\ x_{2n+2} = \frac{N_{2n+3} + y_{-1}N_{2n+2} + x_0y_{-1}N_{2n+4}}{N_{2n+2} + y_{-1}N_{2n+1} + x_0y_{-1}N_{2n+3}}, \\ \\ y_{2n+1} = \frac{N_{2n+2} + y_{-1}N_{2n+1} + x_0y_{-1}N_{2n+3}}{N_{2n+1} + y_{-1}N_{2n} + x_0y_{-1}N_{2n+2}}, \\ y_{2n+2} = \frac{N_{2n+3} + x_{-1}N_{2n+2} + y_0x_{-1}N_{2n+4}}{N_{2n+2} + x_{-1}N_{2n+1} + y_0x_{-1}N_{2n+3}}. \end{cases}$$

■

2.3 Global stability of solutions of system (2.1)

Global stability is one of the most important qualitative properties in the study of systems of difference equations, since it describes the long-term behavior of their solutions. In this section, we investigate the global stability of the solutions of system (2.1) and analyze their convergence behavior.

Lemma 2.3.1 *The system (2.1) has a unique real equilibrium point, given by:*

$$\overline{M} = (\overline{x}, \overline{y}) = (\alpha, \alpha).$$

Proof. Suppose I and $J \subset \mathbb{R}$, and consider the function

$$f : I^2 \times J^2 \longrightarrow I$$

$$g : J^2 \times I^2 \longrightarrow J$$

Defined by

$$\begin{cases} f(x_n, x_{n-1}, y_n, y_{n-1}) = 1 + \frac{1}{y_n x_{n-1}}, \\ g(x_n, x_{n-1}, y_n, y_{n-1}) = 1 + \frac{1}{x_n y_{n-1}}. \end{cases} \quad (2.30)$$

We have

$$\begin{cases} \bar{x} = 1 + \frac{1}{\bar{y}\bar{x}}, \\ \bar{y} = 1 + \frac{1}{\bar{y}\bar{x}}. \end{cases} \quad (2.31)$$

We obtain

$$\bar{x}^2 \bar{y} - \bar{x} \bar{y} - 1 = 0 \quad (2.32)$$

$$\bar{y}^2 \bar{x} - \bar{x} \bar{y} - 1 = 0 \quad (2.33)$$

By using (2.32),(2.33), we get

$$\bar{x} \bar{y} = \frac{1}{\bar{x} - 1}. \quad (2.34)$$

$$\bar{x} \bar{y} = \frac{1}{\bar{y} - 1}. \quad (2.35)$$

We deduce

$$\frac{1}{\bar{x} - 1} = \frac{1}{\bar{y} - 1} \quad (2.36)$$

$$\bar{x} = \bar{y}. \quad (2.37)$$

We substitute (2.37) in to (2.31) we get

$$\bar{x}^3 - \bar{x}^2 - 1 = 0.$$

Since, it is the same characteristic equation (1.28)

Then

$$\bar{x} = \alpha = \bar{y}.$$

$$\overline{M} = (\overline{x}, \overline{y}) = (\alpha, \alpha).$$

■

After determining the equilibrium point of system (2.1), we now study its local asymptotic stability by linearizing the system around the equilibrium point $\overline{M} = (\alpha, \alpha)$.

Theorem 2.3.1 *The equilibrium point $\overline{M}$ is locally asymptotically stable.*

Proof. We linearize the system around the equilibrium point $\overline{M}$, can be written as:

$$w_{n+1} = Aw_n, \quad w_n = (x_n, x_{n-1}, y_n, y_{n-1})^T.$$

A Jacobian matrix is given by:

$$\begin{cases} x_{n+1} = 1 + \frac{1}{y_n x_{n-1}}, \\ x_n = x_n, \\ y_{n+1} = 1 + \frac{1}{x_n y_{n-1}}, \\ y_n = y_n. \end{cases} \quad (2.38)$$

$$A = \begin{pmatrix} 0 & \frac{-1}{\alpha^3} & \frac{-1}{\alpha^3} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{-1}{\alpha^3} & 0 & 0 & \frac{-1}{\alpha^3} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then, the characteristic polynomial of A is

$$P_A(\lambda) = \lambda^4 + \frac{(2\alpha^3 - 1)}{\alpha^6} \lambda^2 + \frac{1}{\alpha^6}.$$

We have $\Delta = -0.071 < 0$, Therefore, the characteristic polynomial admits 4 roots

$$\lambda_{1,2} = 0.159 \pm i0.54,$$

$$\lambda_{3,4} = -0.159 \pm i0.54.$$

We find

$$|\lambda_1| = |\lambda_2| = |\lambda_3| = |\lambda_4| = 0.562 < 1.$$

Then, the equilibrium point is locally asymptotically stable. ■

Theorem 2.3.2 *The equilibrium point $\overline{M}$ is globally asymptotically stable.*

Proof. Let $\{x_n, y_n\}_{n \geq 0}$, be a real solution of the system (3.1), it suffices to prove that $\overline{M} = (\overline{x}, \overline{y})$ is globally attractive.

We have

$$\begin{aligned} \lim_{n \rightarrow +\infty} x_{2n+1} &= \lim_{n \rightarrow +\infty} \frac{N_{2n+2} + x_{-1}N_{2n+1} + y_0x_{-1}N_{2n+3}}{N_{2n+1} + x_{-1}N_{2n} + y_0x_{-1}N_{2n+2}} \\ &= \lim_{n \rightarrow +\infty} \frac{N_{2n+2}(1 + \frac{N_{2n+1}}{N_{2n+2}}x_{-1} + \frac{N_{2n+3}}{N_{2n+2}}y_0x_{-1})}{N_{2n+2}(\frac{N_{2n+1}}{N_{2n+2}} + (\frac{N_{2n+3}-N_{2n+2}}{N_{2n+2}})x_{-1} + y_0x_{-1})} \\ &= \lim_{n \rightarrow +\infty} \frac{1 + \frac{1}{\alpha}x_{-1} + \alpha y_0x_{-1}}{\frac{1}{\alpha} + (\alpha - 1)x_{-1} + y_0x_{-1}} \end{aligned}$$

By using the relations (1.21) we get :

$$\begin{aligned} \frac{1}{\alpha} &= \beta\gamma, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= 0. \end{aligned}$$

So we can write

$$\begin{aligned} \beta\gamma &= -\alpha(\beta + \gamma) \\ &= -\alpha(1 - \alpha) \\ &= \alpha(\alpha - 1). \end{aligned}$$

Then

$$\lim_{n \rightarrow +\infty} x_{2n+1} = \lim_{n \rightarrow +\infty} \frac{\alpha(\frac{1}{\alpha} + (\alpha - 1)x_{-1} + x_{-1}x_0)}{\frac{1}{\alpha} + (\alpha - 1)x_{-1} + x_{-1}x_0}$$

So

$$\lim_{n \rightarrow +\infty} x_{2n+1} = \alpha.$$

By following the same steps, we calculate the following limits.

$$\begin{aligned}
 \lim_{n \rightarrow +\infty} x_{2n+2} &= \lim_{n \rightarrow +\infty} \frac{N_{2n+3} + y_{-1}N_{2n+2} + x_0y_{-1}N_{2n+4}}{N_{2n+2} + y_{-1}N_{2n+1} + x_0y_{-1}N_{2n+3}} \\
 &= \lim_{n \rightarrow +\infty} \frac{N_{2n+3}(1 + \frac{N_{2n+2}}{N_{2n+3}}y_{-1} + \frac{N_{2n+4}}{N_{2n+3}}x_0y_{-1})}{N_{2n+3}(\frac{N_{2n+2}}{N_{2n+3}} + (\frac{N_{2n+4}-N_{2n+3}}{N_{2n+3}})y_{-1} + x_0y_{-1})} \\
 &= \lim_{n \rightarrow +\infty} \frac{1 + \frac{1}{\alpha}y_{-1} + \alpha x_0y_{-1}}{\frac{1}{\alpha} + (\alpha - 1)y_{-1} + x_0y_{-1}} \\
 &= \lim_{n \rightarrow +\infty} \frac{\alpha(\frac{1}{\alpha} + (\alpha - 1)y_{-1} + x_0y_{-1})}{\frac{1}{\alpha} + (\alpha - 1)y_{-1} + x_0y_{-1}} \\
 &= \alpha.
 \end{aligned}$$

So

$$\lim_{n \rightarrow +\infty} x_n = \bar{x}.$$

and

$$\begin{aligned}
 \lim_{n \rightarrow +\infty} y_{2n+1} &= \lim_{n \rightarrow +\infty} \frac{N_{2n+2} + y_{-1}N_{2n+1} + x_0y_{-1}N_{2n+3}}{N_{2n+1} + y_{-1}N_{2n} + x_0y_{-1}N_{2n+2}} \\
 &= \lim_{n \rightarrow +\infty} \frac{N_{2n+2}(1 + \frac{N_{2n+1}}{N_{2n+2}}y_{-1} + \frac{N_{2n+3}}{N_{2n+2}}x_0y_{-1})}{N_{2n+2}(\frac{N_{2n+1}}{N_{2n+2}} + (\frac{N_{2n+3}-N_{2n+2}}{N_{2n+2}})y_{-1} + x_0y_{-1})} \\
 &= \lim_{n \rightarrow +\infty} \frac{1 + \frac{1}{\alpha}y_{-1} + \alpha x_0y_{-1}}{\frac{1}{\alpha} + (\alpha - 1)y_{-1} + x_0y_{-1}} \\
 &= \lim_{n \rightarrow +\infty} \frac{\alpha(\frac{1}{\alpha} + (\alpha - 1)y_{-1} + x_0y_{-1})}{\frac{1}{\alpha} + (\alpha - 1)y_{-1} + x_0y_{-1}} \\
 &= \alpha.
 \end{aligned}$$

and

$$\begin{aligned}
 \lim_{n \rightarrow +\infty} y_{2n+2} &= \lim_{n \rightarrow +\infty} \frac{N_{2n+3} + x_{-1}N_{2n+2} + y_0x_{-1}N_{2n+4}}{N_{2n+2} + x_{-1}N_{2n+1} + y_0x_{-1}N_{2n+3}} \\
 &= \lim_{n \rightarrow +\infty} \frac{N_{2n+3}(1 + \frac{N_{2n+2}}{N_{2n+3}}x_{-1} + \frac{N_{2n+4}}{N_{2n+3}}y_0x_{-1})}{N_{2n+3}(\frac{N_{2n+2}}{N_{2n+3}} + (\frac{N_{2n+4}-N_{2n+3}}{N_{2n+3}})x_{-1} + y_0x_{-1})} \\
 &= \lim_{n \rightarrow +\infty} \frac{1 + \frac{1}{\alpha}x_{-1} + \alpha y_0x_{-1}}{\frac{1}{\alpha} + (\alpha - 1)x_{-1} + y_0x_{-1}} \\
 &= \lim_{n \rightarrow +\infty} \frac{\alpha(\frac{1}{\alpha} + (\alpha - 1)x_{-1} + y_0x_{-1})}{\frac{1}{\alpha} + (\alpha - 1)x_{-1} + y_0x_{-1}} \\
 &= \alpha.
 \end{aligned}$$

So

$$\lim_{n \rightarrow +\infty} y_n = \bar{y}.$$

Then

$$\lim_{n \rightarrow +\infty} (x_n, y_n) = (\bar{x}, \bar{y}). \quad (2.39)$$

Thus, the equilibrium point $\bar{M}$ is globally attractive, therefore it is globally asymptotically stable. ■

2.4 Numerical Examples

In this section, we present some numerical examples to illustrate the behavior of the solutions of system (2.1). Using different initial conditions, we compute the corresponding sequences (x_n) and (y_n) and observe their convergence toward the equilibrium point. The numerical results and graphical representations confirm the theoretical results established in the previous sections concerning the explicit form and the global stability of the solutions.

Example 2.4.1 We consider the following system of difference equations:

$$x_{n+1} = \frac{y_n x_{n-1} + 1}{y_n x_{n-1}}, \quad y_{n+1} = \frac{x_n y_{n-1} + 1}{x_n y_{n-1}}. \quad (2.40)$$

the initial conditions are given as follows:

- Initial conditions : $x_{-1} = 0.2$, $x_0 = 0.8$.

- Initial conditions : $y_{-1} = 0.3$, $y_0 = 0.7$.

The following table presents the computed values of the sequence (x_n) and (y_n) generated by the difference system with the given initial conditions . The values are displayed for $n = 0$ to $n = 30$.

n	1	2	3	4	5	6	7	8	9	10
x_n	8.1429	1.2419	1.1045	1.6966	1.5115	1.3904	1.4816	1.4872	1.4526	1.4620
y_n	5.1667	1.1754	1.1558	1.7703	1.5099	1.3737	1.4763	1.4913	1.4555	1.4616

n	11	12	13	14	15	16	17	18	19	20
x_n	1.4710	1.4653	1.4640	1.4661	1.4659	1.4653	1.4656	1.4656	1.4655	1.4656
y_n	1.4700	1.4651	1.4643	1.4662	1.4658	1.4653	1.4656	1.4657	1.4655	1.4656

n	21	22	23	24	25	26	27	28	29	30
x_n	1.4656	1.4656	1.4656	1.4656	1.4656	1.4656	1.4656	1.4656	1.4656	1.4656
y_n	1.4656	1.4656	1.4656	1.4656	1.4656	1.4656	1.4656	1.4656	1.4656	1.4656

Table 2.1: Computed values of x_n and y_n for $n = 1$ to 30 from the Difference equation System

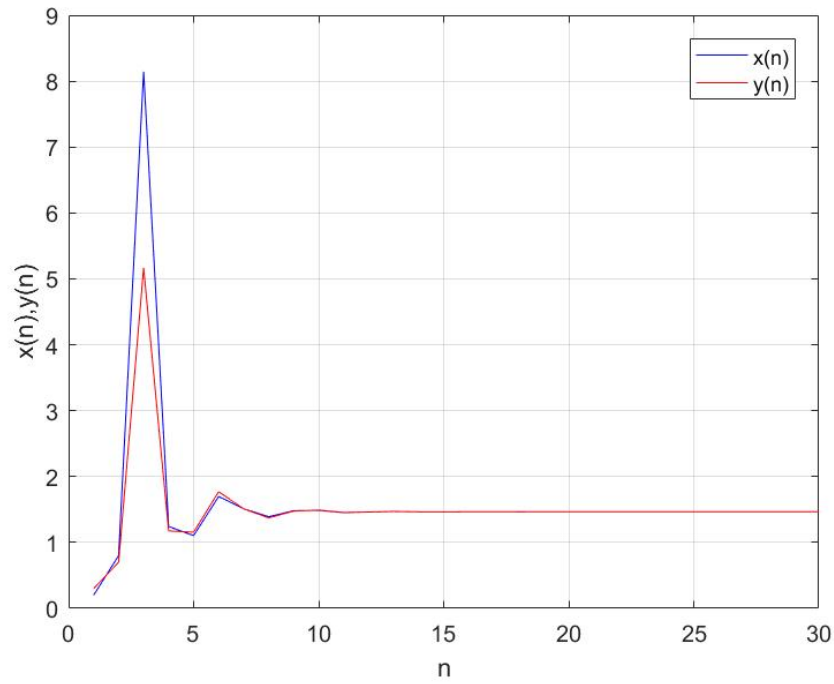


Figure 2.1: Numerical Illustration of the Asymptotic Stability of System (2.40)

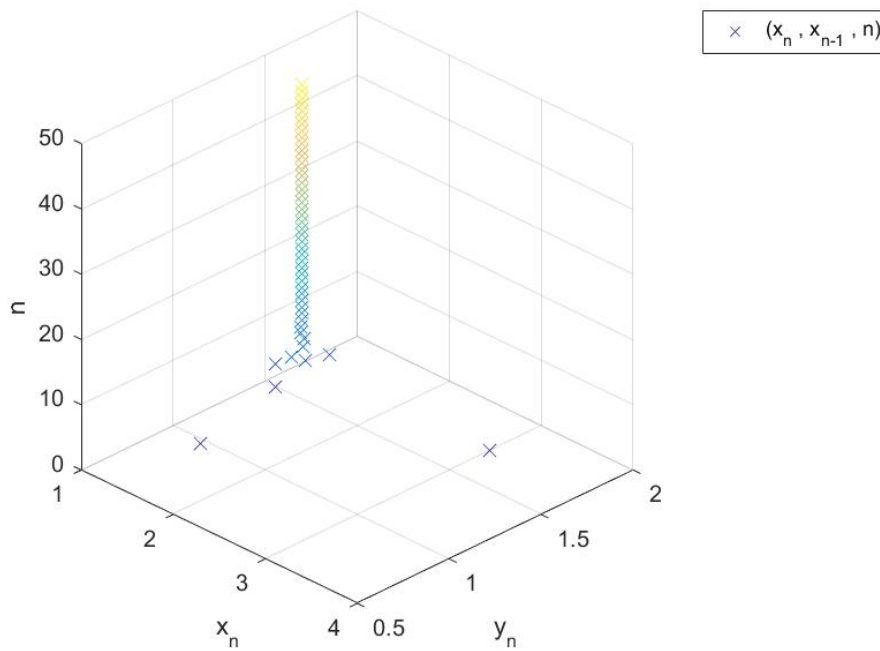


Figure 2.2: 3D phase surface plot of the system (2.40)

Example 2.4.2 *Let the following system of difference equations:*

$$x_{n+1} = \frac{y_n x_{n-1} + 1}{y_n x_{n-1}}, \quad y_{n+1} = \frac{x_n y_{n-1} + 1}{x_n y_{n-1}}. \quad (2.41)$$

Where the initial conditions are chosen as follows:

- *Initial conditions : $x_{-1} = 0.5$, $x_0 = 1.5$.*

- *Initial conditions : $y_{-1} = 1.1$, $y_0 = 0.9$.*

The following table presents the computed values of the sequence (x_n) and (y_n) generated by the difference system with the given initial conditions . The values are displayed for $n = 0$ to $n = 30$.

n	1	2	3	4	5	6	7	8	9	10
x_n	3.2222	1.2308	1.4907	1.5065	1.4576	1.4695	1.4673	1.4594	1.4654	1.4685
y_n	1.6061	1.3448	1.4400	1.6042	1.4658	1.4138	1.4680	1.4813	1.4642	1.4626

n	11	12	13	14	15	16	17	18	19	20
x_n	1.4655	1.4646	1.4656	1.4658	1.4655	1.4656	1.4656	1.4656	1.4656	1.4656
y_n	1.4660	1.4656	1.4655	1.4659	1.4656	1.4654	1.4656	1.4656	1.4656	1.4656

n	21	22	23	24	25	26	27	28	29	30
x_n	1.4656	1.4656	1.4656	1.4656	1.4656	1.4656	1.4656	1.4656	1.4656	1.4656
y_n	1.4656	1.4656	1.4656	1.4656	1.4656	1.4656	1.4656	1.4656	1.4656	1.4656

Table 2.2: Computed values of x_n and y_n for $n = 1$ to 30 from the Difference equation System

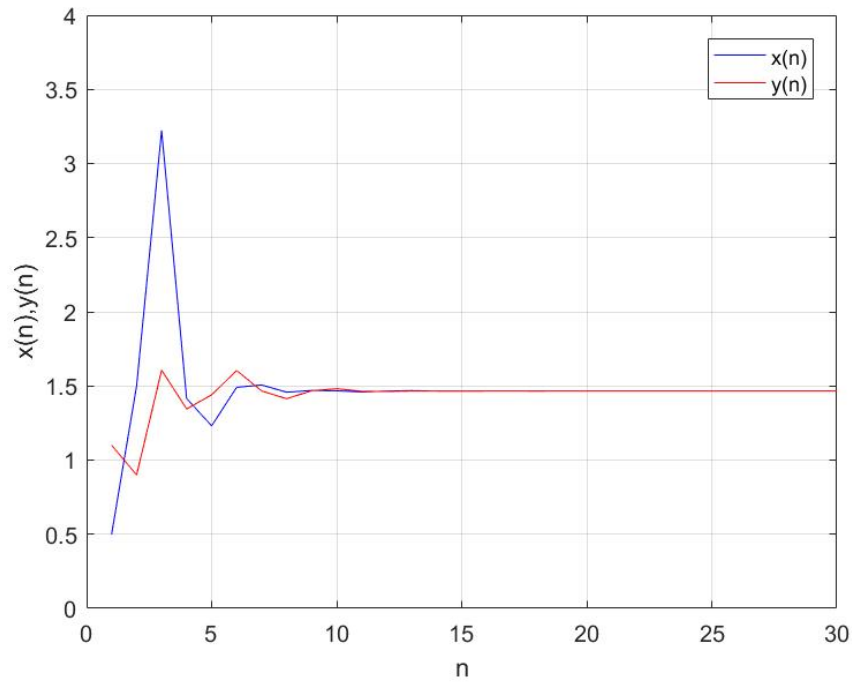


Figure 2.3: Numerical Illustration of the Asymptotic Stability of System (2.41)

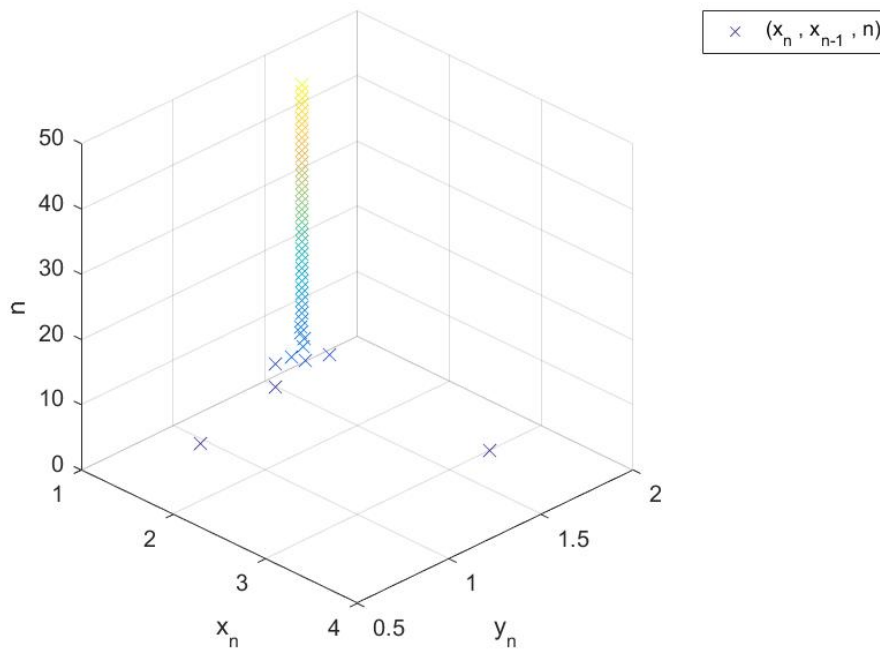


Figure 2.4: 3D phase surface plot of the system (2.41)

2.5 Conclusion

In this chapter, we studied a nonlinear system of difference equations related to the Narayana sequence. By using suitable transformations, we derived explicit formulas for the solutions in terms of Narayana numbers. We also proved that the system admits a unique equilibrium point $M = (\alpha, \alpha)$, which is locally and globally asymptotically stable. Finally, the numerical examples confirmed the theoretical results and illustrated the convergence of the solutions toward the equilibrium point.

Generalized systems of difference equations related to the Narayana sequence

3.1 Introduction

In this chapter, we obtain explicit formulas of the solutions for the following two systems of difference equations

$$\begin{aligned}x_{n+1} &= f^{-1}(g(y_n) + g(y_{n-2})), \\ y_{n+1} &= g^{-1}(f(x_n) + f(x_{n-2})).\end{aligned}\tag{3.1}$$

$$\begin{aligned}x_{n+1} &= f^{-1}\left(1 + \frac{1}{g(y_n)f(x_{n-1})}\right), \\ y_{n+1} &= g^{-1}\left(1 + \frac{1}{f(x_n)g(y_{n-1})}\right).\end{aligned}\tag{3.2}$$

where $n \in \mathbb{N}_0$, and $f, g : D \rightarrow \mathbb{R}$ are continuous one-to-one functions on $D \subseteq \mathbb{R}$ and the initial conditions x_{-i}, y_{-i} , $i = 0, 1, 2$, are real numbers.

In [2], the authors derived explicit formulas for solutions of the systems of difference

equations

$$\begin{aligned}x_{n+1} &= f^{-1}(ag(y_n) + bf(x_{n-1}) + cg(y_{n-2}) + df(x_{n-3})), \\y_{n+1} &= g^{-1}(af(x_n) + bg(y_{n-1}) + cf(x_{n-2}) + dg(y_{n-3})), \\x_{n+1} &= f^{-1}\left(a + \frac{b}{g(y_n)} + \frac{c}{g(y_n)f(x_{n-1})} + \frac{d}{g(y_n)f(x_{n-1})g(y_{n-2})}\right), \\y_{n+1} &= g^{-1}\left(a + \frac{b}{f(x_n)} + \frac{c}{f(x_n)g(y_{n-1})} + \frac{d}{f(x_n)g(y_{n-1})f(x_{n-2})}\right).\end{aligned}$$

We focus on a special case obtained by setting $a=1$, $b=0$, $c=1$, and $d=0$, and then investigate the systems of difference equations(3.1) and (3.2) linked to Narayana numbers.

3.2 Form of the solution of system (3.1)

To understand the behavior of the system and study its properties, in this section, we determine the solutions of Systems (3.1) by using the following lemma, which provides important relations that will be used in the subsequent proofs.

Lemma 3.2.1 *Consider the two homogeneous third order linear autonomous differences equations*

$$R_{n+1} = R_n + R_{n-2}, \quad (3.3)$$

$$S_{n+1} = -S_n - S_{n-2}. \quad (3.4)$$

With the initial conditions $R_{-2}, R_{-1}, R_0, S_{-2}, S_{-1}$ and $S_0 \in \mathbb{R}$.

Hence, for all $n \in N_0$ we have

$$\begin{aligned}R_n &= R_{-2}N_n + R_{-1}N_{n-1} + R_0N_{n+1}, \\S_n &= (-1)^{n+1}(S_{-2}N_n - S_{-1}N_{n-1} + S_0N_{n+1}).\end{aligned}$$

Proof. We can prove this lemma using the same steps as in the proof of lemma 2.2.1.

■

Before presenting Lemma 3.2.2, we use the results of Lemma 3.2.1 to derive explicit formulas for the even and odd terms of the solutions, through a suitable transformation that reduces the system to third-order linear difference equations associated with the Narayana sequence.

Lemma 3.2.2 *Consider the linear system of third order linear autonomous differences equations:*

$$\begin{aligned} u_{n+1} &= v_n + v_{n-2}, \\ v_{n+1} &= u_n + u_{n-2}. \end{aligned} \tag{3.5}$$

With the initial conditions $u_{-2}, u_{-1}, u_0, v_{-2}, v_{-1}$ and $v_0 \in \mathbb{R}$.

Then

$$\begin{cases} u_{2n+1} = N_{2n+1}u_{-2} + N_{2n}v_{-1} + N_{2n+2}u_0, \\ u_{2n+2} = N_{2n+2}v_{-2} + N_{2n+1}u_{-1} + N_{2n+3}v_0, \\ v_{2n+1} = N_{2n+1}v_{-2} + N_{2n}u_{-1} + N_{2n+2}v_0, \\ v_{2n+2} = N_{2n+2}u_{-2} + N_{2n+1}v_{-1} + N_{2n+3}u_0. \end{cases} \tag{3.6}$$

Proof.

We can prove this lemma using the same steps as in the proof of lemma 2.2.2. ■

Before presenting the solution, we introduce the following definition.

Definition 3.2.1 *A solution $\{x_n, y_n\}_{n \geq 0}$ of system (3.1) is said to be well-defined if for all $n \in N_0$ we have*

$$g(y_n) + g(y_{n-2}) \in D_{f^{-1}},$$

and

$$f(x_n) + f(x_{n-2}) \in D_{g^{-1}}.$$

The following theorem gives the explicit form of the solutions of system (3.1). This

result is essential for analyzing the behavior of the system and studying its main properties.

Theorem 3.2.1 *Let $\{x_n, y_n\}_{n \geq -2}$ be a well defined solution of system (3.1), then we have*

$$\begin{cases} x_{2n+1} = f^{-1}(N_{2n+1}f(x_{-2}) + N_{2n}g(y_{-1}) + N_{2n+2}f(x_0)), n \in N, \\ x_{2n+2} = f^{-1}((N_{2n+2}g(y_{-2}) + N_{2n+1}f(x_{-1}) + N_{2n+3}g(y_0)), n \in N_0, \\ y_{2n+1} = g^{-1}(N_{2n+1}g(y_{-2}) + N_{2n}f(x_{-1}) + N_{2n+2}g(y_0)), n \in N, \\ y_{2n+2} = g^{-1}(N_{2n+2}f(x_{-2}) + N_{2n+1}g(y_{-1}) + N_{2n+3}f(x_0)), n \in N_0. \end{cases} \quad (3.7)$$

Proof.

We have

$$\begin{aligned} x_{n+1} &= f^{-1}(g(y_n) + g(y_{n-2})), \\ y_{n+1} &= g^{-1}(f(x_n) + f(x_{n-2})). \end{aligned}$$

Because the function f, g are "1-1", then:

$$\begin{cases} f(x_{n+1}) = g(y_n) + g(y_{n-2}), \\ g(x_{n+1}) = f(x_n) + f(x_{n-2}). \end{cases} \quad (3.8)$$

By the change of variables

$$X_n = f(x_n), \quad Y_n = g(y_n), n \geq -2. \quad (3.9)$$

System (3.8) is transformed to the following one:

$$\begin{cases} X_{n+1} = Y_n + Y_{n-2}, \\ Y_{n+1} = X_n + X_{n-2}. \end{cases} \quad (3.10)$$

It is clear that the system (3.10) is in the form of (3.5) So, from lemma 3.2.1 we get as

follows:

$$\begin{cases} X_{2n+1} = N_{2n+1}X_{-2} + N_{2n}Y_{-1} + N_{2n+2}X_0, \\ X_{2n+2} = N_{2n+2}Y_{-2} + N_{2n+1}X_{-1} + N_{2n+3}Y_0, \\ Y_{2n+1} = N_{2n+1}Y_{-2} + N_{2n}X_{-1} + N_{2n+2}Y_0, \\ Y_{2n+2} = N_{2n+2}X_{-2} + N_{2n+1}Y_{-1} + N_{2n+3}X_0. \end{cases} \quad (3.11)$$

By using (3.9) we write

$$\begin{cases} x_{2n+1} = f^{-1}(N_{2n+1}f(x_{-2}) + N_{2n}g(y_{-1}) + N_{2n+2}f(x_0)), n \in \mathbb{N}, \\ x_{2n+2} = f^{-1}(N_{2n+2}g(y_{-2}) + N_{2n+1}f(x_{-1}) + N_{2n+3}g(y_0)), n \in \mathbb{N}_0, \\ y_{2n+1} = g^{-1}(N_{2n+1}g(y_{-2}) + N_{2n}f(x_{-1}) + N_{2n+2}g(y_0)), n \in \mathbb{N}, \\ y_{2n+2} = g^{-1}(N_{2n+2}f(x_{-2}) + N_{2n+1}g(y_{-1}) + N_{2n+3}f(x_0)), n \in \mathbb{N}_0. \end{cases}$$

■

For more detail and better understanding, we can reinforce Theorem 3.2.1 the following application.

Example. Let

$$f(t) = jt, \quad g(t) = kt, \quad j, k \in \mathbb{N}_0.$$

where

$$D_f = D_g = R.$$

It is clearly that f and g are "1-1". From the system (3.1) we get:

$$\begin{cases} x_{n+1} = \frac{k}{j}(y_n + y_{n-2}), \\ y_{n+1} = \frac{j}{k}(x_n + x_{n-2}), \end{cases} \quad n \in \mathbb{N}_0. \quad (3.12)$$

Then from (3.7) we obtain:

$$\begin{cases} x_{2n+1} = \frac{1}{j} (jN_{2n+1}x_{-2} + kN_{2n}y_{-1} + jN_{2n+2}x_0) n \in N, \\ x_{2n+2} = \frac{1}{j} (kN_{2n+2}y_{-2} + jN_{2n+1}x_{-1} + kN_{2n+3}y_0) n \in N_0, \\ y_{2n+1} = \frac{1}{k} (kN_{2n+1}y_{-2} + jN_{2n}x_{-1} + kN_{2n+2}y_0) n \in N, \\ y_{2n+2} = \frac{1}{k} (jN_{2n+2}x_{-2} + kN_{2n+1}y_{-1} + jN_{2n+3}x_0) n \in N_0. \end{cases}$$

where $\{N_n\}_{n \geq 0}$ is the Narayana sequence.

This example is valid for any values of j and k , To study the stability, we take $j = 5$ and $k = 4$.

3.3 Global stability of solutions of system (3.1)

Global stability is one of the most important qualitative properties in the study of systems of difference equations, since it describes the long-term behavior of their solutions. In this section, we study the global asymptotic stability of the real solutions of the system (3.12).

Theorem 3.3.1 *The equilibrium point (0,0) is unstable.*

Proof.

We can observe that the system is linear therefore, to study its stability, we rewrite it in the following matrix form:

$$X_{n+1} = AX_n, \quad X_n = (x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2})^T.$$

Such that

$$A = \begin{pmatrix} 0 & 0 & 0 & \frac{4}{5} & 0 & \frac{4}{5} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{5}{4} & 0 & \frac{5}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then, the characteristic polynomial of A is

$$P_A(\lambda) = \lambda^6 - \lambda^4 - 2\lambda^2 - 1 = 0.$$

The characteristic polynomial admits 6 roots

$$\lambda_{1,2} = \pm 1.4656,$$

$$\lambda_{3,4} = -0.2328 \pm i0.7926,$$

$$\lambda_{5,6} = 0.2328 \pm i0.7926.$$

We find

$$|\lambda_1| = |\lambda_2| = 1.4656 > 1,$$

Then, the origin is unstable.

■

3.4 Numerical Example

In this section, we provide numerical simulations to illustrate and verify the theoretical results obtained, we consider the following system of difference equations:

$$x_{n+1} = \frac{5}{4}(y_n + y_{n-2}), \quad y_{n+1} = \frac{4}{5}(x_n + x_{n-2}). \quad (3.13)$$

which is obtained from system (3.1) by taking

$$f(t) = 5t, \quad g(t) = 4t.$$

the initial conditions are given as follows:

- Initial conditions : $x_{-2} = 0.2$, $x_{-1} = 0.2$, $x_0 = 0.3$.
- Initial conditions : $y_{-2} = 0.1$, $y_{-1} = 0.2$, $y_0 = 0.15$.

The following table presents the computed values of the sequence (x_n) and (y_n) generated by the difference system with the given initial conditions . The values are displayed for $n = 0$ to $n = 24$.

n	1	2	3	4	5	6	7	8
x_n	0.1000	0.3000	0.3125	0.7500	0.6000	1.5500	1.3250	3.3500
y_n	0.1000	0.2000	0.1500	0.4000	0.3300	0.8400	0.7300	1.8400

n	9	10	11	12	13	14	15	16
x_n	2.8375	7.2000	6.0875	15.4500	13.0875	33.2000	28.1000	71.3000
y_n	1.5400	3.9200	3.3300	8.4400	7.1400	18.1200	15.3400	38.9200

n	17	18	19	20	21	22	23	24
x_n	60.3625	153.1500	129.6500	328.9500	278.4750	706.5500	598.1375	1517.6000
y_n	32.9500	83.6000	70.7700	179.5600	152.0100	385.6800	326.5000	828.4000

Table 3.1: Computed values of x_n and y_n for $n = 1$ to 24 from the Difference equation System(3.1)

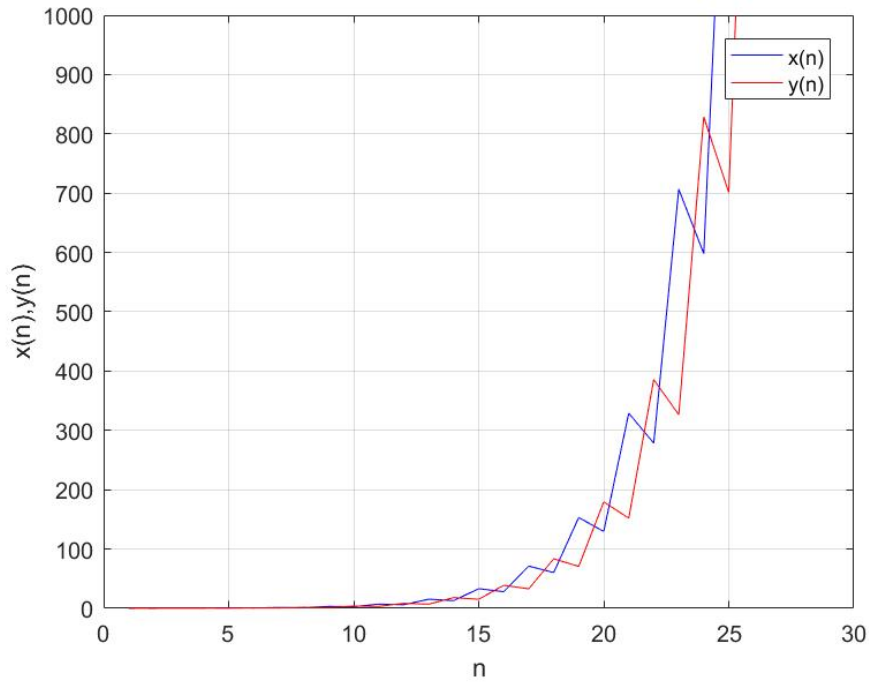


Figure 3.1: Numerical Illustration of the Asymptotic Stability of System (3.13)

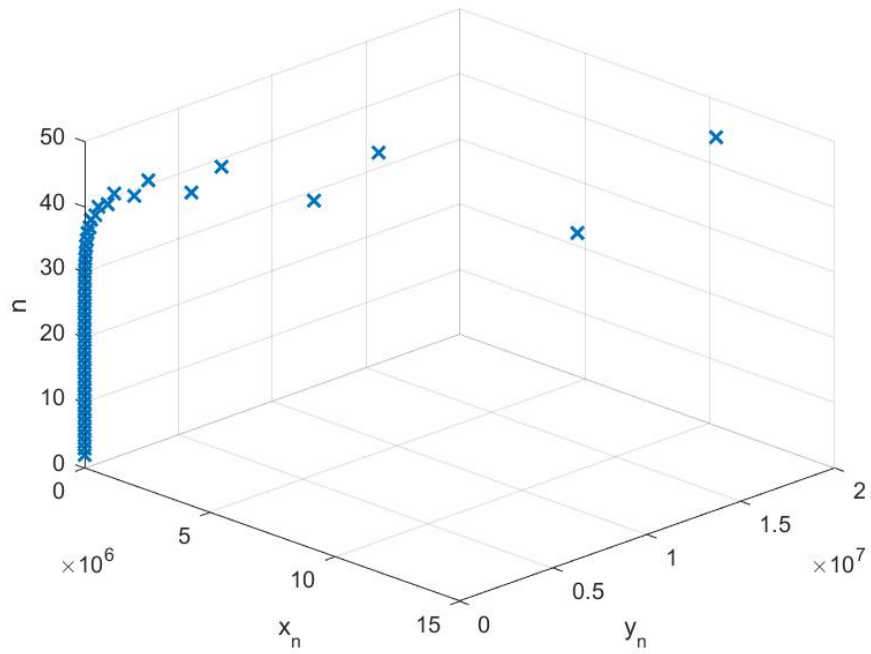


Figure 3.2: 3D phase surface plot of the system (3.13)

3.5 Form of the solution of system (3.2)

To understand the behavior of the system and study its properties, in this section, we determine the solutions of Systems (3.2) by using the previous two lemma (3.2.1, 3.2.2) which provides important relations that will be used in the subsequent proofs.

Before presenting the solution, we introduce the following definition.

Definition 3.5.1 A solution $\{x_n, y_n\}_{n \geq 0}$ of system (3.2) is said to be well-defined if for all $n \in N_0$, we have

$$\frac{g(y_n)f(x_{n-1}) + 1}{g(y_n)f(x_{n-1})} \in D_{f^{-1}},$$

and

$$\frac{f(x_n)g(y_{n-1}) + 1}{f(x_n)g(y_{n-1})} \in D_{g^{-1}}.$$

The following theorem gives the explicit form of the solutions of system (3.2). This result is essential for analyzing the behavior of the system and studying its main properties.

Theorem 3.5.1 Let $\{x_n, y_n\}_{n \geq -1}$ be a well defined solution of system (3.2), then we have

$$\begin{cases} x_{2n+1} = f^{-1}\left(\frac{N_{2n+1} + N_{2n}g(y_{-1}) + N_{2n+2}f(x_0)g(y_{-1})}{N_{2n} + N_{2n-1}g(y_{-1}) + N_{2n+1}f(x_0)g(y_{-1})}\right), \\ x_{2n+2} = f^{-1}\left(\frac{N_{2n+2} + N_{2n+1}f(x_{-1}) + N_{2n+3}g(y_0)f(x_{-1})}{N_{2n+1} + N_{2n}f(x_{-1}) + N_{2n+2}g(y_0)f(x_{-1})}\right), \\ y_{2n+1} = g^{-1}\left(\frac{N_{2n+1} + N_{2n}f(x_{-1}) + N_{2n+2}g(y_0)f(x_{-1})}{N_{2n} + N_{2n-1}f(x_{-1}) + N_{2n+1}g(y_0)f(x_{-1})}\right), \\ y_{2n+2} = g^{-1}\left(\frac{N_{2n+2} + N_{2n+1}g(y_{-1}) + N_{2n+3}f(x_0)g(y_{-1})}{N_{2n+1} + N_{2n}g(y_{-1}) + N_{2n+2}f(x_0)g(y_{-1})}\right). \end{cases} \quad (3.14)$$

Proof.

We have

$$\begin{aligned} x_{n+1} &= f^{-1}\left(1 + \frac{1}{g(y_n)f(x_{n-1})}\right), \\ y_{n+1} &= g^{-1}\left(1 + \frac{1}{f(x_n)g(y_{n-1})}\right). \end{aligned}$$

Because the function f, g are "1-1", then:

$$\begin{cases} f(x_{n+1}) = 1 + \frac{1}{g(y_n)f(x_{n-1})}, \\ g(y_{n+1}) = 1 + \frac{1}{f(x_n)g(y_{n-1})}, \end{cases} \quad n \in \mathbb{N}_0.$$

By using the change of variables (3.9) we get:

$$\begin{cases} X_{n+1} = 1 + \frac{1}{Y_n X_{n-1}}, \\ Y_{n+1} = 1 + \frac{1}{X_n Y_{n-1}}, \end{cases} \quad n \in \mathbb{N}_0. \quad (3.15)$$

We use change of variable.

$$X_n = \frac{u_n}{v_{n-1}}, \quad Y_n = \frac{v_n}{u_{n-1}}, \quad n \geq -2. \quad (3.16)$$

In (3.15) and obtain

$$X_{n+1} = \frac{u_{n+1}}{v_n}, \quad Y_{n+1} = \frac{v_{n+1}}{u_n}.$$

So

$$\begin{aligned} \frac{u_{n+1}}{v_n} &= 1 + \frac{1}{\left(\frac{v_n}{u_{n-1}}\right)\left(\frac{u_{n-1}}{v_{n-2}}\right)} \\ &= \frac{v_n + v_{n-2}}{v_n}. \end{aligned}$$

and

$$\begin{aligned} \frac{v_{n+1}}{u_n} &= 1 + \frac{1}{\left(\frac{u_n}{v_{n-1}}\right)\left(\frac{v_{n-1}}{u_{n-2}}\right)} \\ &= \frac{u_n + u_{n-2}}{u_n}. \end{aligned}$$

Therefore, system (3.15) becomes

$$\begin{aligned} u_{n+2} - v_{n+1} - v_{n-1} &= 0, \\ v_{n+2} - u_{n+1} - u_{n-1} &= 0. \end{aligned} \quad (3.17)$$

That it follows from lemma 3.2.2:

$$\begin{cases} X_{2n+1} = \frac{N_{2n+1} + N_{2n}Y_{-1} + N_{2n+2}X_0Y_{-1}}{N_{2n} + N_{2n-1}Y_{-1} + N_{2n+1}Y_0Y_{-1}}, \\ X_{2n+2} = \frac{N_{2n+2} + N_{2n+1}X_{-1} + N_{2n+3}Y_0X_{-1}}{N_{2n+1} + N_{2n}X_{-1} + N_{2n+2}Y_0X_{-1}}, \\ Y_{2n+1} = \frac{N_{2n+1} + N_{2n}X_{-1} + N_{2n+2}Y_0X_{-1}}{N_{2n} + N_{2n-1}X_{-1} + N_{2n+1}Y_0X_{-1}}, \\ Y_{2n+2} = \frac{N_{2n+2} + N_{2n+1}Y_{-1} + N_{2n+3}X_0Y_{-1}}{N_{2n+1} + N_{2n}Y_{-1} + N_{2n+2}X_0Y_{-1}}. \end{cases} \quad (3.18)$$

Now, by using (3.9)

$$\begin{cases} x_{2n+1} = f^{-1}\left(\frac{N_{2n+1} + N_{2n}g(y_{-1}) + N_{2n+2}f(x_0)g(y_{-1})}{N_{2n} + N_{2n-1}g(y_{-1}) + N_{2n+1}f(x_0)g(y_{-1})}\right), \\ x_{2n+2} = f^{-1}\left(\frac{N_{2n+2} + N_{2n+1}f(x_{-1}) + N_{2n+3}g(y_0)f(x_{-1})}{N_{2n+1} + N_{2n}f(x_{-1}) + N_{2n+2}g(y_0)f(x_{-1})}\right), \\ y_{2n+1} = g^{-1}\left(\frac{N_{2n+1} + N_{2n}f(x_{-1}) + N_{2n+2}g(y_0)f(x_{-1})}{N_{2n} + N_{2n-1}f(x_{-1}) + N_{2n+1}g(y_0)f(x_{-1})}\right), \\ y_{2n+2} = g^{-1}\left(\frac{N_{2n+2} + N_{2n+1}g(y_{-1}) + N_{2n+3}f(x_0)g(y_{-1})}{N_{2n+1} + N_{2n}g(y_{-1}) + N_{2n+2}f(x_0)g(y_{-1})}\right). \end{cases} \quad (3.19)$$

■

For more detail and better understanding, we can reinforce Theorem 3.5.1, we use the previous application.

Example. Let

$$f(t) = jt, \quad g(t) = kt, \quad j, k \in \mathbb{N}_0.$$

where

$$D_f = D_g = R.$$

it is clearly that f and g are "1-1". From the system (3.2) we get:

$$\begin{cases} x_{n+1} = \frac{1}{j} + \frac{1}{kj^2y_nx_{n-1}}, \\ y_{n+1} = \frac{1}{k} + \frac{1}{k^2jx_ny_{n-1}}. \end{cases} \quad (3.20)$$

Then from (3.14) we obtain:

$$\begin{cases} x_{2n+1} = \frac{1}{j} \left(\frac{N_{2n+1} + N_{2n}ky_{-1} + N_{2n+2}jkx_0y_{-1}}{N_{2n} + N_{2n-1}ky_{-1} + N_{2n+1}jkx_0y_{-1}} \right), \\ x_{2n+2} = \frac{1}{j} \left(\frac{N_{2n+2} + N_{2n+1}jx_{-1} + N_{2n+3}jky_0x_{-1}}{N_{2n+1} + N_{2n}jx_{-1} + N_{2n+2}jky_0x_{-1}} \right), \\ y_{2n+1} = \frac{1}{k} \left(\frac{N_{2n+1} + N_{2n}jx_{-1} + N_{2n+2}k jy_0x_{-1}}{N_{2n} + N_{2n-1}jx_{-1} + N_{2n+1}k jy_0x_{-1}} \right), \\ y_{2n+2} = \frac{1}{k} \left(\frac{N_{2n+2} + N_{2n+1}ky_{-1} + N_{2n+3}jkx_0y_{-1}}{N_{2n+1} + N_{2n}ky_{-1} + N_{2n+2}jkx_0y_{-1}} \right). \end{cases}$$

where $\{N_n\}_{n \geq 0}$ is the Narayana sequence.

3.6 Global stability of solutions of system (3.2)

Global stability is one of the most important qualitative properties in the study of systems of difference equations, since it describes the long-term behavior of their solutions. In this section, we study the global asymptotic stability of the real solutions of the system (3.20)

Lemma 3.6.1 *The system (3.20) has a unique real equilibrium point, given by:*

$$\overline{M} = (\overline{x}, \overline{y}) = \left(\frac{\alpha}{j}, \frac{\alpha}{k} \right).$$

Proof. Suppose I and $J \subset \mathbb{R}$, and consider the function

$$f : I^2 \times J^2 \longrightarrow I$$

$$g : J^2 \times I^2 \longrightarrow J$$

Defined by

$$\begin{cases} f(x_n, x_{n-1}, y_n, y_{n-1}) = \frac{1}{j} + \frac{1}{kj^2 y_n x_{n-1}}, \\ g(x_n, x_{n-1}, y_n, y_{n-1}) = \frac{1}{k} + \frac{1}{k^2 j x_n y_{n-1}}. \end{cases} \quad (3.21)$$

We have

$$\begin{cases} \bar{x} = \frac{1}{j} + \frac{1}{kj^2 \bar{y}}, \\ \bar{y} = \frac{1}{k} + \frac{1}{k^2 j \bar{x}}. \end{cases} \quad (3.22)$$

We obtain

$$j^2 k \bar{x}^2 \bar{y} - k j \bar{x} \bar{y} - 1 = 0. \quad (3.23)$$

$$k^2 j \bar{y}^2 \bar{x} - k j \bar{x} \bar{y} - 1 = 0. \quad (3.24)$$

By using (3.23), (3.24), we get

$$\bar{x} \bar{y} = \frac{1}{j^2 k \bar{x} - k j}. \quad (3.25)$$

$$\bar{x} \bar{y} = \frac{1}{k^2 j \bar{y} - k j}. \quad (3.26)$$

We deduce

$$\frac{1}{j^2 k \bar{x} - k j} = \frac{1}{k^2 j \bar{y} - k j} \quad (3.27)$$

$$\bar{x} = \frac{k}{j} \bar{y}. \quad (3.28)$$

We substitute (3.28) in to (3.23) we get

$$k^2 \bar{y}^3 - k^2 \bar{y}^2 - 1 = 0.$$

By

$$r = k\bar{y}. \quad (3.29)$$

We get

$$r^3 - r^2 - 1 = 0.$$

Since, it is the same characteristic equation (1.28)

Then

$$r = \alpha = 1.4657.$$

From (3.6)

$$\bar{M} = (\bar{x}, \bar{y}) = \left(\frac{\alpha}{j}, \frac{\alpha}{k}\right).$$

■

After determining the equilibrium point of system (3.2), we now study its local asymptotic stability by linearizing the system around the equilibrium point $\bar{M} = (\frac{\alpha}{j}, \frac{\alpha}{k})$.

Theorem 3.6.1 *The equilibrium point $\bar{M}$ is locally asymptotically stable.*

Proof. We linearize the system around the equilibrium point $\bar{M}$, can be written as:

$$w_{n+1} = Aw_n, \quad w_n = (x_n, x_{n-1}, y_n, y_{n-1})^T.$$

A Jacobian matrix is given by:

$$\begin{cases} x_{n+1} = \frac{1}{j} + \frac{1}{k^2 j y_n x_{n-1}}, \\ x_n = x_n, \\ y_{n+1} = \frac{1}{k} + \frac{1}{k j^2 x_n y_{n-1}}, \\ y_n = y_n. \end{cases} \quad (3.30)$$

$$A = \begin{pmatrix} 0 & \frac{-1}{\alpha^3} & \frac{-1}{\alpha^3} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{-1}{\alpha^3} & 0 & 0 & \frac{-1}{\alpha^3} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then, the characteristic polynomial of A is

$$P_A(\lambda) = \lambda^4 + \frac{(2\alpha^3 - 1)}{\alpha^6} \lambda^2 + \frac{1}{\alpha^6}.$$

We have $\Delta = -0.071 < 0$, Therefore, the characteristic polynomial admits 4 roots

$$\lambda_{1,2} = 0.159 \pm i0.54,$$

$$\lambda_{3,4} = -0.159 \pm i0.54.$$

We find

$$|\lambda_1| = |\lambda_2| = |\lambda_3| = |\lambda_4| = 0.562 < 1.$$

Then, the equilibrium point is locally asymptotically stable.

■

Theorem 3.6.2 *The equilibrium point $\overline{M}$ is globally asymptotically stable.*

Proof. Let $\{x_n, y_n\}_{n \geq 0}$, be a real solution of the system (3.2), it suffices to prove that $\overline{M} = (\bar{x}, \bar{y})$ is globally attractive.

We have

$$\begin{aligned}
 \lim_{n \rightarrow +\infty} x_{2n+1} &= \lim_{n \rightarrow +\infty} \frac{1}{j} \left(\frac{N_{2n+2} + jx_{-1}N_{2n+1} + k jy_0 x_{-1} N_{2n+3}}{N_{2n+1} + jx_{-1}N_{2n} + k jy_0 x_{-1} N_{2n+2}} \right) \\
 &= \lim_{n \rightarrow +\infty} \frac{1}{j} \left(\frac{N_{2n+2} \left(1 + \frac{N_{2n+1}}{N_{2n+2}} jx_{-1} + \frac{N_{2n+3}}{N_{2n+2}} k jy_0 x_{-1} \right)}{N_{2n+2} \left(\frac{N_{2n+1}}{N_{2n+2}} + \frac{N_{2n+3} - N_{2n+2}}{N_{2n+2}} jx_{-1} + k jy_0 x_{-1} \right)} \right) \\
 &= \lim_{n \rightarrow +\infty} \frac{1}{j} \left(\frac{1 + \frac{1}{\alpha} jx_{-1} + \alpha k jy_0 x_{-1}}{\frac{1}{\alpha} + (\alpha - 1) jx_{-1} + k jy_0 x_{-1}} \right) \\
 &= \lim_{n \rightarrow +\infty} \frac{1}{j} \left(\frac{\alpha \left(\frac{1}{\alpha} + (\alpha - 1) jx_{-1} + k jx_{-1} y_0 \right)}{\frac{1}{\alpha} + (\alpha - 1) jx_{-1} + k jx_{-1} y_0} \right)
 \end{aligned}$$

So

$$\lim_{n \rightarrow +\infty} x_{2n+1} = \frac{\alpha}{j}.$$

By following the same steps, we calculate the following limits.

$$\begin{aligned}
 \lim_{n \rightarrow +\infty} x_{2n+2} &= \lim_{n \rightarrow +\infty} \frac{1}{j} \left(\frac{N_{2n+3} + ky_{-1}N_{2n+2} + k jx_0 y_{-1} N_{2n+4}}{N_{2n+2} + ky_{-1}N_{2n+1} + k jx_0 y_{-1} N_{2n+3}} \right) \\
 &= \lim_{n \rightarrow +\infty} \frac{1}{j} \left(\frac{N_{2n+3} \left(1 + \frac{N_{2n+2}}{N_{2n+3}} ky_{-1} + \frac{N_{2n+4}}{N_{2n+3}} k jx_0 y_{-1} \right)}{N_{2n+3} \left(\frac{N_{2n+2}}{N_{2n+3}} + \left(\frac{N_{2n+4} - N_{2n+3}}{N_{2n+3}} \right) ky_{-1} + k jx_0 y_{-1} \right)} \right) \\
 &= \lim_{n \rightarrow +\infty} \frac{1}{j} \left(\frac{1 + \frac{1}{\alpha} ky_{-1} + \alpha k jx_0 y_{-1}}{\frac{1}{\alpha} + (\alpha - 1) ky_{-1} + k jx_0 y_{-1}} \right) \\
 &= \lim_{n \rightarrow +\infty} \frac{1}{j} \left(\frac{\alpha \left(\frac{1}{\alpha} + (\alpha - 1) ky_{-1} + k jx_0 y_{-1} \right)}{\frac{1}{\alpha} + (\alpha - 1) ky_{-1} + k jx_0 y_{-1}} \right) \\
 &= \frac{\alpha}{j}.
 \end{aligned}$$

So

$$\lim_{n \rightarrow +\infty} x_n = \bar{x}.$$

and

$$\begin{aligned}
 \lim_{n \rightarrow +\infty} y_{2n+1} &= \lim_{n \rightarrow +\infty} \frac{1}{k} \left(\frac{N_{2n+2} + N_{2n+1}ky_{-1} + k j x_0 y_{-1} N_{2n+3}}{N_{2n+1} + N_{2n}ky_{-1} + k j x_0 y_{-1} N_{2n+2}} \right) \\
 &= \lim_{n \rightarrow +\infty} \frac{1}{k} \left(\frac{N_{2n+2}(1 + \frac{N_{2n+1}}{N_{2n+2}}ky_{-1} + \frac{N_{2n+3}}{N_{2n+2}}k j x_0 y_{-1})}{N_{2n+2}(\frac{N_{2n+1}}{N_{2n+2}} + (\frac{N_{2n+3}-N_{2n+2}}{N_{2n+2}})ky_{-1} + k j x_0 y_{-1})} \right) \\
 &= \lim_{n \rightarrow +\infty} \frac{1}{k} \left(\frac{1 + \frac{1}{\alpha}ky_{-1} + \alpha k j x_0 y_{-1}}{\frac{1}{\alpha} + (\alpha - 1)ky_{-1} + k j x_0 y_{-1}} \right) \\
 &= \lim_{n \rightarrow +\infty} \frac{1}{k} \left(\frac{\alpha(\frac{1}{\alpha} + (\alpha - 1)ky_{-1} + k j x_0 y_{-1})}{\frac{1}{\alpha} + (\alpha - 1)ky_{-1} + k j x_0 y_{-1}} \right) \\
 &= \frac{\alpha}{k}.
 \end{aligned}$$

and

$$\begin{aligned}
 \lim_{n \rightarrow +\infty} y_{2n+2} &= \lim_{n \rightarrow +\infty} \frac{1}{k} \left(\frac{N_{2n+3} + j x_{-1} N_{2n+2} + k j y_0 x_{-1} N_{2n+4}}{N_{2n+2} + j x_{-1} N_{2n+1} + k j y_0 x_{-1} N_{2n+3}} \right) \\
 &= \lim_{n \rightarrow +\infty} \frac{1}{k} \left(\frac{N_{2n+3}(1 + \frac{N_{2n+2}}{N_{2n+3}}j x_{-1} + \frac{N_{2n+4}}{N_{2n+3}}k j y_0 x_{-1})}{N_{2n+3}(\frac{N_{2n+2}}{N_{2n+3}} + (\frac{N_{2n+4}-N_{2n+3}}{N_{2n+3}})j x_{-1} + k j y_0 x_{-1})} \right) \\
 &= \lim_{n \rightarrow +\infty} \frac{1}{k} \left(\frac{1 + \frac{1}{\alpha}k x_{-1} + \alpha k j y_0 x_{-1}}{\frac{1}{\alpha} + (\alpha - 1)j x_{-1} + k j y_0 x_{-1}} \right) \\
 &= \lim_{n \rightarrow +\infty} \frac{\alpha(\frac{1}{\alpha} + (\alpha - 1)j x_{-1} + k j y_0 x_{-1})}{\frac{1}{\alpha} + (\alpha - 1)j x_{-1} + k j y_0 x_{-1}} \\
 &= \frac{\alpha}{k}.
 \end{aligned}$$

So

$$\lim_{n \rightarrow +\infty} y_n = \bar{y}.$$

Then

$$\lim_{n \rightarrow +\infty} (x_n, y_n) = (\bar{x}, \bar{y}). \quad (3.31)$$

Thus, the equilibrium point $\bar{M}$ is globally attractive, therefore it is globally asymptotically stable. ■

3.7 Numerical Example

In this section, we provide numerical simulations to illustrate and verify the theoretical results obtained, we consider the following system of difference equations:

$$x_{n+1} = \frac{1}{j} + \frac{1}{kj^2y_nx_{n-1}}, \quad y_{n+1} = \frac{1}{k} + \frac{1}{jk^2x_ny_{n-1}}. \quad (3.32)$$

which is obtained from system (3.2) by taking

$$f(t) = jt, \quad g(t) = kt.$$

The parameteres and the initial conditions are given as follows:

- The parameteres: $j = 5, k = 4$
- Initial conditions : $x_{-2} = 0.2$, $x_{-1} = 0.2$, $x_0 = 0.3$.
- Initial conditions : $y_{-2} = 0.1$, $y_{-1} = 0.2$, $y_0 = 0.15$.

The following table presents the computed values of the sequence (x_n) and (y_n) generated by the difference system with the given initial conditions . The values are displayed for $n = 0$ to $n = 30$.

n	1	2	3	4	5	6	7	8	9	10
x_n	0.2000	0.1000	0.3000	0.8667	0.2727	0.2333	0.3048	0.3059	0.2894	0.2932
y_n	0.1000	0.2000	0.1500	0.4583	0.3462	0.3500	0.4048	0.3672	0.3510	0.3676

n	11	12	13	14	15	16	17	18	19	20
x_n	0.2940	0.2918	0.2930	0.2938	0.2931	0.2929	0.2931	0.2932	0.2931	0.2931
y_n	0.3715	0.3656	0.3653	0.3667	0.3665	0.3663	0.3665	0.3664	0.3663	0.3664

n	21	22	23	24	25	26	27	28	29	30
x_n	0.2931	0.2931	0.2931	0.2931	0.2931	0.2931	0.2931	0.2931	0.2931	0.2931
y_n	0.3664	0.3664	0.3664	0.3664	0.3664	0.3664	0.3664	0.3664	0.3664	0.3664

Table 3.2: Computed values of x_n and y_n for $n = 1$ to 30 from the Difference equation System(3.2)

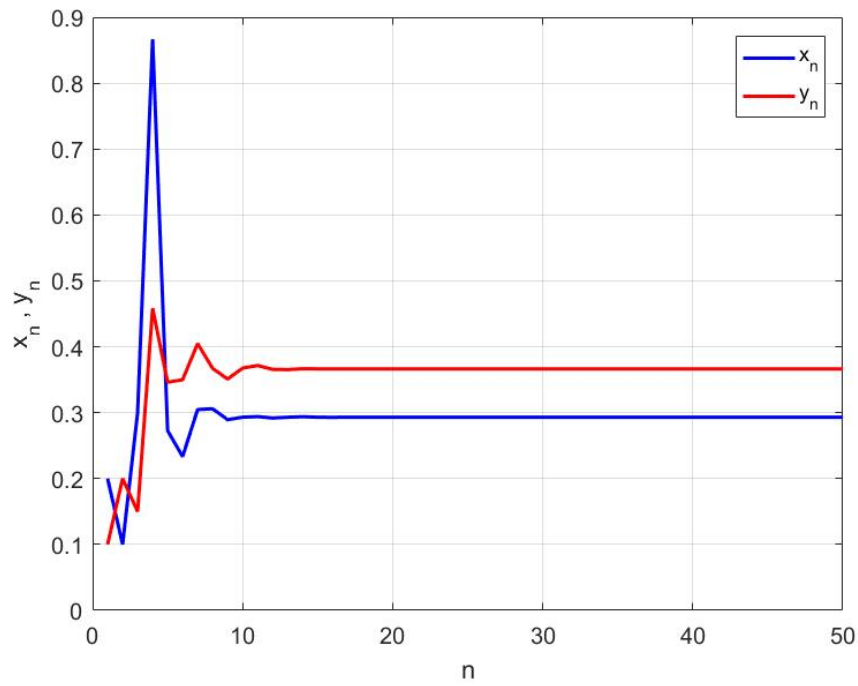


Figure 3.3: Numerical Illustration of the Asymptotic Stability of System (3.32)

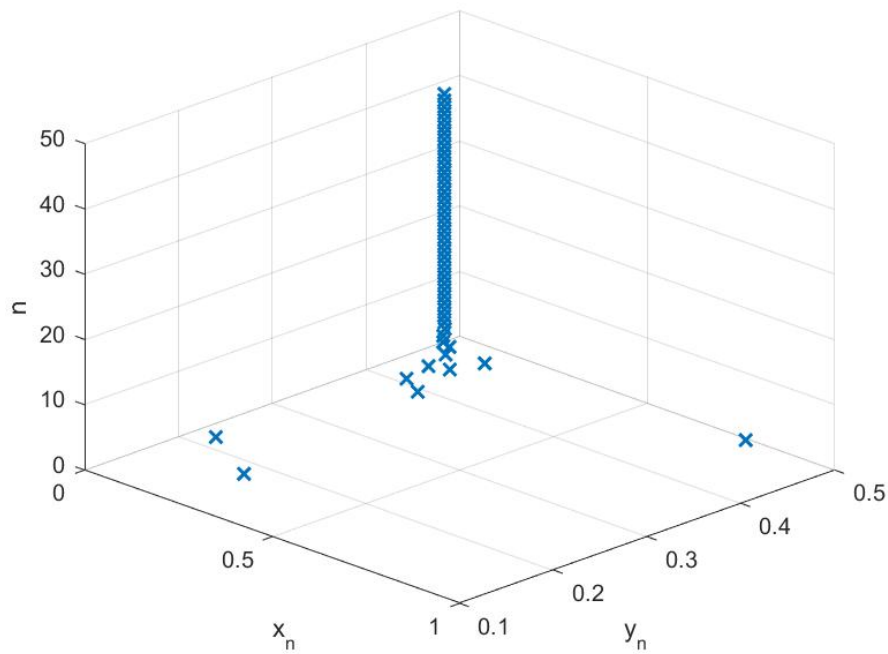


Figure 3.4: 3D phase surface plot of the system (3.32)

3.8 Conclusion

In this chapter, we studied two general systems of difference equations defined by one to one functions f and g , on a set D of real numbers, related to Narayana numbers. By changing variables, we found the explicit formulas for the solutions in terms of Narayana numbers. Let us consider an example involving the function f and g , we found that system (3.12) has a single equilibrium point $(0,0)$ which is unstable. The system (3.20) also possesses a single equilibrium point $(\frac{\alpha}{j}, \frac{\alpha}{k})$, which is locally and globally asymptotically stable. These the theory results were confirmed by using numerical examples.

General Conclusion

In conclusion, this Master's thesis established a strong connection between nonlinear systems of difference equations and well-known integer sequences, particularly the Narayana and Fibonacci sequences. The properties of these sequences played a fundamental role in deriving explicit solution formulas and describing the qualitative behavior of the corresponding nonlinear systems.

Although nonlinear systems of difference equations often appear simple in form, their analysis is generally challenging, and obtaining closed-form solutions is rarely straightforward. In this thesis, several classes of nonlinear systems were investigated. Explicit formulas for their solutions were derived in terms of Narayana numbers, and their qualitative properties, including asymptotic behavior and the stability of equilibrium points, were thoroughly analyzed.

Furthermore, generalized models were studied by employing suitable transformations that reduced the original nonlinear systems to linear recurrence relations associated with the Narayana sequence. The theoretical findings were supported and illustrated through numerical examples, confirming the validity of the obtained results.

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