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Dedication

This thesis is proudly and affectionately dedicated to:

Myself, for the resilience, hard work, and determination to overcome every challenge throughout this journey.

My beloved parents Mousbah and Djama, for their endless love, prayers, and lifelong sacrifices.

My dear brother Houssam, my sisters Khawla and Chaima, and my sister-in-law, for their unwavering support and love.

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Douaa Djouimaa

Dedication

To myself,

I dedicate this achievement in recognition of all the efforts I have made and the challenges I have overcome with determination and perseverance to reach this stage.

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My mother **Loubna** and my father **Saïd** thank you for your love, sacrifices, and unwavering support. It is through your encouragement and prayers that I have reached where I am today.

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Abstract

This work falls within the framework of designing and implementing electronic circuits for chaotic systems, where a Hyperchaotic System of the fifth dimension (5D) characterized by a high degree of dynamic complexity has been proposed. Subsequently, physical representation of the system was achieved by designing an equivalent analog electronic circuit, adopting adaptive synchronization methodology to analyze and adjust its nonlinear behavior. In the final stage, an in-depth comparative study was conducted between the results of numerical simulations obtained using the Python environment, and the results of electronic simulations using Multisim 14.3, where the results showed high compatibility and qualitative match, which confirms the physical feasibility of embodying the proposed system and its practical applicability in secure communications systems.

Keywords: Hyperchaotic System, Circuit Design, Adaptive Synchronization, Secure Communication .

Résumé

Ce travail s'inscrit dans le cadre de la conception et de l'implémentation de circuits électroniques pour des systèmes chaotiques, où un système hyperchaotique de cinquième dimension (5D) caractérisé par un haut degré de complexité dynamique a été proposé. Par la suite, la représentation physique du système a été réalisée en concevant un circuit électronique analogique équivalent, adoptant une méthodologie de synchronisation adaptative pour analyser et ajuster son comportement non linéaire. Dans la dernière étape, une étude comparative approfondie a été menée entre les résultats de simulations numériques obtenues en utilisant l'environnement Python et les résultats de simulations électroniques utilisant Multisim 14.3, où les résultats ont montré une compatibilité élevée et une correspondance qualitative, qui confirme la faisabilité physique de l'incorporation du système proposé et son applicabilité pratique dans des systèmes de communications sécurisés.

Mots-clés: Système hyperchaotique, Conception de circuits, Synchronisation adaptative, Communication sécurisée.

ملخص

يندرج هذا العمل ضمن إطار تصميم وتنفيذ الدارات الإلكترونية للأنظمة الفوضوية، حيث تم اقتراح نظام فائق الفوضى من البعد الخامس (5D) يتميز بدرجة عالية من التعقيد الديناميكي. عقب ذلك، تم تحقيق التمثيل الفيزيائي للنظام عبر تصميم دائرة إلكترونية تناظرية مكافئة، مع اعتماد منهجية المزامنة المتكيفة لتحليل وضبط سلوكه غير الخطي. وفي المرحلة النهائية، أُجريت دراسة مقارنة معمقة بين نتائج المحاكاة العددية المحصل عليها باستخدام بيئة Python، ونتائج المحاكاة الإلكترونية باستخدام Multisim 14.3، حيث أظهرت النتائج توافقًا عاليًا وتطابقًا نوعيًا، مما يؤكد القابلية الفيزيائية لتجسيد النظام المقترح وإمكانية تطبيقه عمليًا في أنظمة الاتصالات الآمنة.

الكلمات الأساسية: نظام فائق الفوضى، تصميم الدارات الإلكترونية، التزامن التكميلي، الاتصال الآمن.

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Introduction

Chaotic systems are an important topic in dynamical systems because of their nonlinear behavior and sensitivity to initial conditions, which makes them difficult to predict. These properties have contributed to expanding their use in many scientific and engineering applications. They are also practically embodied via electronic circuits capable of generating chaotic signals. In the field of information security, these systems are exploited to design effective encryption algorithms. Which makes it an important tool for enhancing data confidentiality and protection.

In 1984, D. Pederson [19], examines the development of computer-aided circuit analysis and design since the 1950s, focusing on the development of circuit simulation within design systems (CAD), the most important advances, and the interaction between the theoretical and applied aspects.

In 2000, Sprott [22] introduced a class of simple chaotic systems based on third-order autonomous differential equations (jerk equations). He showed that piecewise linear nonlinearities are sufficient to produce robust chaotic behavior and allow straightforward electronic implementation using operational amplifiers and diodes.

In 2005, X. Wang [26] was proposed is a secure digital communication system based on chaotic embedding, encryption, and synchronization between chaotic systems. The system relies on sending only one signal between a chaotic sender and receiver over a public channel. Encryption keys are generated by synchronizing the two systems to ensure secure encryption and decryption. Its effectiveness has been verified by numerical simulation and stability has been demonstrated using Lyapunov's theorem.

In 2016, a new 3D chaotic system containing quadratic, cubic and quadrilateral nonlinearities was proposed [28], its dynamic properties were studied, implemented via an electronic circuit, and proven to have complex behavior and be applicable in various en-

gineering and scientific fields.

In 2017, M. Borah and all [5], studies the emergence of chaos in a fractal model of a PMSG generator and proposes a new predictive controller to suppress it and stabilize the system at different rank states. The results are also confirmed by dynamic analysis, simulation and circular implementation consistent with the numerical results.

In 2019, a new 3D chaotic system containing quadratic and quadrilateral nonlinearities was proposed[24], its dynamic properties were studied, stabilized and synchronized using adaptive control, with the model verified theoretically and numerically and implemented via an electronic circuit.

In 2022, a 4D hyperchaotic system was developed based on a 3D system [9], achieved via an electronic circuit and proven to be controllable and synchronous.

Also, K. Benkouider and all [2], a ten-dimensional supercomplex chaotic system capable of generating multiple behaviors such as periodic, chaotic, and hyperchaotic is proposed, demonstrating the property of multistability and coexistence of multiple attractors, and active control laws are designed to achieve synchronization with less distant systems with model verification via an electronic circuit.

-In 2024, a multi-scroll chaotic system was proposed based on a stable linear system with a finite controller [27], which showed complex chaotic behavior that was analyzed mathematically and numerically, with its existence verified via simulation and an electronic circuit.

In 2025, Z. Kohl and all [14], examines the design of third-order Jerk circuits to generate chaotic signals, presenting a systematic method for selecting coefficients rather than trial and error, and transforming the circuit into a simpler, practically executable version with real electronic components.

Moreover, a four-/five-dimensional chaotic system characterized by the presence of a line of equilibrium points was proposed [3], and its dynamic properties such as bifurcation and multistability were studied, with its implementation via an electronic circuit and applied to the FPGA, in addition to using a deep learning model (LSTM) to predict its behavior with high accuracy.

Motivated by the previous study and others, we propose in this work a 5-dimensional hyperchaotic system with higher complexity behavior. When we give an analysis and synchronization, then we propose an electronic circuit design in order to prove the physical existence of the novel system in order to use it in secure communications.

The first chapter, brings together the basic concepts and concepts of the theory of chaotic dynamical systems (fixed point, stability, attractor, and bifurcation), and we also

present the concepts of chaos theory based on the definition of chaos. Next, the properties and scenarios of the transition to chaos are given; thus, a method for detecting chaos (Lyapunov exponent) is given. We will need all of these concepts in the work and in detail in the rest of the memory.

In the **second chapter**, we give the basic concepts of synchronization and control of chaotic dynamical systems, starting from synchronization and its types (complete and adaptive) and then the control and its methods (OGY method, feedback, and adaptive).

In the **third chapter**, we move to a more advanced application stage, where we present the new system that has been adopted, recalling the basic concepts that were addressed in the first chapter, especially with regard to the general characteristics of the system studied. Next, we rely on the theoretical results obtained in chapter 2, which included an analytical study of the system, to design and implement the corresponding electronic circuit. This chapter aims to highlight how to move from a mathematical model to practical application, by representing the system in the form of an electronic circuit and analyzing its behavior, allowing the validity of theoretical results to be verified and the performance of the system to be studied in a realistic environment.

Chaotic Dynamical Systems

1.1 Dynamical System

Definition 1.1. [15]

A dynamical system is a mathematical description of a process that evolves over time, typically represented by a system of n first-order differential equations in space $\mathbb{R}^n$, which determines the temporal behaviour of an evolutionary process.

Let $x = x(t) \in \mathbb{R}^n$, $t \in I \subseteq \mathbb{R}$ be the vector representing the dynamics of a **continuous system** (the ordinary differential equations). The mathematical representation of the system may be written as

$$\dot{x} = \frac{dx}{dt} = f(x, t) \tag{1.1}$$

on the opposite its **discrete system**, An equation is said to be **linear** if the function f is linear. Otherwise, it is called **non-linear**. Also, if its evolution law does not depend on time (the law is then said to be stationary) the system called **autonomous System** and if its evolution law then depends on time, the system is **nonautonomous System**.

Note: The nonautonomous system can be written as a autonomous with variable change $t = x_{n+1}$.

Example: Suppose we have an LCR circuit consisting of a resistor of resistance R , a capacitor of capacitance C , and an inductor of inductance L . In a simple electrical circuit the values of R , C , and L are always nonnegative and are independent of time t . Kirchhoff's current law (the sum of the currents flowing into a node is equal to the sum of the currents flowing out of it) is satisfied if we pass a current I to the closed loop as shown in Fig(1.1).

According to Kirchhoff's voltage law of the circuit (the sum of the potential differences around any closed loop in a circuit is zero), we have the equation

$$V_{12} + V_{23} + V_{31} = 0. \quad (1.2)$$

Here V_{ij} denotes the voltage difference between node i and node j . From Ohm's law, we get the relation

$$V_{31} = IR. \quad (1.3)$$

Also, from the definition of capacitance C , we have

$$C \frac{dV_{12}}{dt} = I. \quad (1.4)$$

Again, Faraday's law of inductance gives

$$L \frac{dI}{dt} = V_{23}. \quad (1.5)$$

Substituting (1.2) and (1.4) into (1.5) and writing $V_{12} = V$, we get

$$V + L \frac{dI}{dt} + IR = 0. \quad (1.6)$$

or,

$$\frac{dI}{dt} = -\frac{R}{L}I - \frac{V}{L}. \quad (1.7)$$

Again, from (1.4)

$$\frac{dV}{dt} = \frac{I}{C}. \quad (1.8)$$

Thus finally we obtain the following equations

$$\frac{dV}{dt} = \frac{I}{C} \quad \text{and} \quad \frac{dI}{dt} = -\frac{R}{L}I - \frac{V}{L}. \quad (1.9)$$

These equations represent a dynamical system of dimension two in the VI plane. This is a simple linear model in LCR circuit. The linear models have undoubtedly had good success, but they also have limitations. Linear models can only produce persistent oscillations of a harmonic (trigonometric) type.

A circuit is called nonlinear when it contains at least one nonlinear circuit element like

a nonlinear resistor, a nonlinear capacitor or a nonlinear inductor. Chua's diode model equation is a simple example of nonlinear electric circuit.

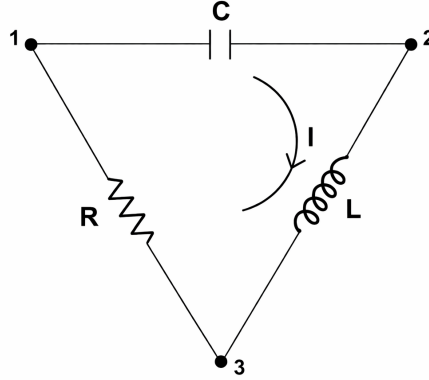


Figure 1.1: Schematic of an RLC circuit.

1.2 Flow and Fixed Point

Consider the autonomous system (1.1), $x \in \mathbb{R}^n$, $f \in C^r(U)$, $U \subset \mathbb{R}^n$

Definition 1.2. [7]

Let $x = x(x_0, t)$, $x_0 \in D$, be a solution of (1.1) with initial condition $x(0) = x_0$. We call the flow, the application

$$\varphi_t : D \rightarrow \mathbb{R}^n \text{ defined by } \varphi_t(x_0) = x(x_0, t).$$

The flow $\varphi_t(x_0)$ has the following properties:

1. $\varphi_t(x_0)$ is of class C^r .
2. $\varphi_0(x_0) = x_0$.
3. $\varphi_{t+s}(x_0) = \varphi_t(\varphi_s(x_0))$.

Definition 1.3. [15] The notion of a fixed point is essential for the study of the local behavior of dynamical systems. A fixed point represents a constant or equilibrium solution of the system. For an autonomous system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n,$$

a point $\bar{x}$ is a fixed point of the flow φ_t if

$$\varphi_t(\bar{x}) = \bar{x} \quad \text{for all } t,$$

which is equivalent to

$$f(\bar{x}) = 0.$$

In non-autonomous systems, a fixed points can be defined over a given time interval. Fixed points are also called equilibrium, critical, or stationary points, and depending on the system, their number may be finite, infinite, or null .

1.2.1 Fixed Point Stability

Definition 1.4. [7]

A fixed point $\bar{x} \in \mathbb{R}^n$ is stable if and only if:

$$\forall \varepsilon > 0, \exists \delta > 0$$

such as

$$\|x(0) - \bar{x}\| < \delta \Rightarrow \|x(t) - \bar{x}\| < \varepsilon,$$

or $\|\cdot\|$ designates the standard norme in $\mathbb{R}^n$. Furthermore, if there exists δ_0 with $0 < \delta_0 < \delta$,

$$\|x(0) - \bar{x}\| < \delta_0 \Rightarrow \lim_{t \rightarrow \infty} x(t) = \bar{x},$$

$\bar{x}$ is asymptotically stable. if $\bar{x}$ is not stable, therefore it is unstable.

Stability in the Lyapunov sense

Definition 1.5. [25] A solution $\phi(t)$ of the system is stable in the Lyapunov sense if for $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for any other given solution $\psi(t)$ of system, for which $|\phi(t_0) - \psi(t_0)| < \delta$, we have $|\phi(t) - \psi(t)| < \varepsilon$, for any $t \geq 0$.

1.2.2 Stability of Linear System

This section describes the stability analysis of a linear system of homogeneous first-order differential equations. The systems with constant coefficients can be written as

$$\dot{x}_i = \sum_{j=1}^n a_{ij}x_j; \quad i = 1, 2, \dots, n \quad (1.10)$$

where a_{ij} ($i, j = 1, 2, \dots, n$) are constants. In matrix notation, (1.10) can be written as

$$\dot{x} = Ax \quad (1.11)$$

where A is an $n \times n$ matrix and $x = (x_1, x_2, \dots, x_n)^T$ is a column vector.

The characteristic equation of (1.11) is

$$\det(A - \lambda I) = 0.$$

Depending upon the roots of the characteristic equation the following cases may arise for stability of solutions of (1.11):

- (i) If all the roots of the characteristic equation of (1.11) have negative real part, then all solutions of ((1.11)) are asymptotically stable. Moreover, the solutions tend to the equilibrium point origin as $t \rightarrow \infty$;
- (ii) If at least one root of the characteristic equation has a positive real part, then all solutions are unstable;
- (iii) If the characteristic equation has simple roots, purely imaginary or zero and the other roots exist and have a negative real part, then all solutions of the system are stable, but not asymptotically.

In case of nonhomogeneous linear systems we prove the following theorem for stability [15].

Theorem 1.1. [15]

The solutions of the nonhomogeneous linear system

$$\dot{x}_i = \sum_{j=1}^n a_{ij}(t)x_j + b_i(t); \quad i = 1, 2, \dots, n$$

are all simultaneously either stable or unstable.

1.2.3 Stability of Non-Linear System

Linearization Methode (Indirect methode)

Theorem 1.2. *(The Hartman—Grobman theorem)[25]*

Let x^* be a hyperbolic fixed point of the diffeomorphism $g : U \rightarrow \mathbb{R}^n$. Then there exist a neighborhood $N \subseteq U$ of the point x^* and a neighborhood $N \subseteq U$ containing the origin such that $g|_N$ is “topologically conjugate” to $Dg(x^*)|_N$.

From the preceding theorems it follows that there are 4^n topological types of hyperbolic fixed points for diffeomorphisms: $g : U \rightarrow \mathbb{R}^n$. They are depicted in the following invariant manifold theorem.

Here, we study the stability properties of equilibrium points of the nonlinear system

$$\dot{x} = f(x). \tag{1.12}$$

The stability of a *hyperbolic equilibrium point* x_0 of system (1.12) is completely determined by the real parts of the eigenvalues λ_j of the Jacobian matrix $Df(x_0)$. In particular, x_0 is *asymptotically stable* if and only if

$$\operatorname{Re}(\lambda_j) < 0, \quad j = 1, \dots, n,$$

that is, if x_0 is a *sink*. A hyperbolic equilibrium point is *unstable* if it is either a source or a saddle.

The following result gives a more precise description of the local behavior near an asymptotically stable equilibrium point.

Theorem 1.3. [20]

Let x_0 be a sink of the nonlinear system (1.12) and suppose that all eigenvalues λ_j of

$Df(x_0)$ satisfy

$$\operatorname{Re}(\lambda_j) < -\alpha < 0.$$

Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in N_\delta(x_0)$, the flow $\varphi_t(x)$ of (1.12) satisfies

$$\|\varphi_t(x) - x_0\| < \varepsilon \quad \text{for all } t \geq 0.$$

Since hyperbolic equilibrium points are either asymptotically stable or unstable, an equilibrium point x_0 of (1.12) can be stable but not asymptotically stable only if $Df(x_0)$ has a zero eigenvalue or a pair of purely imaginary complex-conjugate eigenvalues $\lambda = \pm ib$. In this case, x_0 is said to be *nonhyperbolic*.

The following theorem, provides an additional necessary condition for stability.

Theorem 1.4. [20]

If x_0 is a stable equilibrium point of (1.12), then no eigenvalue of $Df(x_0)$ has a positive real part.

Thus, stable equilibrium points that are not asymptotically stable can occur only at nonhyperbolic equilibrium points. Determining the stability nature of such points is generally a delicate problem.

A powerful approach for addressing this issue is the method introduced by Lyapunov in his 1892 doctoral thesis.

Definition 1.6. Let $E \subset \mathbb{R}^n$ be an open set, let $V \in C^1(E)$, and let φ_t denote the flow of system (1). For $x \in E$, the derivative of $V(x)$ along the trajectory $\varphi_t(x)$ is defined by

$$\dot{V}(x) = \left. \frac{d}{dt} V(\varphi_t(x)) \right|_{t=0} = DV(x)f(x).$$

If $\dot{V}(x) < 0$ in E , then V decreases along trajectories of the system. In $\mathbb{R}^2$, if $V(x) \geq 0$ with equality only at $x = 0$, then for sufficiently small $C > 0$, the level curves $V(x) = C$ form closed curves surrounding the origin, and trajectories cross these curves from the exterior to the interior as time increases. This implies that the origin is asymptotically stable.

Lyapunov's direct method [15]

The Russian mathematician A.M. Lyapunov generalized the stability conditions which are used in analyzing the stability of a system, in particular the stability in the neighborhood of equilibrium points of a system. This is known as Lyapunov's second method or direct method for stability. He nicely introduced a scalar function, $L(x)$ later called it the Lyapunov function, such that $L(\bar{x}) = 0$ and $L(x) > 0$ when $x \neq \bar{x}$ in the neighborhood of the equilibrium point $\bar{x}$ of the system $\dot{x} = f(x)$.

The function $L(x_1, x_2, \dots, x_n)$ is said to be positively (resp. negatively) definite in a domain $D \subset \mathbb{R}^n$ if $L(x) > 0$ (resp. < 0) for all $x \in D$, $x \neq 0$. Similarly, L is called positively (resp. negatively) semi-definite in D if $L(x) \geq 0$ (resp. ≤ 0) for all $x \in D$.

The total derivative or orbital derivative of L in the direction of the vector field $f(x)$ is defined as

$$\frac{dL}{dt} = f \cdot \nabla L = f(x) \frac{\partial L}{\partial x}.$$

Let $D \subset \mathbb{R}^n$ be an open neighborhood of the equilibrium point $\bar{x}$. Then the function $L : D \rightarrow \mathbb{R}$ satisfying the following properties:

- (i) L is continuously differentiable,
- (ii) $L > 0$ for all $x \in D \setminus \{\bar{x}\}$ and $L(\bar{x}) = 0$,

is called a Lyapunov function. Moreover, if

$$\frac{dL}{dt} \leq 0 \text{ in } D,$$

then $\bar{x}$ is stable. This condition implies that the point $x(t)$ moves along a path where $L(x)$ does not increase. Hence, $x(t)$ will remain close to the point $\bar{x}$ and come to $\bar{x}$ if

$$\frac{dL}{dt} = 0.$$

There is no systematic procedure to deduct the Lyapunov function $L(x)$. However, in case of conservative system it L is the energy of the system. In fact, Lyapunov constructed this function on the basis of the principle of energy in mechanics.

Theorem 1.5. (*Lyapunov theorem*)

Suppose that the origin is an equilibrium point of $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ and let $L = L(x_1, x_2, \dots, x_n)$ be a Lyapunov function in a neighborhood D of the origin. If

- (i) *the orbital derivative $\dot{L} \leq 0$ in D , that is, if $\dot{L}$ is negative semi-definite in D , the origin is stable,*
- (ii) *$\dot{L} < 0$ in $D \setminus \{0\}$, that is, if $\dot{L}$ is negative definite in D , then the origin is asymptotically stable,*
- (iii) *$\dot{L} > 0$, that is, positive definite in D , the origin is unstable.*

1.3 Attractor

1.3.1 Definition of Attractor

Definition 1.7 (Attractor – Ruelle 1981). Let A be a closed compact set of the phase-space (or of the flow). It is supposed that A is an invariant set (i.e. $\phi_t(A) = A$, for any t). It is said that A is stable if for any neighborhood U of A , there exists a neighborhood V of A such that any solution $x(x_0, t) = \phi_t(x_0)$ will remain in U if $x_0 \in V$. Moreover:

$$\bigcap_{t \geq 0} \phi_t(V) = A \quad (1.13)$$

and if there exists a dense orbit in A , then A is an attractor.

When A is an attractor, the set $W = \bigcup_{t > 0} \phi_t(V)$ is called “basin of attraction” of A . This set is the set of points whose trajectories asymptotically converge towards A . The simplest attractor is the fixed point. A second type of attractor for a vector field is the limit cycle: it is a closed trajectory which attracts all the close orbits. The third type is often called the strange attractor [25].

1.3.2 Strange Attractor [25]

This attractor was defined by Ruelle and Takens (1971). It is characterized by:

1. In the phase space, the attractor is of null volume,
2. The dimension D of the attractor is fractal (non-integer) $2 < D < n$, where n is the dimension of the phase space.

3. There is “sensitive dependence on initial conditions” (SDIC): two trajectories (of the attractor) initially very close will diverge.

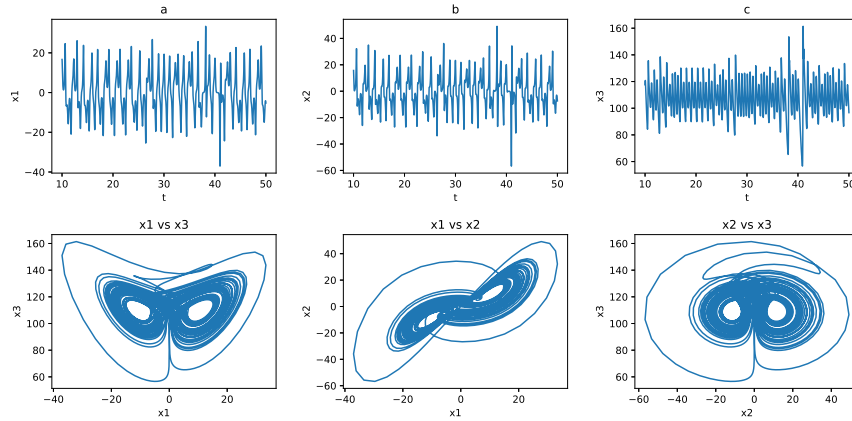


Figure 1.2: Strange attractor in 2D and time evolutions

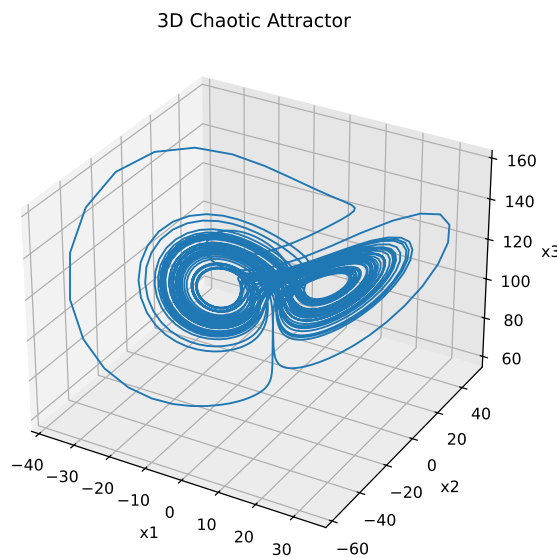


Figure 1.3: 3d Strange attractor of Lorenz

1.3.3 Periodic Orbit

The period of the trajectory $x(t)$, defined for all real t , or equivalently the period of the corresponding orbit, is defined as the smallest positive T such that

$\forall t \in \mathbb{R} :$

$$x(t + T) = x(t).$$

Such a trajectory is called a periodic trajectory or a periodic orbit, notions that we have already encountered without dwelling on them, for example in the case of the harmonic oscillator. A periodic trajectory is also called a cycle [18].

1.4 Bifurcation

Bifurcation is a route to chaos. Loosely speaking a dynamical system is said to be chaotic or irregular or complex if it does not fit any traditional patterns such as monotonic or periodic convergence or divergence, centre or limit cycle and its time series appears to be erratic as if it was a stochastic model although the system is completely deterministic and no random factors are present.[21]

Definition 1.8. A bifurcation is a qualitative change of the system solution x when one modifies the parameter. and more precisely the disappearance or change stability and the emergence of new solutions. Let the system $\dot{x} = f(x, \mu)$ be such that $f \in C^1(U)$, $x \in \mathbb{R}$, $\mu \in \mathbb{R}$. In dimension 1 a non-hyperbolic equilibrium (x^*, μ^*) is characterized by

$$\frac{\partial f}{\partial x}(x^*, \mu^*) = 0.$$

Analysis of bifurcations can usefully be conducted from the Taylor expansion of $f(x, \mu)$ relative to the two variables x and μ :

$$f(x, \mu) = a_{00} + a_{10}\mu + a_{01}x + a_{20}\mu^2 + a_{11}\mu x + a_{02}x^2 + \dots$$

[17].

1.4.1 Local bifurcations in one dimension[15]

a) Saddle-node bifurcation

The saddle-node bifurcation corresponds to $a_{10} = -1$ and $a_{02} = 1$, with all other a_{ij} equal to zero.

We obtain:

$$\dot{x} = x^2 - \mu$$

Then:

- For $\mu < 0$: there is no equilibrium point.
- For $\mu = 0$: 0 is the only equilibrium point and it is unstable.
- For $\mu > 0$: there are two equilibrium points, the stable equilibrium $-\sqrt{\mu}$ and the unstable equilibrium $\sqrt{\mu}$.

Thus, before the bifurcation value, two fixed points appear, one stable and one unstable. Hence, a saddle-node bifurcation occurs.

b) Transcritical bifurcation

The transcritical bifurcation corresponds to $a_{11} = -1$ and $a_{02} = 1$, with all other a_{ij} equal to zero.

We obtain:

$$\dot{x} = x^2 - \mu x$$

Then:

- For $\mu < 0$: there are two equilibrium points, the stable equilibrium μ and the unstable equilibrium 0.
- For $\mu = 0$: 0 is the only equilibrium point and it is unstable.
- For $\mu > 0$: there are two equilibrium points, the stable equilibrium 0 and the unstable equilibrium μ .

Thus, before the bifurcation value, there is an exchange of stability between the two fixed points. After the bifurcation, the stable fixed point becomes unstable and the unstable one becomes stable. Hence, a transcritical bifurcation occurs.

c) Subcritical pitchfork bifurcation

The subcritical pitchfork bifurcation corresponds to $a_{11} = -1$ and $a_{03} = 1$, with all other a_{ij} equal to zero.

We obtain:

$$\dot{x} = x^3 - \mu x$$

Then:

- For $\mu < 0$: 0 is the only equilibrium point and it is unstable.
- For $\mu = 0$: we have $\dot{x} = x^3$, so 0 is an unstable equilibrium point.
- For $\mu > 0$: 0 is a stable equilibrium point and $\pm\sqrt{\mu}$ are two unstable equilibrium points.

Thus, before the bifurcation value there is one unstable fixed point. After the bifurcation, the unstable fixed point becomes stable and two new unstable fixed points appear. Hence, a subcritical pitchfork bifurcation occurs.

d) Supercritical pitchfork bifurcation

The supercritical pitchfork bifurcation corresponds to $a_{11} = 1$ and $a_{03} = -1$, with all other a_{ij} equal to zero.

We obtain:

$$\dot{x} = -x^3 + \mu x$$

Then:

- For $\mu < 0$: 0 is the only equilibrium point and it is stable.
- For $\mu = 0$: we have $\dot{x} = -x^3$, so 0 is an unstable equilibrium point.
- For $\mu > 0$: 0 is an unstable equilibrium point and $\pm\sqrt{\mu}$ are two stable equilibrium points.

Thus, before the bifurcation value there is one stable fixed point. After the bifurcation, the stable fixed point becomes unstable and two stable fixed points are created. Hence, a supercritical pitchfork bifurcation occurs.

We now present a type of bifurcation phenomenon that appears only in dimension two or higher.

1.4.2 Hopf Bifurcation

The Hopf bifurcation occurs when a stable equilibrium point loses its stability and gives rise to a limit cycle, and vice versa. There are two types of Hopf bifurcations, namely, super-critical and sub-critical Hopf bifurcations. We will now illustrate these two types of Hopf Bifurcation [15].

Supercritical Hopf bifurcation:

Consider the following two-dimensional system with a parameter $\mu \in \mathbb{R}$:

$$\begin{cases} \dot{x} = \mu x - y - x(x^2 + y^2) \\ \dot{y} = x + \mu y - y(x^2 + y^2) \end{cases} \quad (1.14)$$

The origin is the unique fixed point of the system. In polar coordinates, the system can be written as:

$$\begin{cases} \dot{r} = \mu r - r^3 \\ \dot{\theta} = 1 \end{cases} \quad (1.15)$$

The phase portraits for $\mu < 0$ and $\mu > 0$ are shown in Figure (1.4).

When $\mu < 0$, the fixed point $r = 0$ is a stable focus and all trajectories are attracted. to this point in the direct direction of the trigonometric circle.

For $\mu = 0$ the origin is always a stable focus.

For $\mu > 0$, the origin is an unstable focus, and in this case, there is a stable limit cycle which corresponds to $r = \sqrt{\mu}$.

Now we are interested in seeing how eigenvalues behave when the parameter varies.

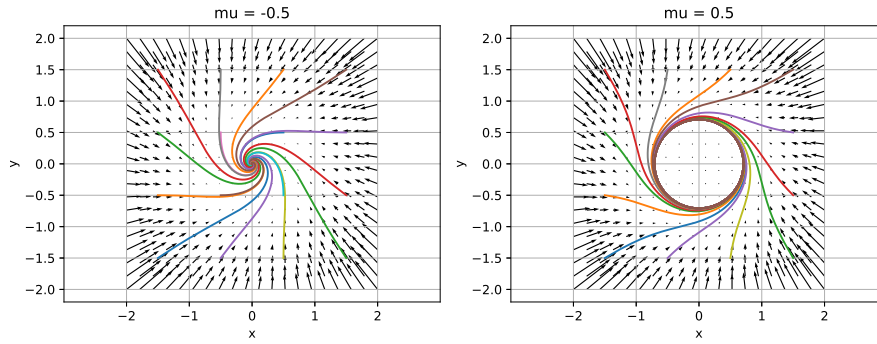


Figure 1.4: Phase portrait for $\mu = -0.5$ on the left and $\mu = 0.5$ on the right.

The original Jacobian matrix is:

$$Df(0,0) = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix} \quad (1.16)$$

which has the eigenvalues $(\mu \pm i)$, so the origin is a stable focus when $\mu < 0$ and an unstable focus when $\mu > 0$.

Therefore, as expected, the eigenvalues cross the imaginary axis from left to right when the parameter changes from negative to positive values.

Thus, we see that a super-critical Hopf bifurcation occurs when a stable focus transforms into an unstable focus surrounded by a limit cycle.

Subcritical Hopf bifurcation:

Consider the following two-dimensional system with a parameter $\mu \in \mathbb{R}$:

$$\begin{cases} \dot{x} = -\mu x - y + x(x^2 + y^2) \\ \dot{y} = x - \mu y + y(x^2 + y^2) \end{cases} \quad (1.17)$$

The system admits the origin $(0,0)$ as its unique equilibrium. The Jacobian matrix is:

$$Df(x,y) = \begin{pmatrix} -\mu + 3x^2 + y^2 & -1 + 2xy \\ 1 + 2x & -\mu + x^2 + 3y^2 \end{pmatrix} \quad (1.18)$$

At the equilibrium point $(0, 0)$ we have:

$$Df(0, 0) = \begin{pmatrix} -\mu & -1 \\ 1 & -\mu \end{pmatrix} \quad (1.19)$$

The eigenvalues of the matrix $Df(0, 0)$ are:

$$\lambda_{1,2} = -\mu \pm i$$

For $\mu < 0$:

$$\Re(\lambda_{1,2}) = -\mu > 0$$

so the equilibrium point $(0, 0)$ is an unstable focus.

For $\mu > 0$:

$$\Re(\lambda_{1,2}) = -\mu < 0$$

so the equilibrium point $(0, 0)$ is a stable focus.

For $\mu = 0$, the system has a pair of purely imaginary complex conjugate eigenvalues.

In polar coordinates, system (1.17) can be written as:

$$\begin{cases} \dot{r} = -\mu r + r^3 \\ \dot{\theta} = 1 \end{cases} \quad (1.20)$$

Thus, for $\mu = 0$ we have $\dot{r} = r^3 > 0$, so the origin is an unstable focus, and in this case there exists an unstable limit cycle corresponding to $r = \sqrt{\mu}$.

We observe that a subcritical Hopf bifurcation occurs since an unstable focus transforms into a stable one surrounded by a limit cycle.

1.5 Detection of Chaos

The term chaos associated with an interval map was first formally introduced by Li and Yorke in 1975, where they established a simple criterion for chaos in one dimensional difference equations (ref. to Li and Yorke 1975)[25].

A) Chaos in the sense of Li–Yorke can be given by:

Definition 1.9. (Chaos in the sense of Li–Yorke.[25]) A continuous map $f : I \rightarrow I$, where I is the unit interval ($I = [0, 1]$), is a chaos in the sense of Li–Yorke if there is an uncountable set $S \subset I$ such that trajectories of any two distinct points x, y in S are proximal and not asymptotic, i.e.

$$\liminf_{n \rightarrow \infty} d(f^n(x), f^n(y)) = 0$$

and

$$\limsup_{n \rightarrow \infty} d(f^n(x), f^n(y)) > 0.$$

The requirement of uncountability of S in this definition (i.e. for continuous maps of the interval, but not in a general compact metric space) is equivalent to the condition that S contains two points, or that S is a perfect set (i.e. nonempty, compact and without isolated points).

B)Chaos in the sense of Devaney (1989):

Definition 1.10. (Devaney’s definition of chaos– 1989.[25]) Let V be a set. A continuous map $f : V \rightarrow V$ is said to be chaotic on V if

1. **Topological transitivity:** for any pair of open non-empty sets $U, W \subset V$ there exists a $k > 0$ such that $f^k(U) \cap W \neq \emptyset$.
2. **Density of periodic points:** of f are dense in V .
3. **Sensitive dependence on initial conditions:** there exists a $\delta > 0$ such that, for any $x \in V$ and any neighborhood N of x , there exists a $y \in N$ and an $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$.

1.5.1 Characteristics of Chaos [23]

the characteristics of chaotic systems are summarized as follows:

1. Extremely sensitive to initial conditions. Sensitivity to the initial value is a typical characteristic of chaotic systems. With a small change in the initial value, the system’s state variables will change significantly after a period of evolution. As the system continues to evolve, it will appear greater changes. This property indicates that the state of a chaotic system is difficult to predict, and its trajectory is challenging to estimate after running for a long time from an initial condition. Therefore, the field of spread spectrum communication is to use the sensitivity of chaotic signals to improve the system’s security.

2. Internal randomness. Chaos is an uncertain behavior produced by a deterministic nonlinear system. It has intrinsic randomness and has nothing to do with external factors of the system. Although the equation of the nonlinear system is deterministic, the dynamic behavior generated by it is difficult to determine. In any region of its attractors, the probability distribution density function is not zero, which is the randomness generated by the deterministic nonlinear system. In fact, the unpredictability of chaos and the extreme sensitivity to initial values lead to the internal randomness of chaotic systems, which also shows that the chaotic system is locally unstable.
3. Determinacy. The determinism of chaotic systems means that the equations describing chaotic systems are deterministic. That is, the chaotic phenomenon is produced by the deterministic nonlinear system, which has the characteristics of randomness and is unpredictable. Hence, the chaotic system is the unity of determinism and randomness.
4. Boundedness. The evolutionary trajectory of chaos is limited to a specific region called the chaotic attraction region. No matter how unstable the chaotic system is, its evolution trajectory will never exceed that region. The region is deterministic and bounded, called boundedness, so the whole chaotic system is stable.
5. Ergodicity. The chaotic variables generated by the chaotic system are ergodic in the domain of attraction. That is to say, the chaotic orbit will experience each state point in the chaotic attraction domain in a limited time, and the orbit will not appear repetitively, which is called ergodicity.
6. Dimensionality. The motion trajectory of chaotic systems is the result of infinite stretching and folding. The motion is carried out in a finite interval, with rich levels of motion state and similar structure. Unlike the motion generated by a general linear system, this infinite number of stretching and folding can only be represented by a fractional dimension. The fractal dimension also depicts the self-similarity of the motion form of chaotic systems.
7. Universality. Universality refers to some common characteristics of different nonlinear systems when they tend to a chaotic state. It does not change according to specific system equations or system parameters, manifested as universal constants of chaotic systems, such as the Feigenbaum constant. Universality is an embodiment of the inherent law of chaos.

8. **Positive Lyapunov exponent.** It refers to the quantitative description of the overall effect of mutual approximation or separation of motion trajectories generated by nonlinear systems. A positive Lyapunov exponent means that adjacent orbits are separated exponentially. In addition, the positive Lyapunov exponent also indicates the loss of information of adjacent points. The larger the Lyapunov exponent is, the more serious the loss of information and implies a higher degree of chaos.

1.5.2 Transition to Chaos Scenario [25]

We know that there exist different kinds of transition to reach chaos known as canonical:

1. **Intermittency route to chaos:** is a development in experimental systems where behaviour appears as quasi-periodic motion interrupted by occasional irregular bursts, and the time between these bursts is statistically distributed even though the system is completely deterministic. As the control parameter is moved away from the periodic domain, the bursts become more frequent until the system becomes completely chaotic.
2. **Period-doubling:** in unidimensional or multidimensional models by means of control parameter(s) change (SDP: sensitive dependence on parameters) (see Schuster 1989, p.63). In the case of two control parameters, the parameter(s) change is related to the codimension notion.
3. **Quasiperiodic route to chaos:** is a fundamental intermediate step in the transition to chaos (see Ruelle and Takens 1971; Landau 1944; Newhouse, Ruelle and Takens 1978; Lorentz 1963). This approach dates back to Landau's work in Physics. Then, Ruelle-Takens have considered (unlike Landau) that a small number of bifurcations was sufficient to produce turbulent phenomena and chaos.

1.6 Lyapunov Exponent Method

Alexander Lyapunov designed a measure to evaluate the gap between trajectories that are initially close, this measure is called "Lyapunov exponent" and is frequently used to judge whether a system has a chaotic character.

1.6.1 Case of a one-dimensional [15]

Consider a one-dimensional function $x_{n+1} = f(x_n)$, where x_0 and $x_0 + \varepsilon$ are two close initial conditions, with ε representing a very small error. We analyze the behavior of the two trajectories that pass through x_0 and $x_0 + \varepsilon$. It is assumed that there is an average exponential divergence, that is to say, there exists a real number λ such that, after N iterations, we have:

$$f^N(x_0 + \varepsilon) - f^N(x_0) \approx \varepsilon e^{N\lambda}$$

Then:

$$\lambda = \frac{1}{N} \ln \left| \frac{f^N(x_0 + \varepsilon) - f^N(x_0)}{\varepsilon} \right|$$

As $N \rightarrow +\infty$, we obtain:

$$\lambda = \lim_{N \rightarrow +\infty} \frac{1}{N} \ln \left| \frac{f^N(x_0 + \varepsilon) - f^N(x_0)}{\varepsilon} \right|$$

For $\varepsilon \rightarrow 0$:

$$\lambda = \lim_{N \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \frac{1}{N} \ln \left| \frac{f^N(x_0 + \varepsilon) - f^N(x_0)}{\varepsilon} \right|$$

Thus:

$$\lambda = \lim_{N \rightarrow +\infty} \frac{1}{N} \ln \left| \frac{df^N(x_0)}{dx_0} \right|$$

Finally, we have:

$$\lambda = \lim_{N \rightarrow +\infty} \frac{1}{N} \ln |(f^N)'(x_0)|$$

Using $x_{n+1} = f(x_n)$, for all $n \in \mathbb{N}$, we obtain:

$$f^N(x_0) = f(f^{N-1}(x_0))$$

By successive application, we get:

$$\begin{aligned} f^N(x_0) &= f'(f^{N-1}(x_0)) \cdot f^{N-1}(x_0) \\ &= f'(f^{N-1}(x_0)) \cdot f'(f^{N-2}(x_0)) \cdot f^{N-2}(x_0) \\ &\quad \dots \\ &= f'(f^{N-1}(x_0)) \cdot f'(f^{N-2}(x_0)) \cdot f'(f^{N-3}(x_0)) \cdots f'(f(x_0)) \cdot f'(x_0) \end{aligned}$$

That is:

$$(f^N)'(x_0) = \underbrace{f'(f^{N-1}(x_0))}_{x_{N-1}} \cdot \underbrace{f'(f^{N-2}(x_0))}_{x_{N-2}} \cdot \underbrace{f'(f^{N-3}(x_0))}_{x_{N-3}} \cdots \underbrace{f'(f(x_0))}_{x_1} \cdot f'(x_0)$$

Then:

$$(f^N)'(x_0) = \prod_{i=0}^{N-1} f'(x_i) \quad (1.21)$$

Substituting (1.21) into (1.22), we obtain:

$$\lambda = \lim_{N \rightarrow +\infty} \frac{1}{N} \ln \left| \prod_{i=0}^{N-1} f'(x_i) \right| \quad (1.22)$$

Finally:

$$\lambda = \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{i=0}^{N-1} \ln |f'(x_i)| \quad (1.23)$$

1.6.2 Case of n-dimensional [15]

Consider an n -dimensional dynamical system defined by:

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad x_{n+1} = f(x_n), \quad x_n \in \mathbb{R}^n \quad (1.24)$$

Let x_0 be a point in the phase portrait of system (1.24), and let $v(x_0)$ be a small neighborhood of this point. Consider another point $x_0 + \Delta x_0$ within this neighborhood, where Δx_0 represents the initial separation between the two nearby points. These two points generate two distinct trajectories of the system.

Denote the successive points of the two trajectories by:

$$x_0, x_1, x_2, \dots, x_N, \dots$$

and

$$x_0 + \Delta x_0, x_1 + \Delta x_1, x_2 + \Delta x_2, \dots, x_N + \Delta x_N, \dots$$

where:

$$x_i = f(x_{i-1}), \quad x_i + \Delta x_i = f(x_{i-1} + \Delta x_{i-1}), \quad i = 1, 2, \dots$$

Thus, the separations evolve as:

$$\begin{aligned}
 \Delta x_1 &= f(x_0 + \Delta x_0) - f(x_0) \approx Df(x_0) \Delta x_0 \\
 \Delta x_2 &= f(x_1 + \Delta x_1) - f(x_1) \approx Df(x_1) \Delta x_1 \\
 \Delta x_3 &= f(x_2 + \Delta x_2) - f(x_2) \approx Df(x_2) \Delta x_2 \\
 &\vdots \\
 \Delta x_N &= f(x_{N-1} + \Delta x_{N-1}) - f(x_{N-1}) \approx Df(x_{N-1}) \Delta x_{N-1}
 \end{aligned}$$

where $Df(x_i)$ is the $n \times n$ Jacobian matrix of f at x_i .

Therefore:

$$\Delta x_N = Df(x_{N-1}) \cdots Df(x_1) Df(x_0) \Delta x_0 = D_N \Delta x_0 \quad (1.25)$$

where

$$D_N = Df(x_{N-1}) \cdots Df(x_1) Df(x_0)$$

is an $n \times n$ matrix depending on N and x_0 .

Thus, the separation at iteration N is fully determined by D_N and Δx_0 .

Since the phase space is multidimensional, we introduce the eigenvectors of D_N as a natural basis.

The eigenvectors $e_i(N)$ satisfy:

$$D_N e_i(N) = v_i(N) e_i(N), \quad i = 1, 2, \dots, n \quad (1.26)$$

The eigenvalues $v_i(N)$ are obtained from:

$$\det(D_N - v(N)I) = 0 \quad (1.27)$$

where I is the identity matrix of order n .

Assume:

$$\Delta x_0 = \sum_{i=1}^n c_i e_i(N) \quad (1.28)$$

Substituting into (1.25):

$$\Delta x_N = \sum_{i=1}^n c_i v_i(N) e_i(N) \quad (2.10)$$

Thus, the separation along the i -th direction is governed by $v_i(N)$.

For large N :

$$|v_i(N)| \sim e^{\lambda_i N}$$

Hence:

$$\lambda_i = \lim_{N \rightarrow \infty} \frac{1}{N} \ln (|v_i(N)|), \quad i = 1, 2, \dots, n \quad (1.29)$$

For a multidimensional system:

- Equation (1.29) gives an estimate of the Lyapunov exponents.
- The number of Lyapunov exponents equals the dimension of the phase space.
- Lyapunov exponents can be positive, negative, or zero. A positive exponent indicates chaotic behavior.

1.7 Kaplan–Yorke Dimension [11]

Let j_0 be a positive integer such that:

$$\sum_{i=1}^{j_0} \lambda_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{j_0+1} \lambda_i < 0. \quad (1.30)$$

We then define the Kaplan–Yorke dimension by:

$$D_{KY} = j_0 + \frac{\sum_{i=1}^{j_0} \lambda_i}{|\lambda_{j_0+1}|}. \quad (1.31)$$

Synchronization and Control of Chaotic Dynamical Systems

Introduction

In this chapter, we first introduce the concept of synchronization of chaotic dynamical systems and present its main types. Then, we discuss the fundamental principles of chaos control and the most commonly used control techniques. Special attention is given to feedback control methods and adaptive control approaches. The main objective of this chapter is to provide a theoretical framework for the synchronization and control of chaotic systems, which is essential for practical applications and for the circuit implementation presented in the following chapter. Finally, this chapter is organized as follows: we begin with the definition of synchronization and some types, Then, we present specific control techniques, including feedback and adaptive methods.

2.1 Synchronization of Chaotic Dynamical System

Definition 2.1. [23] Consider two nonlinear systems as follows:

$$\dot{x} = F(t, x), \tag{2.1}$$

$$\dot{y} = F'(t, y) + U(t, x, y), \quad (2.2)$$

where state variables $x, y \in \mathbb{R}^n$, and $F, F' : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are nonlinear maps. $U : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the synchronization control variable of chaotic system. $\mathbb{R}^+$ is the nonnegative real number set.

If we have $D(t_0) \subseteq \mathbb{R}^n$, $\forall x_0, y_0 \in D(t_0)$, and

$$\lim_{t \rightarrow \infty} (x(t; t_0, x_0) - y(t; t_0, y_0)) = 0, \quad (2.3)$$

then systems (2.1) and (2.2) are synchronous. We call system (2.1) the drive system, and system (2.2) the response system, and $D(t_0)$ is the synchronization region. If $D(t_0)$ belongs to the whole space $\mathbb{R}^n$, then the synchronization is defined as global (complete) synchronization. If $D(t_0)$ is a subset of $\mathbb{R}^n$, then the synchronization is defined as local (partial) synchronization. If $F = F'$, then it is called identical synchronization. If we set the error vector $e = x - y$, then the synchronization error system of systems (2.1) and (2.2) is

$$\dot{e} = F(t, x) - F'(t, y) - U(t, x, y) = G(t, e, x), \quad (2.4)$$

Similarly, based on the stability of error system (2.4) at the origin, the stability conditions of the system are derived.

In the study of chaos synchronization, the stability of the synchronization system is determined mainly by the Lyapunov stability principle or by the condition Lyapunov exponent (CLE) criterion proposed by Pecora and Carroll. The former is the main criterion, and the latter is a necessary condition for synchronization. But the linear stability theorem shows that the necessary condition is also a sufficient condition for most dynamical systems, and it is used as a criterion during synchronization research.

The above synchronization definition is suitable for synchronization of not only chaotic systems, but also nonchaotic systems. It is suitable for both autonomous and nonautonomous systems, and it can be extended to hyperchaotic systems and spatiotemporal chaotic systems [23].

2.1.1 Some Types of Synchronization

2.1.1.1 Complete Synchronization [16]

Pecora and Carroll consider chaotic systems of the form:

$$\dot{x} = f(x), \quad (2.5)$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a nonlinear function.

The system is decomposed into two subsystems: a driver system and a response system. The driver system is given by:

$$\dot{x} = d(x(t)), \quad (\text{driver}) \quad (2.6)$$

and the response system is defined as:

$$\dot{y} = r(y(t), x(t)), \quad (\text{response}) \quad (2.7)$$

where $x \in \mathbb{R}^k$, $y \in \mathbb{R}^m$, and $k + m = n$.

The vector $x(t)$ represents the driving signal. Some of the outputs of the driver system are used to drive the response system, allowing the two systems to achieve synchronization under appropriate conditions.

2.1.1.2 Adaptive Synchronization [23]

Consider an autonomous system

$$\dot{x} = f(x, \omega). \quad (2.8)$$

where the state variable $x \in \mathbb{R}^n$, the system parameter $\omega \in \mathbb{R}^m$, and nonlinear function $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$.

The given system that generates the desired orbit is

$$\dot{y} = f(y, \omega). \quad (2.9)$$

where $y \in \mathbb{R}^n$, $\omega \in \mathbb{R}^m$. The variable y starts from the initial and evolves with the known parameter ω^* , and one orbit $O(y)$ is obtained, which is called as a desired orbit or reference

orbit. System (2.9) that produces the orbit is called a target system or a reference system. The adaptive control synchronization method is to introduce a control mechanism for the parameter ω in system (2.8), so that the orbit $O(x)$ of system (2.8) from any initial value follows the orbit $O(y)$ of the target system. System (2.8) is called a controlled system or response system. If $\lim(y - x) = 0$, then the controlled system and the target system are synchronized.

The parameter ω_j in system (2.8) can be obtained by introducing a certain amount of perturbation to modify the evolution of its orbit, which can make the $O(x)$ and $O(y)$ realize the synchronization. For parameter ω_j , the control law is

$$\dot{\omega}_j = -\rho h(x_i - y_i) \operatorname{sgn}\left(\frac{\partial f_i}{\partial \omega_j}\right) - \delta g(\omega_j - \omega_j^*). \quad (2.10)$$

where $i = 1, 2, \dots, L$, $L \leq n$, $j = 1, 2, \dots, k$, $k \leq m$. ω_j^* is the parameter value to generate the desired orbit. $h(\cdot, \cdot)$ and $g(\cdot)$ are control laws, respectively, and ρ, δ are positive control constants. When the control law is selected, by changing ρ and δ , the combined system from systems (2.10) and (2.8) is

$$\begin{cases} \dot{x} = f(x, \omega), \\ \dot{\omega}_j = -\rho h(x_i - y_i) \operatorname{sgn}\left(\frac{\partial f_i}{\partial \omega_j}\right) - \delta g(\omega_j - \omega_j^*). \end{cases} \quad (2.11)$$

If all the CLEs are less than zero, the synchronization between the two autonomous chaotic systems is realized.

2.2 Control of Chaotic Dynamical System

Definition 2.2. [4]

It is the process of directing or modifying the behavior of a continuous dynamic system (described by nonlinear differential equations) that exhibits chaotic behavior (Chaos), by applying external influences or control inputs that are studied and of specific value.

The primary goal of this control is to suppress random and unexpected chaotic disturbance, divert the system toward regular and periodic behavior, or stabilize it completely at a stable equilibrium state that was originally unstable, with the least possible control effort or cost.

2.2.1 Some Methods of Control

This part presents a unified development of the parameter-dependent approach within a relatively general framework. The control strategy for a nonlinear dynamical system consists in stabilizing one of its unstable periodic orbits that is embedded within a chaotic attractor. This stabilization is achieved through small, time-dependent perturbations of an accessible system parameter. The underlying idea is based on the observation that a chaotic attractor typically contains a dense set of unstable periodic orbits. From a heuristic standpoint, it follows that one of these orbits can be selected and stabilized by applying carefully designed, continuously adjusted small perturbations. These perturbations act so as to guide the system trajectory toward the desired (generally unstable) periodic orbit and maintain it in its vicinity.

2.2.1.1 The OGY Chaos Control Method [6]

We consider a continuous-time autonomous dynamical system

$$\dot{x} = f(x(t), p) \quad (2.12)$$

where p is an adjustable scalar parameter. It is assumed that for $p = p^*$, the system exhibits chaotic behavior. The objective is to stabilize an unstable saddle-type periodic orbit embedded within the chaotic attractor by applying small parameter perturbations. A small neighborhood around p^* is defined such that the chaotic attractor and the target periodic orbit persist (i.e., no bifurcation occurs). The system is analyzed through a Poincaré map P , reducing the continuous-time dynamics to a discrete map on a surface of cross section. The selected unstable periodic orbit satisfies

$$\xi_f^* = P(\xi_f^*, p^*). \quad (2.13)$$

To study the local behavior near this orbit, the map is linearized, leading to

$$\Delta \xi_{k+1} \approx J_k \Delta \xi_k + W_k \Delta p_k \quad (2.14)$$

where J_k is the Jacobian matrix and W_k represents the parameter sensitivity. The Jacobian has stable and unstable eigenvalues ($\lambda_s < 1 < \lambda_u$), defining stable and unstable manifolds. The key idea of the OGY method is to allow the chaotic trajectory naturally

approaches the target orbit, then apply a small parameter perturbation Δp_k that forces the next iterate onto the local stable manifold. The control law is derived by imposing orthogonality to the unstable direction, resulting in

$$\Delta p_k = -\frac{\lambda_u g_u^T \Delta \xi_k}{g_u^T W_k}. \quad (2.15)$$

The perturbation is applied only if it remains within the allowable bound. If the trajectory lies on the stable manifold, no perturbation is applied. Because chaotic systems are ergodic and contain dense unstable periodic orbits, the trajectory will repeatedly return near the target, allowing iterative application of the control until stabilization is achieved.

Unlike classical control approaches, the OGY method exploits intrinsic properties of chaos such as ergodicity and structural stability to achieve stabilization using minimal parameter variation.

2.2.1.2 Feedback Control of Chaotic Systems [4]

[23] Feedback control is one of the fundamental approaches used to stabilize unstable equilibrium points in chaotic dynamical systems. The main idea consists of adding a control input that depends on the current state of the system in order to transform an unstable chaotic behavior into a stable one.

Mathematical Formulation

Consider a nonlinear dynamical system described by

$$\dot{x} = f(x), \quad (2.16)$$

where $x \in \mathbb{R}^n$ is the state vector and $f(x)$ is a nonlinear vector function.

Let x^* be an equilibrium point satisfying

$$f(x^*) = 0. \quad (2.17)$$

In chaotic systems, such equilibrium points are typically unstable.

Feedback Control Law

To stabilize the equilibrium point, a control input u is introduced:

$$\dot{x} = f(x) + u. \quad (2.18)$$

The control law is chosen as a state feedback of the form

$$u = -K(x - x^*), \quad (2.19)$$

where K is the gain matrix and $(x - x^*)$ represents the state error.

Linearization Around the Equilibrium

To analyze stability, the system is linearized around the equilibrium point x^* . Define the error vector as

$$e = x - x^*. \quad (2.20)$$

Using first-order Taylor expansion around x^* , the linearized system becomes

$$\dot{e} = Ae, \quad (2.21)$$

where

$$A = \left. \frac{\partial f}{\partial x} \right|_{x=x^*} \quad (2.22)$$

is the Jacobian matrix evaluated at the equilibrium point.

Closed-Loop System

After applying the feedback control law, the closed-loop system is given by

$$\dot{e} = Ae - Ke, \quad (2.23)$$

which can be written as

$$\dot{e} = (A - K)e. \quad (2.24)$$

Thus, the stability of the controlled system depends on the matrix $(A - K)$.

Stability Condition

The closed-loop system is asymptotically stable if all eigenvalues of the matrix $(A - K)$ have negative real parts:

$$\text{Re}(\lambda_i(A - K)) < 0. \quad (2.25)$$

Under this condition, the error $e(t)$ converges to zero as $t \rightarrow \infty$, ensuring that the state $x(t)$ converges to the desired equilibrium point x^* .

Selection of the Gain Matrix

The gain matrix K can be determined using several approaches, including:

- Pole placement technique,
- Linear Quadratic Regulator (LQR) method,
- Direct gain selection in simple systems (e.g., $K = kI$).

Physical Interpretation

From a physical perspective, feedback control acts as a corrective mechanism. Whenever the system state deviates from the equilibrium point, the control input generates a restoring action proportional to the deviation, driving the system back toward the desired state.

In circuit implementation, this mechanism can be realized using operational amplifiers that multiply the error signal by an appropriate gain.

2.2.1.3 The adaptive method for control of chaos [4]

The basic idea:

Many alternative approaches to the OGY method have been proposed for the stabilization of the unstable periodic orbits (UPO) of a chaotic dynamics. In general the strategies for the control of chaos can be classified into two main classes, namely: closed loop or feedback methods and open loop or non feedback methods.

The first class includes those methods which select the perturbation based upon a knowledge of the state of the system, and oriented to control a prescribed dynamics. Among them, we here recall (besides OGY) the so called occasional proportional feedback (OPF) simultaneously introduced by Hunt and Showalter, the method of Huebler, and the method introduced by Pyragas, which apply a delayed feedback on one of the system variables. All these methods are model independent, in the sense that the knowledge on the system necessary to select the perturbation can be done by simply observing the system for a suitable learning time.

The second class includes those strategies which consider the effect of external perturbations (independent on the knowledge of the actual dynamical state) on the evolution of the system. Periodic or stochastic perturbations have been seen to produce drastic changes in the dynamics of chaotic systems, leading eventually to the stabilization of some periodic behavior. These approaches, however, are in general limited by the fact that their action is not goal oriented, i.e. the final periodic state cannot be decided by the operator.

In what follows, we will try to summarize one of the possible closed-loop approach for the stabilization of UPO. Although it belongs to the same class of OGY, it can be considered as alternative, insofar as it consists in perturbing a state variable of the system, instead of a control parameter. This may result preferable in all cases in which the control parameters are strongly influenced by the environmental conditions, and a variation of

them is not easy to be performed.

The algorithm for adaptive chaos control: [4]

In the following we will then discuss the general problem of forcing a dynamical system

$$\dot{x} = F(x, k) \quad (2.26)$$

toward a desired goal dynamics $u(t)$, by the use of a feedback perturbation $x_i(t)$ acting on one of its state variables. Here dot denotes temporal derivative, k is a vector of control parameters, $x(t)$, $(x_1(t), x_2(t), \dots, x_m(t))$ is a m -dimensional vector of the state variables, F is a suitable nonlinear function, and $u(t)$, $(g_1(t), g_2(t), \dots, g_m(t))$ is the m -dimensional desired dynamics.

The first attempt to solve this problem was provided by Pyragas, who introduced an additive feedback perturbation in the equation for the observable variable x_1 given by

$$x_i(t) = K(g_1(t) - x_1(t)) = K(x_1(t - T) - x_1(t)) \quad (2.27)$$

The adaptive algorithm can be summarized as follows. At each time t_n at which the perturbation must be applied, one measures the distance $d(t_n)$ between the dynamics of the observable $x_1(t_n)$ and the goal dynamics $g_1(t_n)$

$$d(t_n) = x_1(t_n) - g_1(t_n) \quad (2.28)$$

Then, one evaluates the local variation rate

$$j(t_n) = \log_K \left(\frac{d(t_n)}{d(t_{n-1})} \right) \quad (2.29)$$

and selects the perturbation as

$$\xi(t_n) = K(t_n)(g_1(t_n) - x_1(t_n)) \quad (2.30)$$

where

$$\frac{1}{K(t_n)} = \frac{1}{K_0} (1 - \tanh(p j(t_n))), \quad p > 0, K_0 > 0 \quad (2.31)$$

A further remark is in order. Equation can be rewritten as

$$K(t_n) = \frac{1}{q(t_n)}, q(t_n) \neq 0 \quad (2.32)$$

$$q(t_n) = q_0 (1 - \tanh(p j(t_n))) \quad (2.33)$$

Since the perturbation is additive on the right-hand side of the model equation, $q(t_n)$ and q_0 must be time intervals. In that case, the evolution equation for $q(t_n)$ was

$$q(t_n) = q(t_{n-1}) (1 - \tanh(p j(t_n))) \quad (2.34)$$

that is, substituting in Eq (2.33) a fixed reference time interval q_0 with a recursive search in the time intervals. The obtained irregular sequence of time intervals was used to extract suitable chaotic indicators, such as the periods of the UPOs and a discriminator between chaos and stochasticity.

It is important to remark that the recursive formula implies a direct choice of the optimal q in Eq (2.33), thus of the optimal K in Eq (2.31), while the selection of a fixed reference time interval q_0 implies an overconstraint to the search for the optimal perturbation. However, the former mechanism can be only applied when the switch on of the process is done close to the UPO to be controlled. In practice, this would imply a preliminary learning task for an initial targeting procedure. We will come back to this point in the following of the present Report.

When, instead, one wants to switch on the control on an arbitrarily selected region of the attractor, then the latter strategy would be preferable, insofar as the former one can have limitations due to initial large fluctuation of the q_n which can eventually drive the q sequence away from the controllability range. This, of course, can be cured by a reliable choice of the sensitivity, but the whole process may result quite complicated to implement. This is the main reason for which we will focus on the latter strategy, which, though being less complete, has the merit of being easily implementable.

2.3 Example: Complete Synchronization Using Feedback Control

Consider the drive (master) Lorenz system given by:

$$\dot{x}_1 = x_2 - x_1 \quad (2.35)$$

$$\dot{x}_2 = rx_1 - x_2 - x_1x_3 \quad (2.36)$$

$$\dot{x}_3 = x_1x_2 - bx_3 \quad (2.37)$$

The response (slave) system is defined as:

$$\dot{y}_2 = x_1y_3 + rx_1 - y_2 + u_2 \quad (2.38)$$

$$\dot{y}_3 = x_1y_2 - by_3 + u_3 \quad (2.39)$$

2.3.1 Controller Design

To achieve complete synchronization, feedback control inputs are added to the response system:

$$\dot{y}_2 = x_1y_3 + rx_1 - y_2 + u_2 \quad (2.40)$$

$$\dot{y}_3 = x_1y_2 - by_3 + u_3 \quad (2.41)$$

The control laws are chosen as:

$$u_2 = -k_2(y_2 - x_2) \quad (2.42)$$

$$u_3 = -k_3(y_3 - x_3) \quad (2.43)$$

where $k_2 > 0$ and $k_3 > 0$ are control gains.

2.3.2 Error System

Define the synchronization errors as:

$$e_2 = y_2 - x_2, \quad e_3 = y_3 - x_3 \quad (2.44)$$

The error dynamics are obtained as:

$$\dot{e}_2 = x_1 e_3 - (1 + k_2) e_2 \quad (2.45)$$

$$\dot{e}_3 = x_1 e_2 - (b + k_3) e_3 \quad (2.46)$$

2.3.3 Stability Analysis

Consider the Lyapunov function:

$$V = \frac{1}{2}(e_2^2 + e_3^2) \quad (2.47)$$

Its time derivative is:

$$\dot{V} = e_2 \dot{e}_2 + e_3 \dot{e}_3 \quad (2.48)$$

Substituting the error system:

$$\dot{V} = -(1 + k_2)e_2^2 - (b + k_3)e_3^2 + 2x_1 e_2 e_3 \quad (2.49)$$

2.3.4 Synchronization Condition

By choosing sufficiently large control gains k_2 and k_3 , the negative quadratic terms dominate the cross term, ensuring:

$$\dot{V} < 0 \quad (2.50)$$

Thus, the error system is asymptotically stable:

$$e_2 \rightarrow 0, \quad e_3 \rightarrow 0 \quad (2.51)$$

2.3.5 Conclusion

Therefore, the proposed feedback control guarantees complete synchronization between the drive and response Lorenz systems.

Adaptive Synchronization and Circuit Validation of a Novel 5-D Hyperchaotic System

3.1 Description of the Proposed System

Consider the real parameters $\alpha, \beta, \gamma, \delta, \theta$, and ρ to be positive, and the variables x_1, x_2, x_3, x_4 , and x_5 . The following five-dimensional dynamical system [8] is obtained by extending a modified Lorenz-type system [1] and adding additional nonlinear coupling and damping terms:

$$\begin{cases} \frac{dx_1}{dt} = \alpha(x_2 - x_1), \\ \frac{dx_2}{dt} = \gamma x_1 - x_2 - x_1 x_3, \\ \frac{dx_3}{dt} = \beta x_1 x_2 x_3 - x_2 - \beta x_3, \\ \frac{dx_4}{dt} = -\delta(x_1^2 + x_4), \\ \frac{dx_5}{dt} = -\rho x_5 + \theta x_1 x_2. \end{cases} \quad (3.1)$$

When $\alpha = 10, \beta = 3, \gamma = 35, \delta = 10, \theta = 10$, and $\rho = 10$, the system (3.1) displays complex dynamical behavior. The presence of two positive Lyapunov exponents confirms the existence of a hyperchaotic attractor generated by the system in numerical simulations. This indicates a high sensitivity to initial conditions and a richer dynamical structure compared to classical three-dimensional chaotic systems.

To further illustrate this behavior, the phase portraits of the system are depicted through two-dimensional projections onto different planes (x_i, x_j) , where $i, j = 1, \dots, 5$ and $i \neq j$. These projections reveal a complex geometrical structure with trajectories confined to a bounded region of the phase space.

In particular, the projections shown in Figures (3.7) (3.8) (3.9) (3.10) (3.11) (3.12) (3.13) (3.14) (3.15) and (3.16) exhibit intricate and irregular patterns that do not fill the space, which is a characteristic feature of a hyperchaotic attractor. The observed structures suggest a fractal-like geometry, confirming that the attractor has zero volume in the phase space.

3.2 Equilibrium Points and Stability Analysis

Equilibrium Points

To determine the equilibrium points of the system, we solve: states of (3.1) = 0

$$\text{From the first equation: } [\alpha(x_2 - x_1) = 0 \Rightarrow x_2 = x_1.]$$

$$\text{Substituting into the second equation: } [\gamma x_1 - x_1 - x_1 x_3 = 0 \Rightarrow x_1(\gamma - 1 - x_3) = 0.]$$

This leads to two cases:

Case 1: $x_1 = 0$ Then $x_2 = 0$, and substituting into the remaining equations yields: $[x_3 = 0, x_4 = 0, x_5 = 0.]$

Hence, the trivial equilibrium point is: $[E_1 = (0, 0, 0, 0, 0).]$

Case 2: $x_3 = \gamma - 1$ Substituting into the third equation: $[\beta x_1^2(\gamma - 1) - x_1 - \beta(\gamma - 1) = 0,]$

$$\text{which gives: } [x_1 = \frac{1 \pm \sqrt{1 + 4\beta^2(\gamma - 1)^2}}{2\beta(\gamma - 1)}.]$$

$$\text{Thus: } [x_2 = x_1, x_3 = \gamma - 1, x_4 = -x_1^2, x_5 = \frac{\theta}{\rho} x_1^2.]$$

$$\text{The equilibrium points are: } [E_1 = (0, 0, 0, 0, 0),][E_{\pm} = (x_1, x_1, \gamma - 1, -x_1^2, \frac{\theta}{\rho} x_1^2).]$$

For $\gamma = 35, \beta = 3, \theta = \rho = 10$, we obtain: $[E_2 \approx (1.0049, 1.0049, 34, -1.0098, 1.0098),][E_3 \approx (-0.9951, -0.9951, 34, -0.9902, 0.9902).]$

Jacobian Matrix

The Jacobian matrix of the system is given by:

$$J(x_1, x_2, x_3, x_4, x_5) = \begin{pmatrix} -\alpha & \alpha & 0 & 0 & 0 \\ \gamma - x_3 & -1 & -x_1 & 0 & 0 \\ \beta x_2 x_3 & \beta x_1 x_3 - 1 & \beta x_1 x_2 - \beta & 0 & 0 \\ -2\delta x_1 & 0 & 0 & -\delta & 0 \\ \theta x_2 & \theta x_1 & 0 & 0 & -\rho \end{pmatrix}.$$

Stability Analysis

Equilibrium E_1 The eigenvalues of $J(E_1)$ are: $[\lambda_1 = -24.7419, \lambda_2 = 13.7419, \lambda_3 = -3, \lambda_{4,5} = -10.]$

Since one eigenvalue is positive, E_0 is unstable. Moreover, all eigenvalues have non-zero real parts, hence E_0 is hyperbolic.

Equilibrium E_2 The eigenvalues are: $[\lambda_1 = -14.0399,][\lambda_{2,3} = 1.5053 \pm 11.9300i, \lambda_{4,5} = -10.]$

Since the real parts of the complex eigenvalues are positive, E_2 is unstable and hyperbolic.

Equilibrium E_3 The eigenvalues are: $[\lambda_1 = -14.0852,][\lambda_{2,3} = 1.5574 \pm 11.9633i, \lambda_{4,5} = -10.]$

Thus, E_3 is also unstable and hyperbolic.

Application of the Hartman–Grobman Theorem

Since all equilibrium points are hyperbolic (i.e., none of the eigenvalues has zero real part), the Hartman–Grobman theorem applies. Therefore, the nonlinear system is locally topologically conjugate to its linearization near each equilibrium point.

In all cases, the presence of eigenvalues with positive real parts implies that the equilibria are unstable and exhibit saddle-type behavior.

3.3 Dissipativity Analysis of the System

The divergence of the vector field g is examined to determine whether the system is dissipative or conservative. A dynamical system is considered dissipative if the divergence is negative ($\nabla \cdot g < 0$), indicating that the phase space contracts towards an attractor.

Calculating the Divergence Given the system of equations (3.1):

$$\begin{aligned} f_1(x_1, x_2, x_3, x_4, x_5) &= \alpha(x_2 - x_1) \\ f_2(x_1, x_2, x_3, x_4, x_5) &= \gamma x_1 - x_2 - x_1 x_3 \\ f_3(x_1, x_2, x_3, x_4, x_5) &= \beta x_1 x_2 x_3 - x_2 - \beta x_3 \\ f_4(x_1, x_2, x_3, x_4, x_5) &= -\delta(x_1^2 + x_4) \\ f_5(x_1, x_2, x_3, x_4, x_5) &= -\rho x_5 + \theta x_1 x_2 \end{aligned}$$

The divergence $\nabla \cdot \mathbf{g}$ is defined as the sum of the partial derivatives with respect to each state variable:

$$\begin{aligned} \nabla \cdot \mathbf{g} &= \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} + \frac{\partial f_4}{\partial x_4} + \frac{\partial f_5}{\partial x_5} \\ &= -\alpha - 1 + (\beta x_1 x_2 - \beta) - \delta - \rho \\ &= -(\alpha + \beta + \delta + \rho + 1) + \beta x_1 x_2 \end{aligned} \tag{3.2}$$

Substituting the given parameter values ($\alpha = 10, \beta = 3, \delta = 10, \rho = 10$):

$$\nabla \cdot \mathbf{g} = -(10 + 3 + 10 + 10 + 1) + 3x_1 x_2 \tag{3.3}$$

$$\nabla \cdot \mathbf{g} = -34 + 3x_1 x_2 \tag{3.4}$$

The system is dissipative if $\nabla \cdot \mathbf{g} < 0$, which occurs when:

$$3x_1 x_2 < 34 \implies x_1 x_2 < \frac{34}{3} \approx 11.33 \tag{3.5}$$

The phase space volume shrinks exponentially because the system's trajectories satisfy $x_1 x_2 < 11.33$. Thus, the asymptotic motion settles onto an attractor, confirming the dissipative nature of the new system (3.1).

3.4 Lyapunov Exponents Analysis

The purpose of this subsection is to demonstrate the chaotic behavior of the proposed system (3.1) by computing its Lyapunov exponents. These exponents are widely used to characterize chaotic dynamics, as they provide a quantitative measure of the system's sensitivity to initial conditions.

Numerical simulations were performed using Python to achieve this objective.

The obtained results are as follows:

$$\lambda_1 = 3.8012, \quad \lambda_2 = 1.4723, \quad \lambda_3 = -10.4779, \quad \lambda_4 = -10.5663, \quad \lambda_5 = -15.0921.$$

The sum of the Lyapunov exponents is given by:

$$\sum_{i=1}^5 \lambda_i = -30.8628.$$

From these values, it can be observed that two Lyapunov exponents are positive, while the remaining ones are negative. The presence of more than one positive exponent indicates that the system exhibits hyperchaotic behavior. Furthermore, the negative sum of the Lyapunov exponents confirms that the system is dissipative.

Therefore, the proposed system is classified as a hyperchaotic system with a complex dynamical structure high sensitivity to initial conditions and a richer dynamical behavior compared to classical three-dimensional chaotic systems.

The Kaplan–Yorke dimension is defined as:

$$D_{KY} = 2 + \frac{\lambda_1 + \lambda_2}{|\lambda_3|}.$$

First, we compute:

$$\lambda_1 + \lambda_2 = 3.8012 + 1.4723 = 5.2735.$$

Thus, we obtain:

$$D_{KY} = 2 + \frac{5.2735}{10.4779}.$$

$$D_{KY} \approx 2.5032.$$

This value indicates that the system possesses a fractal attractor structure with a dimension between 2 and 3, confirming the existence of a complex geometric structure in the phase space.

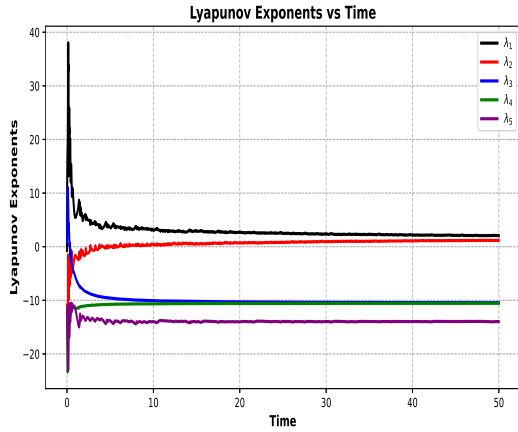


Figure 3.1: Evaluation of Lyapunov Exponent vs Time

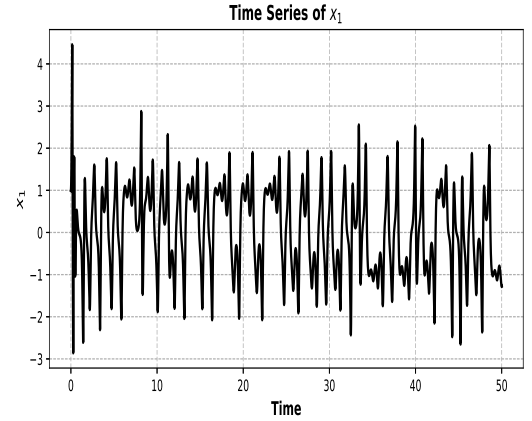


Figure 3.2: $x_1(t)$

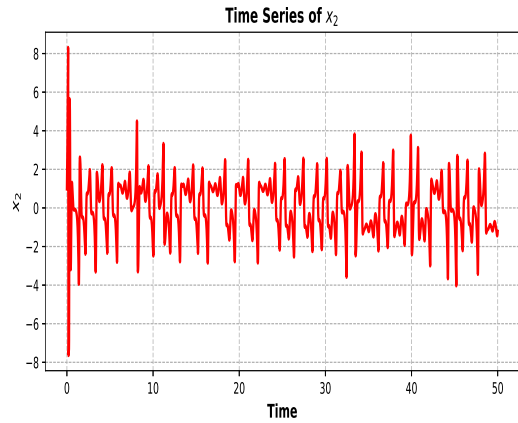


Figure 3.3: $x_2(t)$

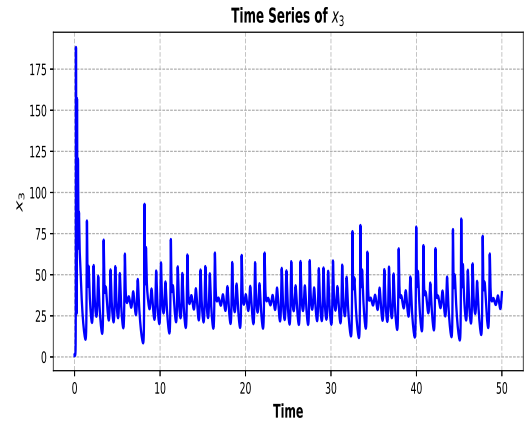


Figure 3.4: $x_3(t)$

3.5 Dynamics of the System

The dynamics of our system will be assessed in this part using the bifurcations diagrams and the evolution of the LLE vs the system parameters. The system (3.1) progresses

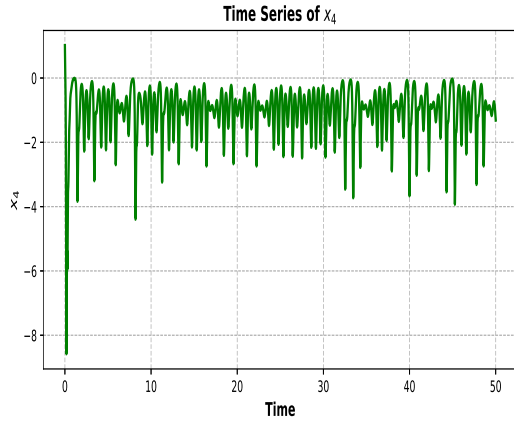


Figure 3.5: $x_4(t)$

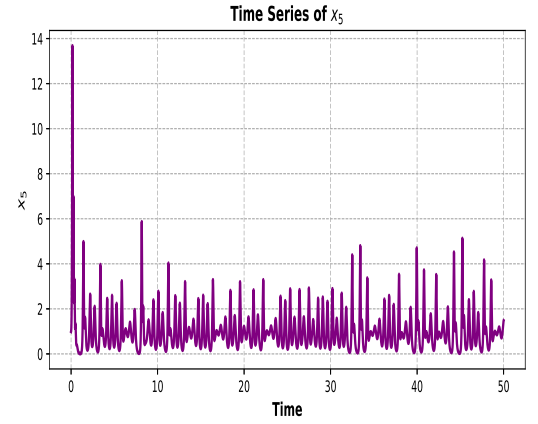


Figure 3.6: $x_5(t)$

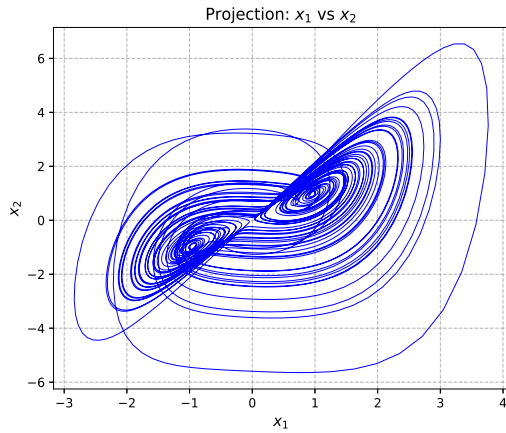


Figure 3.7: x_1 vs x_2

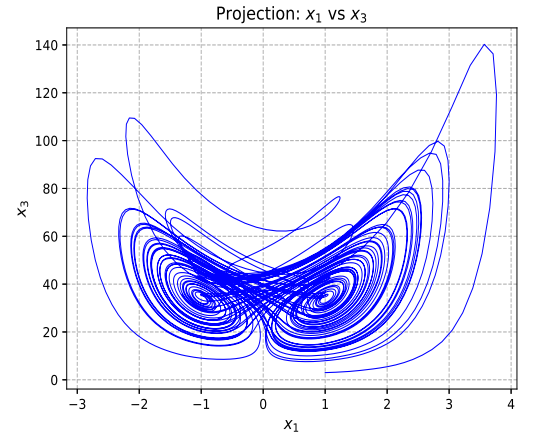


Figure 3.8: x_1 vs x_3

through its periodic, quasiperiodic, regular, and chaotic states.

Bifurcation diagrams versus Largest Lyapunov exponent

The bifurcation diagrams and the largest Lyapunov exponents (LLEs) are plotted in the following figure to demonstrate that they yield equivalent results and provide consistent information about the dynamics of system (3.1). The results of the numerical simulations are presented in Fig (3.17), (3.18), (3.19), (3.20), (3.21) and (3.22) .

Analyse of paramater α The spectrum of the largest Lyapunov exponents of system (3.1) with respect to the parameter α is shown in Figure (3.17). It is evident that system (3.1) exhibits chaotic, periodic, or convergent behavior toward equilibrium points for $\alpha \in [0, 10]$. for $\alpha \in [0, 1.31]$, $[1.5]$, $[1.62, 1.98]$, $[2.01, 2.66]$ and $[2.75, 10]$, the max-

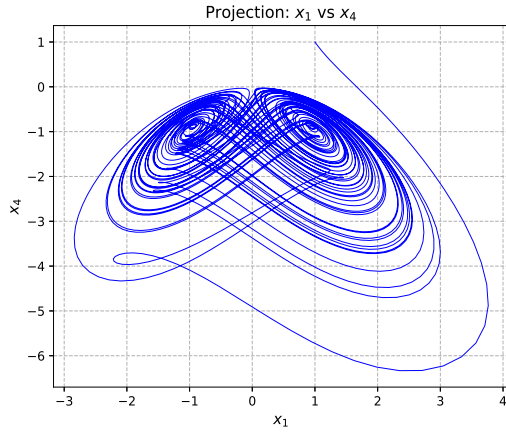


Figure 3.9: x_1 vs x_4

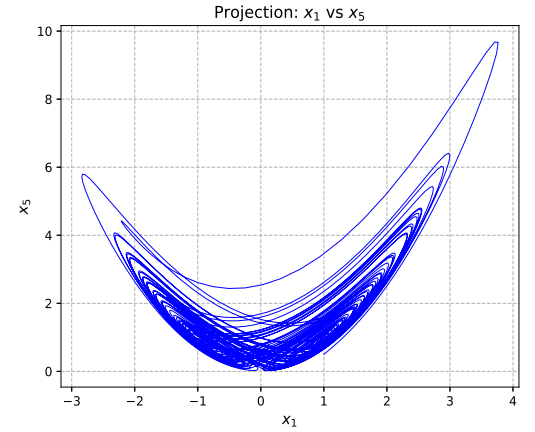


Figure 3.10: x_1 vs x_5

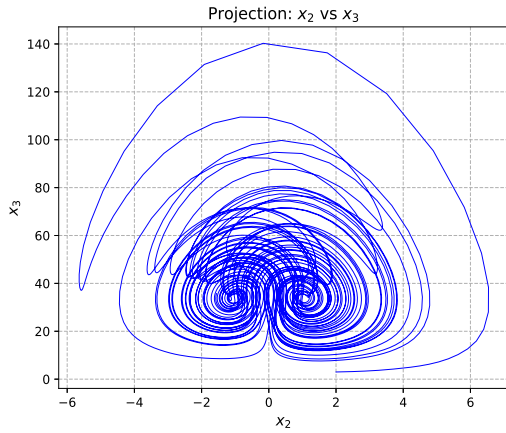


Figure 3.11: x_2 vs x_3

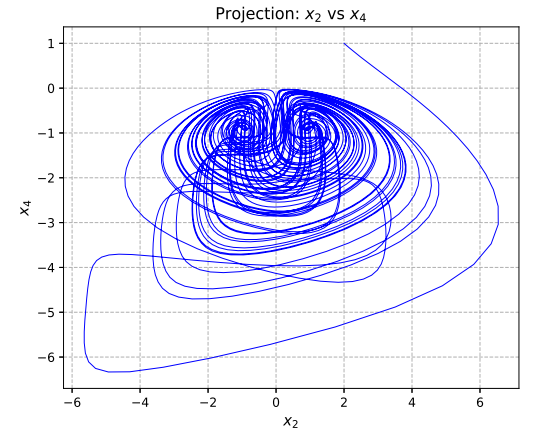


Figure 3.12: x_2 vs x_4

imum Lyapunov exponent is positive, indicating that the system behaves chaotically in this range. for $\alpha \in [1.32, 1.46, 1.55, 2.02, 2.72]$, the maximum Lyapunov exponent is approximately zero, indicating the existence of periodic orbits. Finally, for $\alpha \in [1.33, 1.45]$ and $[1.6]$, the maximum Lyapunov exponent becomes negative, which implies that the trajectories converge to equilibrium points. This behavior is further confirmed by the bifurcation diagram shown in the same figure (3.17).

Analyse of paramater γ The spectrum of the largest Lyapunov exponents of system (3.1) with respect to the parameter γ is shown in Figure (3.18). It is evident that system (3.1) exhibits chaotic, periodic, or convergent behavior toward equilibrium points for $\gamma \in [0, 20]$. for $\gamma \in [1, 1.9]$ and $[1.36, 20]$, the maximum Lyapunov exponent is positive, indicating that the system behaves chaotically in this range. for $\gamma \in [1.10, 1.14, 1.35]$, the maximum Lyapunov exponent is approximately zero, indicating the existence of periodic orbits. Finally, for $\gamma \in [0, 0.99]$ and $[1.15, 1.35]$, the maximum Lyapunov exponent

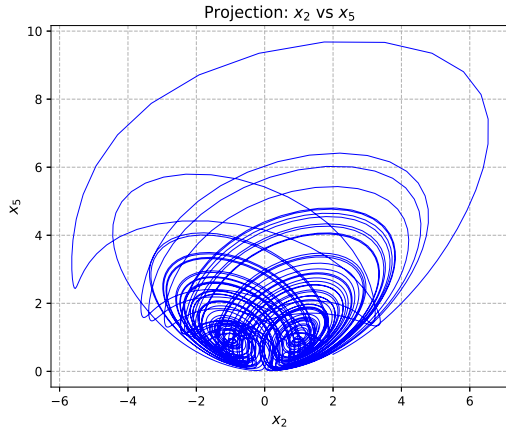


Figure 3.13: x_2 vs x_5

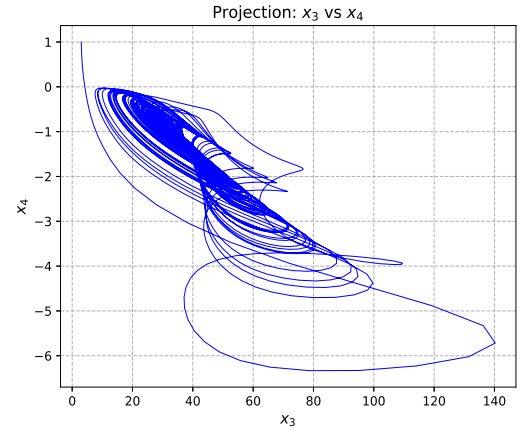


Figure 3.14: x_3 vs x_4

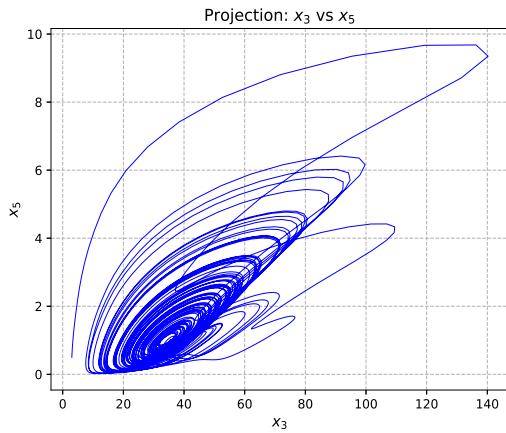


Figure 3.15: x_3 vs x_5

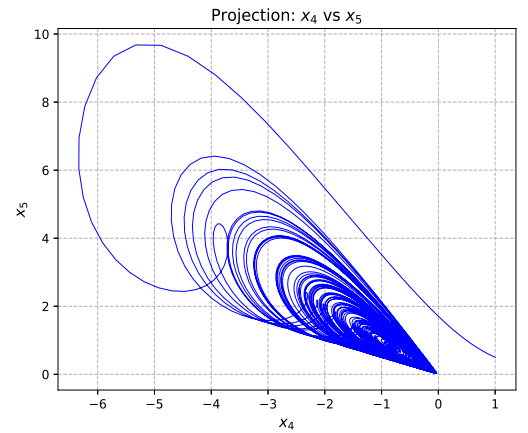


Figure 3.16: x_4 vs x_5

becomes negative, which implies that the trajectories converge to equilibrium points. This behavior is further confirmed by the bifurcation diagram shown in the same figure (3.18).

Analyse of paramater β The spectrum of the largest Lyapunov exponents of system (3.1) with respect to the parameter β is shown in Figure (3.19). It is evident that system (3.1) exhibits chaotic, periodic, or convergent behavior toward equilibrium points for $\beta \in [0.1, 2]$. for $\beta \in [0.23, 0.28]$, $[0.29, 0.31]$, $[0.36, 0.45]$ and $[0.46, 0.52]$, $[0.54, 0.57]$, $[0.59, 0.62]$, $[0.65, 1.06]$ and $[1.08, 2]$, the maximum Lyapunov exponent is positive, indicating that the system behaves chaotically in this range. for $\beta \in [0.22, 0.29, 0.3, 0.35, 0.464, 0.542], 0.64, 1.067]$, the maximum Lyapunov exponent is approximately zero, indicating the existence of periodic orbits. Finally, for $\beta \in [0.1, 0.234]$, $[0.288]$, $[0.32, 0.35]$ and $[0.456, 0.53, 0.62, 1.07]$, the maximum Lyapunov exponent becomes negative, which implies that the trajectories converge to equilib-

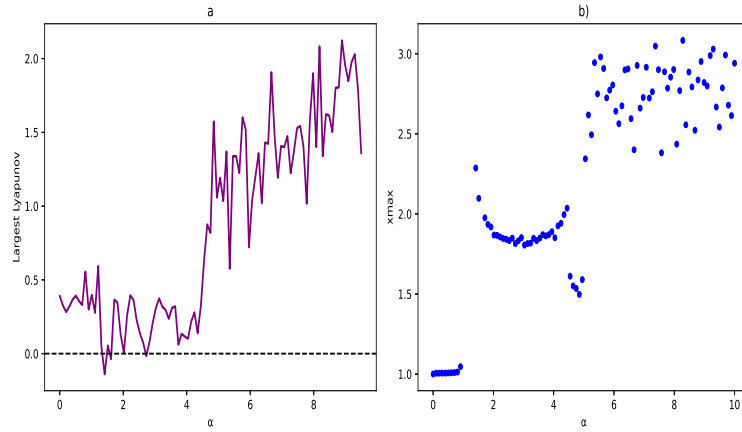


Figure 3.17: Bifurcation diagram and Variation of the largest Lyapunov exponent of system (3.1) versus the parameter $\alpha \in [0, 10]$

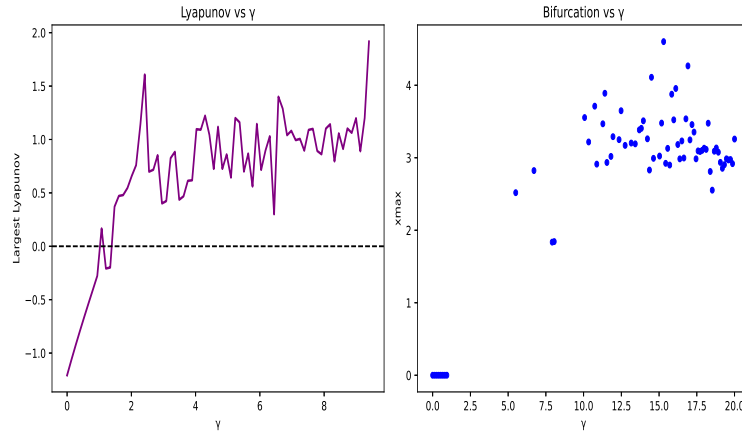


Figure 3.18: Bifurcation diagram and Variation of the largest Lyapunov exponent of system (3.1) versus the parameter $\gamma \in [0, 20]$

rium points. This behavior is further confirmed by the bifurcation diagram shown in the same figure (3.19) .

Analyse of paramater δ

The spectrum of the largest Lyapunov exponents of system (3.1) with respect to the parameter δ is shown in Figure (3.20). It is evident that system (3.1) exhibits chaotic, periodic, or convergent behavior toward equilibrium points for $\delta \in [0, 3]$. for $\delta \in [0, 3]$ the maximum Lyapunov exponent is positive, indicating that the system behaves chaotically in this range. This behavior is further confirmed by the bifurcation diagram shown in the same figure (3.20).

Analyse of paramater ρ

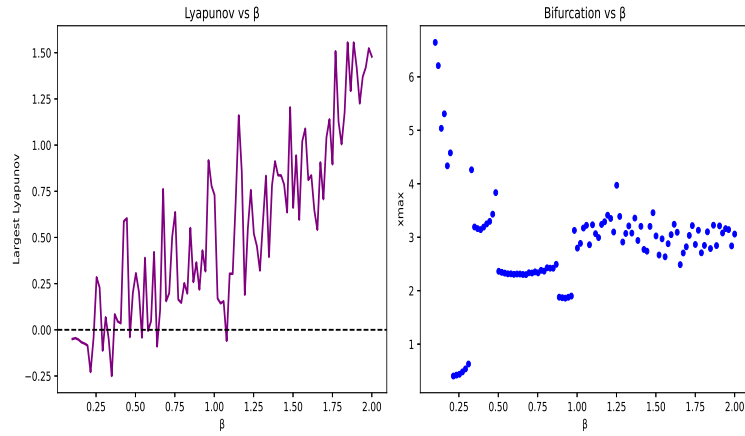


Figure 3.19: Bifurcation diagram and Variation of the largest Lyapunov exponent of system (3.1) versus the parameter $\beta \in [0.1, 2]$

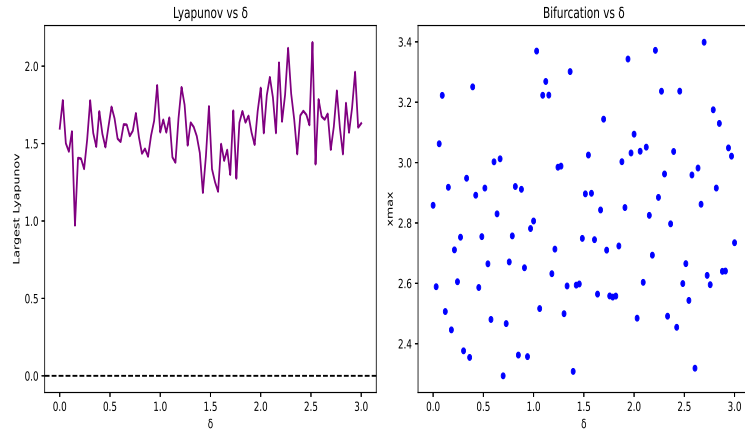


Figure 3.20: Bifurcation diagram and Variation of the largest Lyapunov exponent of system (3.1) versus the parameter $\delta \in [0, 3]$

The spectrum of the largest Lyapunov exponents of system (3.1) with respect to the parameter ρ is shown in Figure (3.21). It is evident that system (3.1) exhibits chaotic, periodic, or convergent behavior toward equilibrium points for $\rho \in [0, 3]$. for $\rho \in [0, 3]$ the maximum Lyapunov exponent is positive, indicating that the system behaves chaotically in this range. This behavior is further confirmed by the bifurcation diagram shown in the same figure (3.21).

Analyse of paramater θ

The spectrum of the largest Lyapunov exponents of system (3.1) with respect to the parameter θ is shown in Figure (3.22). It is evident that system (3.1) exhibits chaotic, periodic, or convergent behavior toward equilibrium points for $\theta \in [0, 10]$. for $\rho \in [0, 3]$ the maximum Lyapunov exponent is positive, indicating that the system behaves chaotically

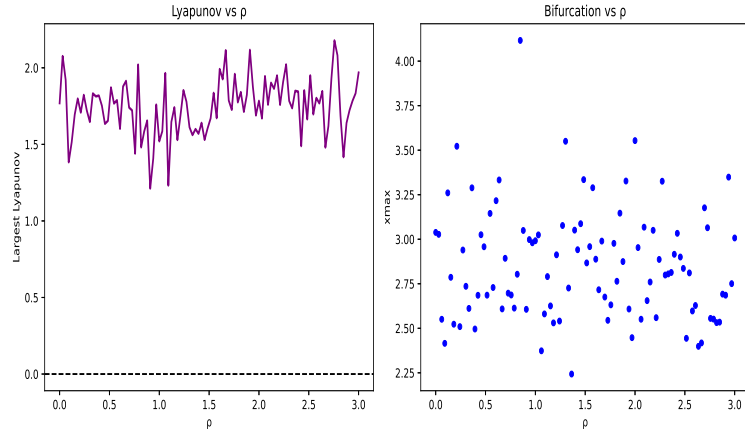


Figure 3.21: Bifurcation diagram and Variation of the largest Lyapunov exponent of system (3.1) versus the parameter $\rho \in [0,3]$

in this range. This behavior is further confirmed by the bifurcation diagram shown in the same figure (3.22).

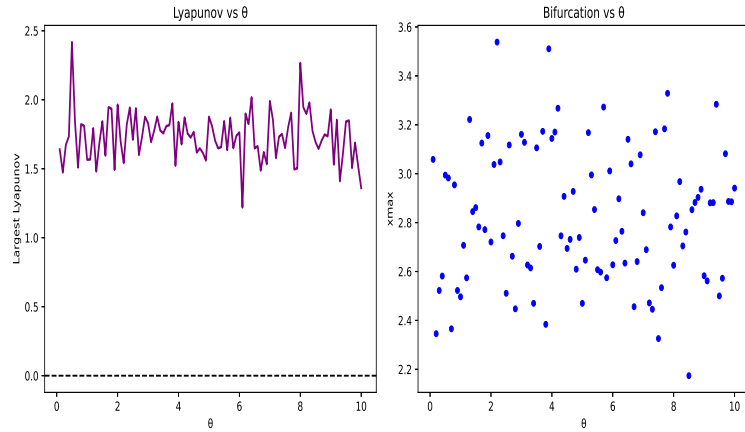


Figure 3.22: Bifurcation diagram and Variation of the largest Lyapunov exponent of system (3.1) versus the parameter $\theta \in [0,10]$

3.6 Adaptive synchronization of the 5-D new chaotic system

In this section, we develop an adaptive control law aimed at achieving global and exponential synchronization of two identical five-dimensional novel chaotic systems with unknown parameters. Accordingly, the master system is described by the dynamics of the proposed chaotic system as follows:

$$\begin{cases} \frac{dx_1}{dt} = \alpha(x_2 - x_1), \\ \frac{dx_2}{dt} = \gamma x_1 - x_2 - x_1 x_3, \\ \frac{dx_3}{dt} = \beta x_1 x_2 x_3 - x_2 - \beta x_3, \\ \frac{dx_4}{dt} = -\delta(x_1^2 + x_4), \\ \frac{dx_5}{dt} = -\rho x_5 + \theta x_1 x_2. \end{cases} \quad (3.6)$$

The slave system is given by:

$$\begin{cases} \frac{dy_1}{dt} = \alpha(y_2 - y_1) + u_1, \\ \frac{dy_2}{dt} = \gamma y_1 - y_2 - y_1 y_3 + u_2, \\ \frac{dy_3}{dt} = \beta y_1 y_2 y_3 - y_2 - \beta y_3 + u_3, \\ \frac{dy_4}{dt} = -\delta(y_1^2 + y_4) + u_4, \\ \frac{dy_5}{dt} = -\rho y_5 + \theta y_1 y_2 + u_5. \end{cases} \quad (3.7)$$

In equation (3.6) and (3.7), the system parameters are assumed to be unknown. The objective is to design adaptive feedback control inputs u_1, u_2, u_3, u_4 and u_5 based on the system states x_1, x_2, x_3, x_4 and x_5 , along with their corresponding parameter estimates $\alpha_1(t), \gamma_1(t), \beta_1(t), \delta_1(t), \rho_1(t), \theta_1(t)$ of the unknown parameters $\alpha, \gamma, \beta, \delta, \rho, \theta$, respectively.

The synchronization error between the novel chaotic systems (3.6) and (3.7) is defined as:

$$\begin{cases} e_1 = y_1 - x_1, \\ e_2 = y_2 - x_2, \\ e_3 = y_3 - x_3, \\ e_4 = y_4 - x_4, \\ e_5 = y_5 - x_5. \end{cases} \quad (3.8)$$

(3.8) implies:

$$\begin{cases} \dot{e}_1 = \dot{y}_1 - \dot{x}_1, \\ \dot{e}_2 = \dot{y}_2 - \dot{x}_2, \\ \dot{e}_3 = \dot{y}_3 - \dot{x}_3, \\ \dot{e}_4 = \dot{y}_4 - \dot{x}_4, \\ \dot{e}_5 = \dot{y}_5 - \dot{x}_5. \end{cases} \quad (3.9)$$

Thus, the synchronization error dynamics is obtained as:

$$\begin{cases} \dot{e}_1 = \alpha(e_2 - e_1) + u_1, \\ \dot{e}_2 = \gamma e_1 - e_2 - (y_1 y_3 - x_1 x_3) + u_2, \\ \dot{e}_3 = \beta(y_1 y_2 y_3 - x_1 x_2 x_3) - e_2 - \beta e_3 + u_3, \\ \dot{e}_4 = -\delta[(y_1^2 - x_1^2) + e_4] + u_4, \\ \dot{e}_5 = -\rho e_5 + \theta(y_1 y_2 - x_1 x_2) + u_5. \end{cases} \quad (3.10)$$

We define the adaptive control inputs as:

$$\begin{cases} u_1 = -\alpha_1(e_2 - e_1) - k_1 e_1, \\ u_2 = -\gamma_1 e_1 + e_2 + (y_1 y_3 - x_1 x_3) - k_2 e_2, \\ u_3 = -\beta_1(y_1 y_2 y_3 - x_1 x_2 x_3) + e_2 + \beta_1 e_3 - k_3 e_3, \\ u_4 = \delta_1[(y_1^2 - x_1^2) + e_4] - k_4 e_4, \\ u_5 = \rho_1 e_5 - \theta_1(y_1 y_2 - x_1 x_2) - k_5 e_5. \end{cases} \quad (3.11)$$

where $k_i > 0$.

Substituting (3.10) into (3.11), we obtain the closed-loop error dynamics as:

$$\begin{cases} \dot{e}_1 = (\alpha - \alpha_1)(e_2 - e_1) - k_1 e_1, \\ \dot{e}_2 = (\gamma - \gamma_1)e_1 - k_2 e_2, \\ \dot{e}_3 = (\beta - \beta_1)(y_1 y_2 y_3 - x_1 x_2 x_3) - (\beta - \beta_1)e_3 - k_3 e_3, \\ \dot{e}_4 = -(\delta - \delta_1)[(y_1^2 - x_1^2) + e_4] - k_4 e_4, \\ \dot{e}_5 = -(\rho - \rho_1)e_5 + (\theta - \theta_1)(y_1 y_2 - x_1 x_2) - k_5 e_5. \end{cases} \quad (3.12)$$

The parameter estimation errors are defined as:

$$\begin{cases} e_\alpha(t) = \alpha - \alpha_1(t), \\ e_\beta(t) = \beta - \beta_1(t), \\ e_\gamma(t) = \gamma - \gamma_1(t), \\ e_\theta(t) = \theta - \theta_1(t), \\ e_\delta(t) = \delta - \delta_1(t), \\ e_\rho(t) = \rho - \rho_1(t). \end{cases} \quad (3.13)$$

Differentiating (3.13) with respect to t , we obtain:

$$\begin{cases} \frac{de_\alpha(t)}{dt} = -\frac{d\alpha_1(t)}{dt}, \\ \frac{de_\beta(t)}{dt} = -\frac{d\beta_1(t)}{dt}, \\ \frac{de_\gamma(t)}{dt} = -\frac{d\gamma_1(t)}{dt}, \\ \frac{de_\theta(t)}{dt} = -\frac{d\theta_1(t)}{dt}, \\ \frac{de_\delta(t)}{dt} = -\frac{d\delta_1(t)}{dt}, \\ \frac{de_\rho(t)}{dt} = -\frac{d\rho_1(t)}{dt}. \end{cases} \quad (3.14)$$

By using (3.13), we rewrite the closed-loop system (3.12) as:

$$\begin{cases} \dot{e}_1 = e_\alpha(e_2 - e_1) - k_1 e_1, \\ \dot{e}_2 = e_\gamma e_1 - k_2 e_2, \\ \dot{e}_3 = e_\beta(y_1 y_2 y_3 - x_1 x_2 x_3) - e_\beta e_3 - k_3 e_3, \\ \dot{e}_4 = -e_\delta[(y_1^2 - x_1^2) + e_4] - k_4 e_4, \\ \dot{e}_5 = -e_\rho e_5 + e_\theta(y_1 y_2 - x_1 x_2) - k_5 e_5. \end{cases} \quad (3.15)$$

We consider the quadratic Lyapunov function given by:

$$V = \frac{1}{2} \left(\sum_{i=1}^5 k_i e_i^2 + e_\alpha^2 + e_\beta^2 + e_\gamma^2 + e_\delta^2 + e_\rho^2 + e_\theta^2 \right). \quad (3.16)$$

Differentiating V along the trajectories of the systems gives:

$$\begin{aligned}
 \dot{V} = & - \sum_{i=1}^5 k_i e_i^2 + e_\alpha \left[e_1(e_2 - e_1) - \frac{d\alpha_1(t)}{dt} \right] \\
 & + e_\gamma \left[e_1 e_2 - \frac{d\gamma_1(t)}{dt} \right] \\
 & + e_\beta \left[e_3(y_1 y_3 - x_1 x_3) + e_3^2 - \frac{d\beta_1(t)}{dt} \right] \\
 & + e_\delta \left[-e_4(y_1^2 - x_1^2) - \frac{d\delta_1(t)}{dt} \right] \\
 & + e_\rho \left[-e_5^2 - \frac{d\rho_1(t)}{dt} \right] \\
 & + e_\theta \left[e_5(y_1 y_2 - x_1 x_2) - \frac{d\theta_1(t)}{dt} \right].
 \end{aligned} \tag{3.17}$$

In view of (3.17), we take the parameter update law as follows:

$$\begin{cases}
 \frac{d\alpha_1}{dt} = e_1(e_2 - e_1), \\
 \frac{d\gamma_1}{dt} = e_1 e_2, \\
 \frac{d\beta_1}{dt} = e_3(y_1 y_2 y_3 - x_1 x_2 x_3) + e_3^2, \\
 \frac{d\delta_1}{dt} = -e_4(y_1^2 - x_1^2) + e_4, \\
 \frac{d\rho_1}{dt} = -e_5^2, \\
 \frac{d\theta_1}{dt} = e_5(y_1 y_2 - x_1 x_2).
 \end{cases} \tag{3.18}$$

Substituting (3.18) into (3.17), we get:

$$\dot{V} = - \sum_{i=1}^5 k_i e_i^2 \tag{3.19}$$

which is a negative definite function on $\mathbb{R}^5$. Hence, according to Lyapunov stability theory [12], it follows that $e_i(t) \rightarrow 0$ as $t \rightarrow \infty$.

Hence, we have proved the following theorem:

Theorem 3.1. *The five-dimensional novel chaotic systems (3.6) and (3.7) with unknown parameters are globally and exponentially synchronized for all initial conditions under the proposed adaptive feedback control law (3.11) and the parameter update law (3.18), where k_1, k_2, k_3, k_4, k_5 are positive constants.*

3.6.1 Numerical simulations

The nonlinear differential equations under the adaptive control scheme were numerically integrated using the Runge–Kutta method implemented via the `solve_ivp` solver.

The parameters of the proposed five-dimensional chaotic system are chosen as $\alpha = 10$, $\beta = 3$, $\gamma = 35$, $\delta = 10$, $\rho = 10$, and $\theta = 10$.

For $i = 1, 2, 3, 4, 5$, the positive gain constants in the adaptive control law are selected as $k_i = 2$.

The initial conditions of the master and slave systems are given, respectively, by $x_1(0) = 1$, $x_2(0) = 2$, $x_3(0) = 3$, $x_4(0) = 1$, $x_5(0) = 0.5$, and $y_1(0) = 0.5$, $y_2(0) = 1$, $y_3(0) = 2$, $y_4(0) = 2$, $y_5(0) = 0.75$.

Furthermore, the initial estimates of the unknown parameters are taken as $a_1(0) = 20$, $b_1(0) = 10$, $g_1(0) = 40$, $d_1(0) = 5$, $r_1(0) = 5$, and $t_1(0) = 5$.

When the adaptive control law (3.11) together with the parameter update law (3.18) is applied, synchronization between the slave system (3.7) and the master system (3.6) is achieved, as illustrated in Figs. 3.23–3.27.

The time histories of the estimated parameters α , γ , β , δ , ρ , and θ are depicted in Fig. 3.28.

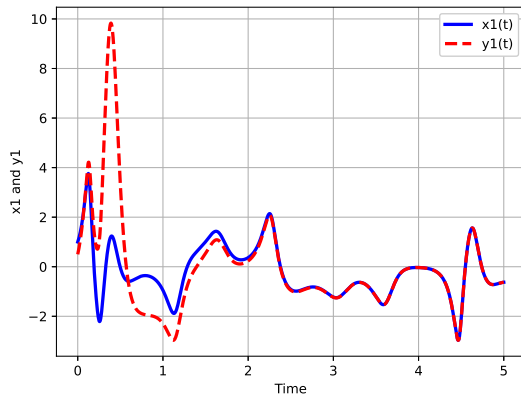


Figure 3.23: Synchronization of the states x_1 and y_1

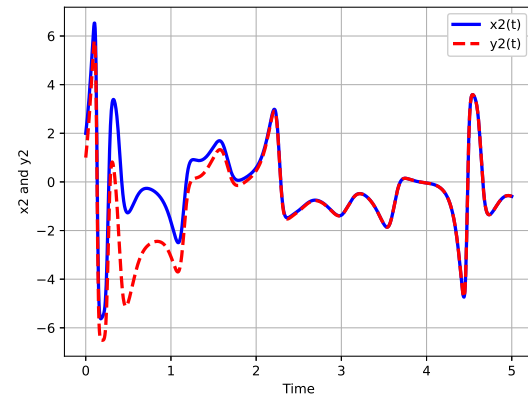


Figure 3.24: Synchronization of the states x_2 and y_2

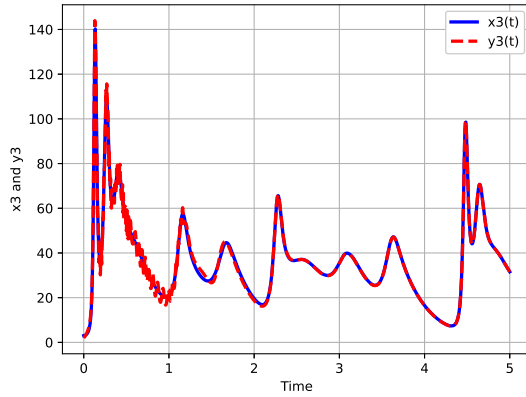


Figure 3.25: Synchronization of the states x_3 and y_3

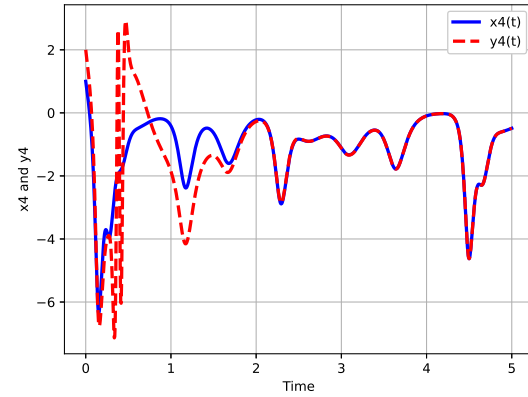


Figure 3.26: Synchronization of the states x_4 and y_4

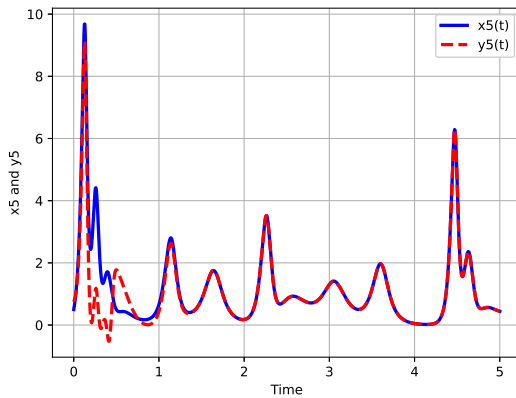


Figure 3.27: Synchronization of the states x_5 and y_5

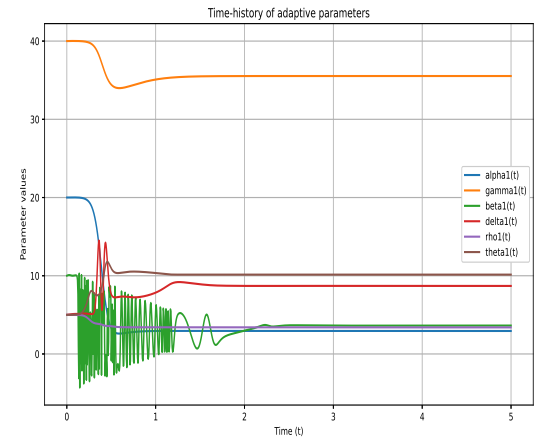


Figure 3.28: Time-history of the parameter estimates $\alpha_1(t)$, $\gamma_1(t)$, β_1 , δ_1 , $\rho_1(t)$ and $\theta_1(t)$

3.7 Circuit Design of the Proposed Hyperchaotic System

3.7.1 Deailed Methode of Circuit Design

Our proposed chaotic system can be compared to an electronic circuit presented in this section, the electronic circuit is depicted in the Figure (3.29) and his design was created with the use of Multisim software. Capacitors, resistors, analog multipliers AD633, and operational amplifiers TL084ACN are used to implement the analog circuit of the proposed 5-D chaotic system (3.1). Steps are given to design an analog circuit that is equivalent to

a system (3.1). The system's initial equation (3.1) reveals:

$$\frac{dx_1}{dt} = -10x_1 + 10x_2 \quad (3.20)$$

Equation (3.20) can be obtained by applying Kirchhoff's law:

$$\begin{aligned} x_1 &= \int [-10x_1 + 10x_2] dt \\ &= \int [10(-x_1) + 10x_2] dt \\ &= -\frac{1}{c_1} \int \left[\frac{-x_1}{R_1} + \frac{x_2}{R_2} \right] dt \\ &= \frac{1}{c_1} \int \left[\frac{x_1}{R_1} + \frac{R_9}{R_8} \frac{-x_2}{R_2} \right] dt \end{aligned} \quad (3.21)$$

The system's second equation (3.1) reveals:

$$\frac{dx_2}{dt} = 35x_1 - x_2 - x_1x_3 \quad (3.22)$$

Equation (3.22) can be obtained by applying Kirchhoff's law:

$$\begin{aligned} x_2 &= \int [35x_1 - x_2 - x_1x_3] dt \\ &= \int [35x_1 + (-x_2) + (-x_1)x_3] dt \\ &= -\frac{1}{c_2} \int \left[\frac{x_1}{R_5} + \frac{-x_2}{R_6} + \frac{-x_1x_3}{10R_7} \right] dt \\ &= \frac{1}{c_2} \int \left[\frac{-x_1}{R_5} + \frac{x_2}{R_6} + \frac{x_1x_3}{10R_7} \right] dt \\ &= \frac{1}{c_2} \int \left[\frac{R_4}{R_3} \frac{-x_1}{R_5} + \frac{x_2}{R_6} + \frac{x_1x_3}{10R_7} \right] dt \end{aligned} \quad (3.23)$$

The system's third equation (3.1) reveals:

$$\frac{dx_3}{dt} = 3x_1x_2x_3 - x_2 - 3x_3 \quad (3.24)$$

Equation (3.24) can be obtained by applying Kirchhoff's law:

$$\begin{aligned}
 x_3 &= \int [3x_1x_2x_3 - x_2 - 3x_3] dt \\
 &= \int [3x_1x_2x_3 + (-x_2) + 3(-x_3)] dt \\
 &= -\frac{1}{c_3} \int [x_1x_2x_3 + (-x_2) + (-x_3)] dt \\
 &= \frac{1}{c_3} \int [x_1x_2(-x_3) + x_2 + x_3] dt \\
 &= \frac{1}{c_3} \int \left[\frac{R_{14}}{R_{13}} \frac{x_1x_2(-x_3)}{10R_{10}} + \frac{x_2}{R_{11}} + \frac{x_3}{R_{12}} \right] dt
 \end{aligned} \tag{3.25}$$

The system's fourth equation (3.1) reveals:

$$\frac{dx_4}{dt} = -10x_1^2 - 10x_4 \tag{3.26}$$

Equation (3.26) can be obtained by applying Kirchhoff's law:

$$\begin{aligned}
 x_4 &= \int [-10x_1^2 - 10x_4] dt \\
 &= \int [10(-x_1^2) + 10(-x_4)] dt \\
 &= -\frac{1}{c_4} \int [(-x_1^2) + (-x_4)] dt \\
 &= \frac{1}{c_4} \int [x_1^2 + (x_4)] dt \\
 &= \frac{1}{c_4} \int \left[\frac{x_1^2}{10R_{18}} + \frac{x_4}{R_{16}} \right] dt
 \end{aligned} \tag{3.27}$$

The system's fifth equation (3.1) reveals:

$$\frac{dx_5}{dt} = -10x_5 + 10x_1x_2 \tag{3.28}$$

Equation (3.28) can be obtained by applying Kirchhoff's law:

$$\begin{aligned}
 x_5 &= \int [-10x_5 + 10x_1x_2] dt \\
 &= \int [10(-x_5) + 10x_1x_2] dt \\
 &= -\frac{1}{c_5} \int [(-x_5) + x_1x_2] dt \\
 &= \frac{1}{c_5} \int [x_5 + x_1(-x_2)] dt \\
 &= \frac{1}{c_5} \int \left[\frac{x_5}{R_{15}} + \frac{R_9}{R_8} \cdot \frac{x_1(-x_2)}{R_{19}} \right] dt
 \end{aligned} \tag{3.29}$$

We acquire the electrical system that is equivalent to the system (3.1) :

$$\begin{cases}
 x_1 = \frac{1}{c_1} \int \left[\frac{x_1}{R_1} + \frac{R_9}{R_8} \frac{(-x_2)}{R_2} \right] dt \\
 x_2 = \frac{1}{c_2} \int \left[\frac{R_4}{R_3} \frac{(-x_1)}{R_5} + \frac{x_2}{R_6} + \frac{x_1x_3}{10R_7} \right] dt \\
 x_3 = \frac{1}{c_3} \int \left[\frac{R_{14}}{R_{13}} \frac{x_1x_2(-x_3)}{10R_{10}} + \frac{x_2}{R_{11}} + \frac{x_3}{R_{12}} \right] dt \\
 x_4 = \frac{1}{c_4} \int \left[\frac{x_1^2}{10R_{18}} + \frac{4x_3}{R_{16}} \right] dt \\
 x_5 = \frac{1}{c_5} \int \left[\frac{x_5}{R_{15}} + \frac{R_9}{R_8} \frac{x_1(-x_2)}{R_{19}} \right] dt
 \end{cases} \tag{3.30}$$

The values of the circular components (3.30) are selected after matching between (3.1) and (3.30):

$$\begin{aligned}
 R_1 &= R_2 = R_7 = R_{16} = R_{15} = R_{18} = 1 \text{ k}\Omega, \\
 R_5 &= 0.285714 \text{ k}\Omega, \quad R_{12} = 3.333 \text{ k}\Omega, \\
 R_3 &= R_4 = R_8 = R_9 = R_{13} = R_{14} = 100 \text{ k}\Omega, \\
 R_6 &= R_{11} = 10 \text{ k}\Omega, \quad R_{10} = 0.333 \text{ k}\Omega, \\
 R_{19} &= 0.1 \text{ k}\Omega, \\
 C_1 &= C_2 = C_3 = C_4 = C_5 = 100 \text{ nF}.
 \end{aligned} \tag{3.31}$$

3.7.2 Comparison of the resault obtained from numericall simulation and cicuit design in multisim of the proposed chaotic system

The results obtained using *NI Multisim 14.3* and *Python* are compared in the following figures ((3.30), (3.31), (3.32), (3.33), (3.34), (3.35), (3.36), (3.37), (3.38) and (3.39)). A clear and strong agreement between the two approaches is observed, which confirms the correctness of the developed model. Additionally, the results exhibit chaotic behavior in the studied system (3.1), confirming its accurate representation. Overall, these findings support the physical validity of the system under the adopted assumptions.

3.8 Secure Communication

Figure (3.40) shows the electronic circuit design for the secure communication strategy created for the new chaotic system. We can see that two signals are transmitted from the transmitter to the receiver: the first one is the driving variable x_5 so as to complete synchronization between the master system and the slave system, and the second signal transmitted is the encrypted message M .

The encrypted message M is obtained using the electronic circuit schematic depicted in Figure (3.41). In the transmitter stage, a microphone (*XLV2*) is utilized to capture the audio source as the clear message m . Where x_1, x_2, x_3 , and x_4 represent the master system's state variables, m is the clear message, and w is a wrong message assumed to be known, which is included to increase the complexity of the encrypted message.

When the synchronization is achieved, the decrypted message $\bar{m}$ is reconstructed in the receiver using the electronic circuit schematic depicted in Figure (3.42). Finally, a speaker (*XLV1*) is implemented at the receiver's output to play back the reconstructed signal, where y_1, y_2, y_3 , and y_4 represent the slave system's state variables, M is the received encrypted message, and w is the same wrong message as in the transmitter.

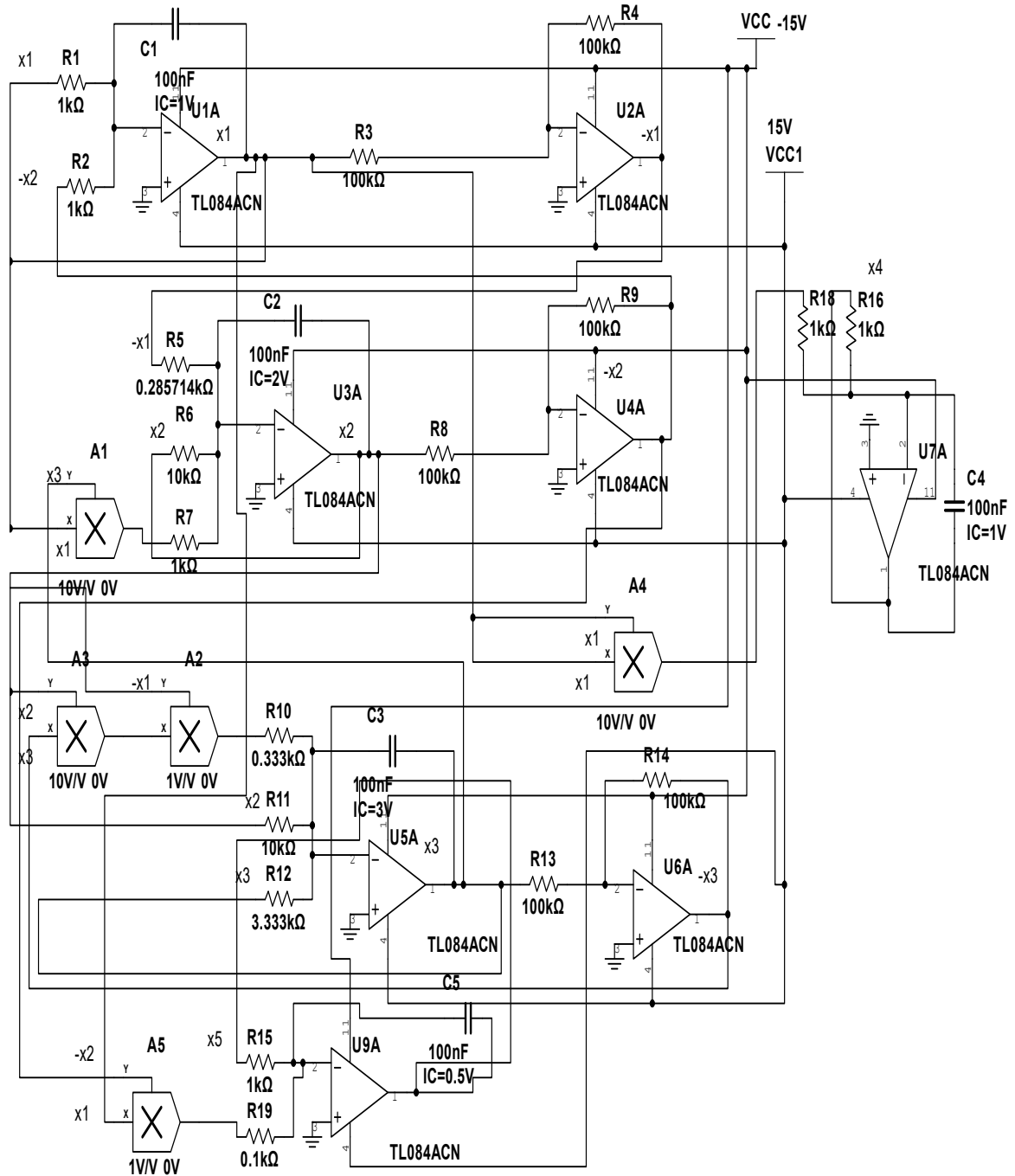


Figure 3.29: Cicuit design in multisim of the 5D chaotic system

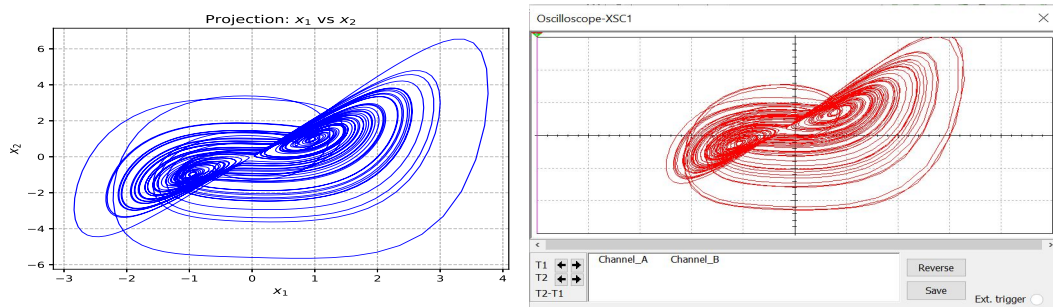


Figure 3.30: Comparison of the resault obtained from numericall simulation and cicuit design in multisim of the proposed chaotic system in x_1x_2

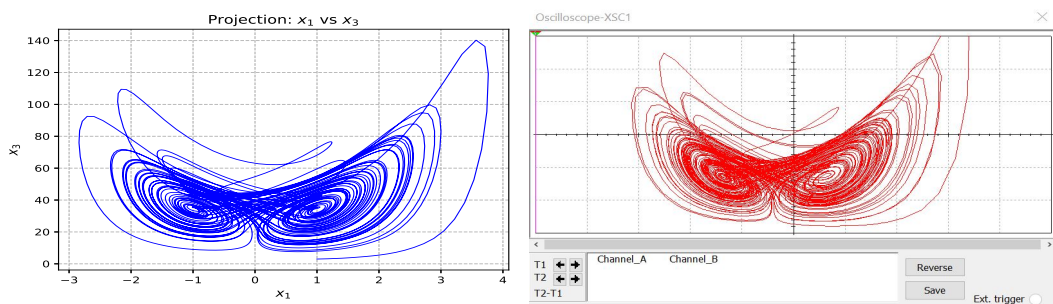


Figure 3.31: Comparison of the resault obtained from numericall simulation and cicuit design in multisim of the proposed chaotic system in x_1x_3

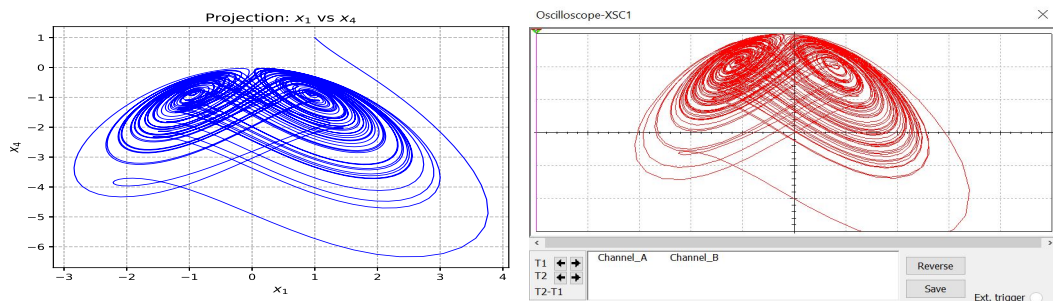


Figure 3.32: Comparison of the resault obtained from numericall simulation and cicuit design in multisim of the proposed chaotic system in x_1x_4

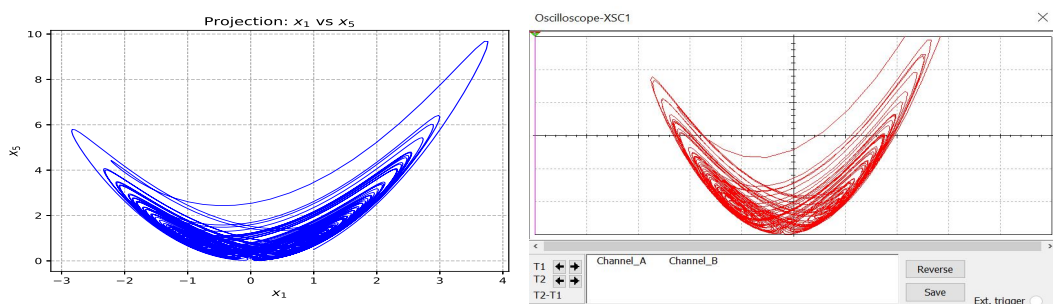


Figure 3.33: Comparison of the resault obtained from numericall simulation and cicuit design in multisim of the proposed chaotic system in x_1x_5

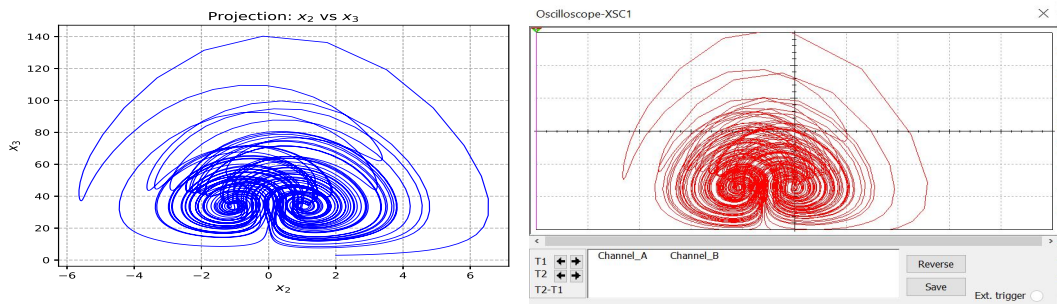


Figure 3.34: Comparison of the resault obtained from numericall simulation and cicuit design in multisim of the proposed chaotic system in x_2x_3

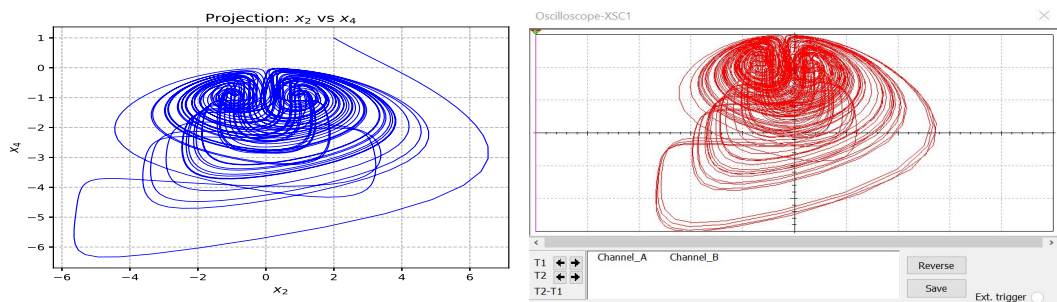


Figure 3.35: Comparison of the resault obtained from numericall simulation and cicuit design in multisim of the proposed chaotic system in x_2x_4

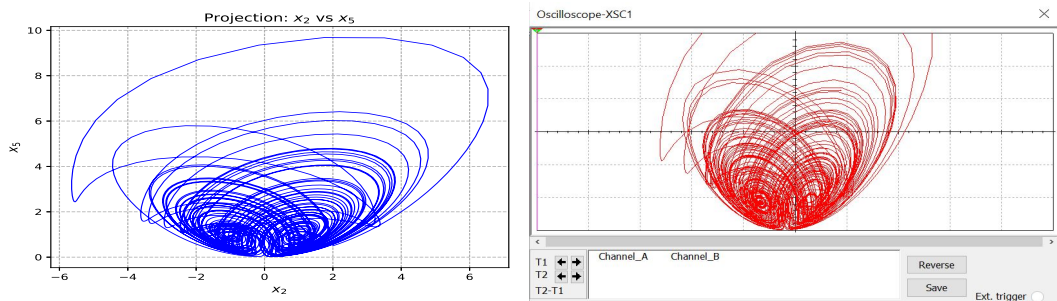


Figure 3.36: Comparison of the resault obtained from numericall simulation and cicuit design in multisim of the proposed chaotic system in x_2x_5

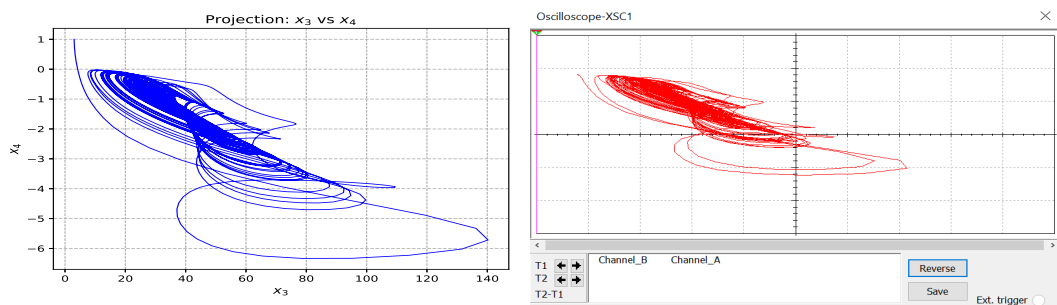


Figure 3.37: Comparison of the resault obtained from numericall simulation and cicuit design in multisim of the proposed chaotic system in x_3x_4

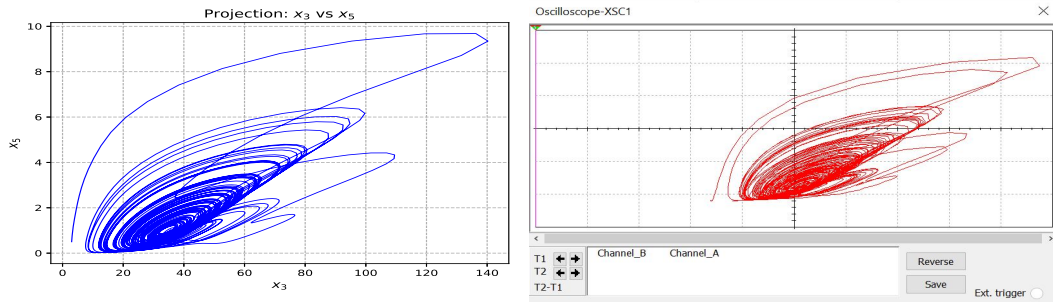


Figure 3.38: Comparison of the resault obtained from numerically simulation and circuit design in multisim of the proposed chaotic system in x_3x_5

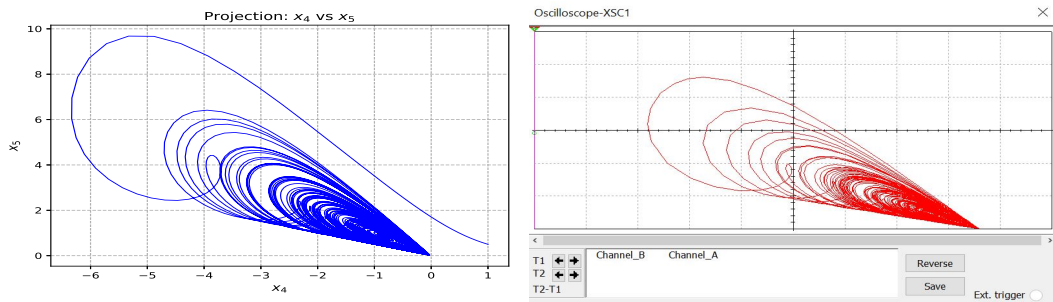


Figure 3.39: Comparison of the resault obtained from numerically simulation and circuit design in multisim of the proposed chaotic system in x_4x_5

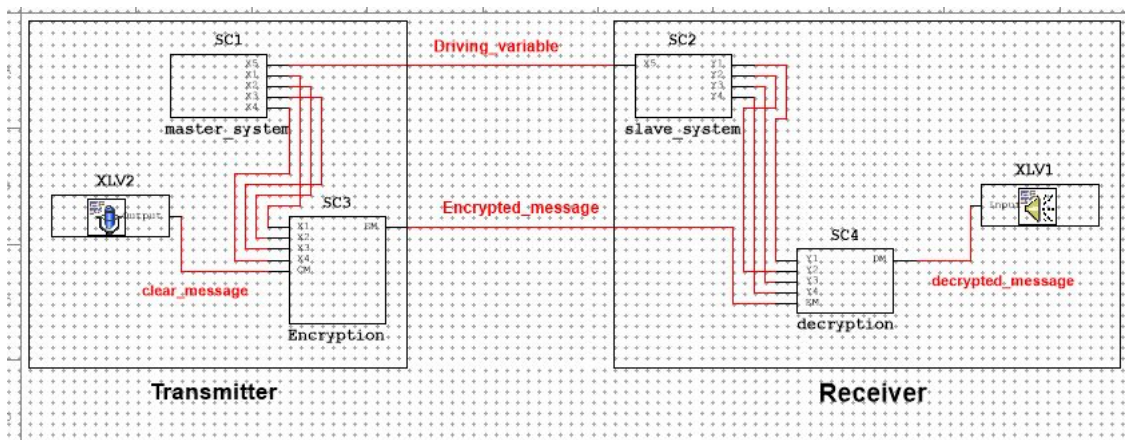


Figure 3.40: Electronic circuit of the secure communication scheme.

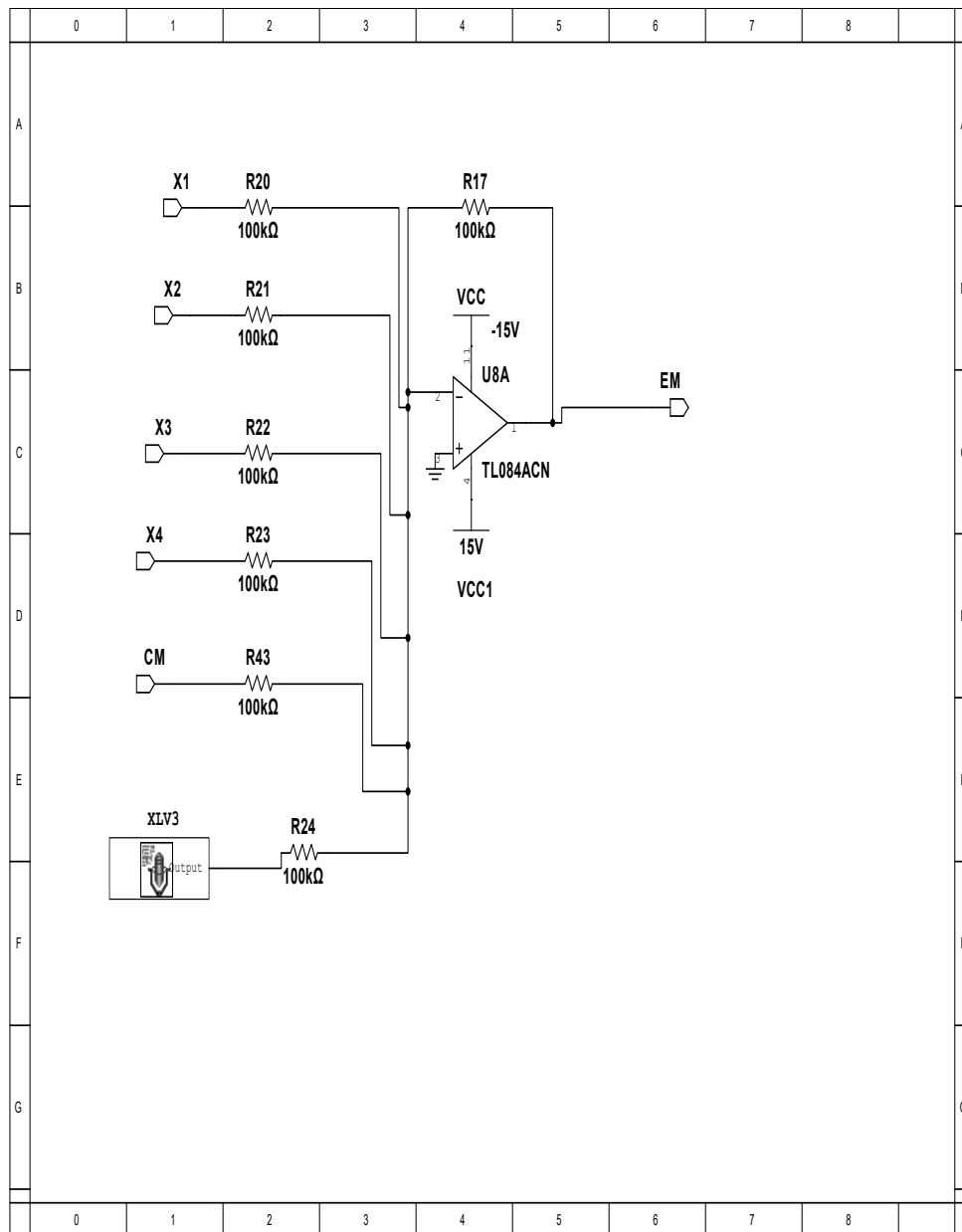


Figure 3.41: Electronic circuit schematic of the encryption process.

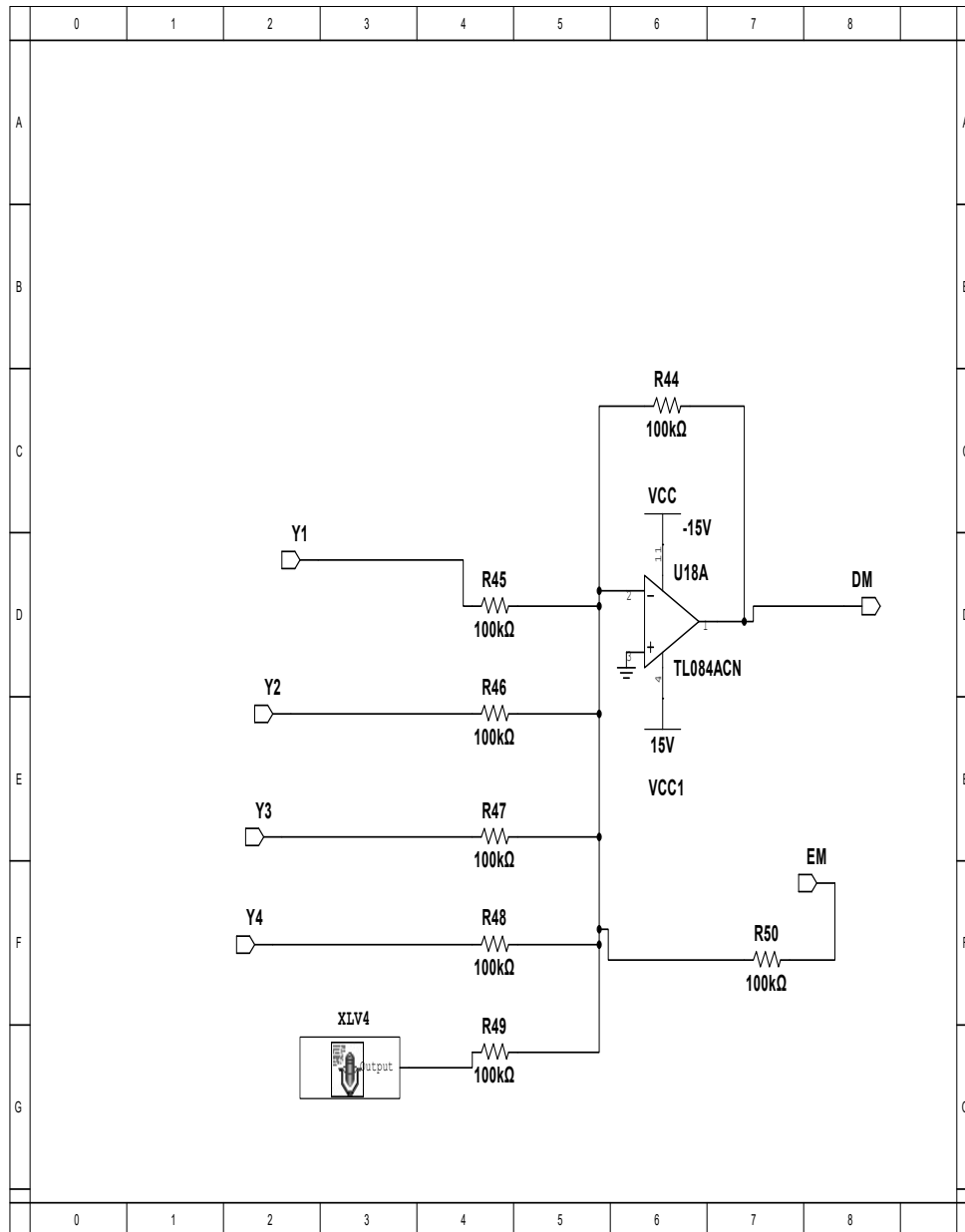


Figure 3.42: Electronic circuit schematic of the decryption process.

Conclusion and Perspective

In this work, it was found that the proposed five-dimensional hyperchaotic system is characterized by complex dynamical behavior resulting from its various nonlinearities. This behavior was studied through dynamical analysis involving phase diagrams and Lyapunov exponents values that confirm the presence of chaos. The results also showed the possibility of achieving synchronization (adaptive synchronization) between the two identical hyperchaotic systems using appropriate control techniques. In addition, the physical existence of the system was verified by designing and implementing The physical circuit realization of the proposed mathematical model, comprising resistors, capacitors, analog multipliers, a direct current power source (V_{CC}), and ground. Which enhances its potential for use in secure communications and information encryption applications.

Despite the positive results obtained, applying the model practically via the electronic circuit faced some technical challenges in Multisim 14.3 related to adjusting the components and achieving dynamic stability of the system, including: Although the circuit design was electronically correct, the practical results did not exactly match the simulation results in Python in terms of signal shape (for example, a straight line). We also faced a problem related to the stability of the practical implementation of the electronic circuit, as the results were correct at first simulation, but were not reproducible with the same accuracy after a period of time or when the experiment was repeated.

In the future studies, we will use this proposed hyperchaotic fractional systems in the encryption and transmission of the information. We also move on to realistically implement these electronic circuits by installing them in a laboratory.

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