



N° Réf :.....

Abdelhafid Boussouf
University Center– Mila

Institute of Mathematics and Computer Sciences

Department of Mathematics

Submitted for the degree of Master
In: Mathematics
Specialty: Applied Mathematics

Chaos in 3D Discrete Fractional Systems

Presented by: - Assala Khellaf
- Rabah Ghetaham

Defended on 25/06 /2025

In front of the jury

Smail Kaouache	MCA	Abdelhafid Boussouf U. C. of Mila	Chairman
Rabah Bououden	MCA	Abdelhafid Boussouf U. C. of Mila	Supervisor
Boujema Labeled	MCB	Abdelhafid Boussouf U. C. of Mila	Examiner

Academic year: 2024/2025

Thanks

We wish to express our gratitude to all those who contributed, directly or indirectly, to the completion of this thesis.

Special thanks to our supervisor **Rabeh Bououden** for his invaluable advice, patience, and guidance throughout this work.

We also thank the jury members, Mr **Smail Kaouache** and Mr **Boujema Labeled** for agreeing to

preside

To the teachers of the department of Mathematics and Computer Science for their knowledge and availability, and to the jury members for their efforts in evaluating this work.

Special thanks to **Amani**, our families, and friends for their unconditional support and encouragement during this demanding period.

Finally, we thank all those who participated directly or indirectly in enriching this research.

Assala, Rabeh

ملخص

تتناول هذه المذكرة دراسة الانظمة الديناميكية الخطية و غير الخطية, مع تركيز خاص على تحليل التشعب و الفوضى في خريطة هينون ثلاثية الابعاد.

يتضمن الفصل الاول مفاهيم اساسية مثل النقاط الثابتة, المدارات الدورية, الاستقرار, و التشعب (مثل تشعب الطي, الالتفاف, و نيمارك-ساكر), بالاضافة الى نظريات رئيسية كنظرية لي-يورك.

اما الفصل الثاني فيطبق هذه المفاهيم على نموذج معمم لخريطة هينون ثلاثية الابعاد, حيث يتم تحليل شروط استقرار النقاط الثابتة و انواع التشعب عبر محاكاة عددية و رسوم بيانية.

الكلمات المفتاحية:

الانظمة الديناميكية المنفصلة, التشعب, الفوضى, خريطة هينون, المحاكاة العددية.

ABSTRACT

This thesis investigates linear and nonlinear discrete dynamical systems, focusing on bifurcation analysis and chaos in a three-dimensional Hénon map.

The first chapter introduces fundamental concepts such as fixed points, periodic orbits, stability, and topological equivalence, alongside key theorems like Li-York's and bifurcation types (pli, flip, Neimark-Sacker).

The second chapter applies these concepts to a generalized 3D Hénon map, analyzing fixed-point stability conditions and demonstrating bifurcations through numerical simulations and phase portraits.

Key words :

Discrete dynamical systems, bifurcation, chaos, Hénon map, numerical simulation.

RÉSUMÉ

Ce mémoire explore les systèmes dynamiques discrets linéaires et non linéaires, en mettant l'accent sur l'analyse des bifurcations et du chaos dans la carte de Hénon tridimensionnelle.

Le premier chapitre introduit des concepts fondamentaux tels que les points fixes, les orbites périodiques, la stabilité, et les équivalences topologiques, il aborde également les théorèmes clés comme celui de Li-Yorke et les bifurcations de pli, flip, et Neimark-Sacker.

Le deuxième chapitre applique ces concepts à une généralisation tridimensionnelle de la carte de Hénon, en étudiant les conditions de stabilité des points fixes et en démontrant des bifurcations via des simulations numériques et des diagrammes de phase.

Mots-clés :

Discrete dynamical systems, bifurcations, chaos, carte de Hénon, simulation numérique.

TABLE OF CONTENTS

Introduction	1
1 Preliminaries on linear and nonlinear discrete dynamical systems	3
1.1 Definitions	3
1.2 First-Order One-Dimensional Discrete Dynamical Systems	7
1.2.1 Graphical Study of One-Dimensional Dynamical Systems	7
1.2.2 Linear Systems	8
1.2.3 One-Dimensional Nonlinear Dynamical Systems	9
1.3 Stability of Two-Dimensional Linear Dynamical Systems	13
1.4 Stability of Two-Dimensional Nonlinear Discrete Dynamical Systems . .	20
1.5 Bifurcation and Chaos	22
1.5.1 Bifurcation Theory	22
1.5.2 The Chaos	23
2 Bifurcation analysis of three- dimensional Henon map	37
2.1 Three-dimensional Henon map	38
2.2 Codimension one bifurcations	40
2.2.1 Bifurcation de Pli	40
2.2.2 Flip bifurcation	45

2.2.3	Neimark-Sacker bifurcation	49
2.3	Numerical simulations	53
2.3.1	Numerical simulations of stability and codimension one bifurcations	54
2.3.2	More numerical simulations of system (2.1)	56
	Conclusion	65

LIST OF FIGURES

1.1	Phase portrait of (1.20) if $ \lambda_1 < \lambda_2 < 1$ (a). Type 1 stable node $0 < \lambda_2 < \lambda_1$. (b) Type 2 stable node $0 < \lambda_1 < \lambda_2 < 0$. (c) Type 3 stable node $\lambda_1 < 0$ and $\lambda_2 < 0$	29
1.2	Phase portrait of (1.20) if $ \lambda_1 < \lambda_2 < 1$ (a). Type 1 unstable node $0 < \lambda_2 < \lambda_1$. (b) Type 2 unstable node $0 < \lambda_1 < \lambda_2 < 0$. (c) Type 3 stable node $\lambda_1 < 0$ and $\lambda_2 < 0$	30
1.3	Phase portrait of (1.20) if $ \lambda_1 < 1, \lambda_2 > 1$ (a). Type 1 col $0 < \lambda_2, \lambda_1 > 0$. (b) Type 2 col $0 < \lambda_1 > 0$ and $\lambda_2 < 0$. (c) Type 3 col $\lambda_1 < 0$ and $\lambda_2 < 0$	31
1.4	Phase portrait of (1.20) if $ \lambda_1 = \lambda_2 < 1$ (a). Type 1 stable star node $0 < \lambda_2 = \lambda_1$. (b) Type 2 stable star node $\lambda_1 = \lambda_2$. (c) Type 3 stable star node $\lambda_1 = \lambda_2 < 0$	32
1.5	Phase portrait of (1.20) if $ \lambda_1 = \lambda_2 < 1$ (a). Type 1 unstable star node $0 < \lambda_2 = \lambda_1$. (b) Type 2 unstable star node $\lambda_1 = \lambda_2$. (c) Type 1 unstable star node $\lambda_1 = \lambda_2 < 0$	33
1.6	Phase portrait of (1.20) if $\lambda_1 = \lambda_2 = \lambda$ (a). Stable improper node $0 < \lambda < 1$. (b) Unstable improper node $1 < \lambda < \infty$	34
1.7	Geometric representation of the eigenvalues $\lambda_{1,2} = a \pm i = r \exp^{\pm i\theta}$	34
1.8	Fixed point center $r = 1$, (a) $b > 0$, (b) $b < 0$	35
1.9	Stable focus, (a) $b > 0$, (b) $b < 0$	36

1.10	Characterization of solutions as a function of $tr(A)$ and $det(A)$	36
2.1	The stability region and bifurcation region of system (2.1) in the (b, a) -plane.	54
2.2	(A)-(B) bifurcation diagrams of system (2.1) in the plane (a, x) : (A) ($b = -0.6$), and (B) ($b = 0.4$); (C) bifurcation diagrams of system (2.1) in the plane (b, x) avec ($b \in (-0.8, 0, 8)$) and ($a = 0.2$), here, the fold bifurcation, Flip bifurcation and Neimark-Sacker bifurcation are labeled as 'SN', 'PD' and 'NS', respectively	55
2.3	Bifurcation diagrams of system (2.1) in the threedimensional (a, b, x) space.	56
2.4	(A) Bifurcation diagrams of system (2.1) in the plan (a, x) ($a \in (0, 0.4)$) for ($b = -0.6$); (B) maximum Lyapunov exponent corresponding to (A); (C) the local amplified bifurcation diagram of (A) for (A) for ($a \in (0.22, 0, 0.32)$)	57
2.5	(A)-(H) phase portraits for various values of a corresponding to Figure 4 (A).	60
2.6	(A)-(C) phase portraits for ($a = 0.385$) in the (x, y) , the plan (y, z) , and the (x, z)	61
2.7	(A) Bifurcation diagram of system (2.1) in the plan (a, x) ($a \in (0, 1)$) for ($b = 0.4$); (B) maximum Lyapunov exponent corresponding to (A); (C) the local amplified bifurcation diagram of (A) for ($a \in (0.81, 0, 0.85)$); (D) maximum Lyapunov exponent corresponding to (C); (E)-(F) chaotic attractors for ($a = 0, 835$) and ($a = 0, 8445$), respectively.	62
2.8	(A) bifurcation diagram of system (2.1) in the plan (x, y) with ($b \in (-0, 8, , 0, 8)$) and ($a = 0, 23$); maximum Lyapunov exponent corresponding to (A); (C) the local amplified bifurcation diagram of (A) for ($b \in (-0, 75, -0, 5)$); (D) maximum Lyapunov exponent corresponding to (C); (E) the local amplified bifurcation diagram of (A) for ($b \in (0, 64, 0, 7)$); (F) maximum Lyapunov exponent corresponding to (E).	63
2.9	In (A)-(C), phase portraits corresponding to Figure 8 (C): (A) $b = -0, 72$, (B) $b = -0, 6$, and (C) $b = -0, 53$, In (D)-(F), phase portraits corresponding to Figure 8 (E): (D) $b = 0, 66$, (E) $b = 0, 675$, and (F) $b = 0, 691$	64

INTRODUCTION

This thesis, titled “Chaos in 3D Discrete Fractional Systems” explores discrete, linear, and nonlinear dynamical systems, with a specific focus on the 3D generalisation of the Hénon map, the primary objective is to study complex dynamical behaviors, particularly bifurcations and chaos, using advanced theoretical and numerical tools. Discrete dynamical systems play a fundamental role in modeling natural and artificial phenomena, ranging from physics to economics, among them, the Hénon map, introduced by Michel Hénon in 1976 as a simplified model of the Poincaré section of the Lorenz system, has become a paradigm for the study of chaos, its two-dimensional version is extensively documented, but multidimensional extensions, such as the three-dimensional Hénon map studied here, offer richer and less explored dynamics, this thesis aims to fill this gap by analyzing the stability properties, bifurcations, and attractors of this generalization.

This thesis is organized into two main chapters:

the first chapter introduces the basic concepts necessary for understanding discrete dynamical systems: fundamental definitions, one-dimensional systems, two-dimensional systems, bifurcations and chaos.

The second chapter applies the theoretical concepts to the system:

$$\begin{cases} x_{n+1} &= a + by_n - x_n^2 \\ y_{n+1} &= x_n + z_n \\ z_{n+1} &= -x_n \end{cases} .$$

This thesis makes a qualitative contribution to the understanding of nonlinear dynamics and paves the way for future research exploring the hidden complexities of applied mathematics and systems theory.

CHAPTER 1

PRELIMINARIES ON LINEAR AND NONLINEAR DISCRETE DYNAMICAL SYSTEMS

The goal of this first chapter is to introduce numerous fundamental notions and techniques of dynamical system theory within the simplest possible framework.

General notation for discrete dynamical system

Definition 1.1.1 :

Let f be a C^1 class function on an open set $D \subset \mathbb{R}_m$, a discrete dynamical system, denoted $(D, \mathbb{N}, f)$, is a relation of the form

$$x_{n+1} = f(x_n). \quad (1.1)$$

Thus, representing the composition of applications, we have

$$x_n = f^n(x_0) \quad (1.2)$$

where

$$f^n(x) = \underbrace{f \circ f \circ \dots \circ f(x)}_{n \text{ fois}}, \forall \in \mathbb{N} \text{ and } f^0 = Id.$$

The application f is called the recurrence, iteration, or point transformation.

Definition 1.1.2 : (Autonomous and non-autonomous discrete dynamical systems)

When the function f in (1.1) explicitly depends on time, the system is said to be non-autonomous, otherwise, the system is called autonomous.

Definition 1.1.3 : (Trajectory)

Given an initial point x_0 , the orbit (or trajectory) of system (1.1) is the sequence

$$\mathcal{O}(x_0) = \{x(0) = x_0, x(1) = f(x(0)), \dots, x(n+1) = f(x(n)), \dots\}.$$

Fixed points and periodic orbits

Fixed points and periodic orbits are particularly simple trajectories that play a central role in the study of dynamical systems. **Fixed point and their stability** The fixed point is the simplest trajectory.

Definition 1.1.4 : (Fixed points)

A point x^* is a fixed point of system (1.1) si

$$f(x^*) = x^*.$$

Sometimes, these points are called stationary points or equilibrium points.

Definition 1.1.5 :

A fixed point x^* is stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|x_0 - x^*\| \leq \delta \text{ implise } \|x_n - x^*\| \leq \epsilon \text{ for all } n \geq 1. \quad (1.3)$$

Definition 1.1.6 :

A fixed point x^* that is not stable is said to be unstable.

Periodic orbits (p -cycle) and their stability

Definition 1.1.7 :(p -cycle)

By definition, a p -cycle est un p -tuple $(x_0, x_1, \dots, x_p)$ such that $k \in \{1, 2, \dots, p\}$ for all $f(x_k) = x_k$ and $f(x_p) = x_p = x_0$, with p being the smallest integer greater than or equal to 1 possessing this property, any point of the cycle is said to be p -periodic.

Remark 1.1.1 :

- 1- A fixed point is a periodic point of period 1.
- 2- A point x est called a p -periodic point of a transformation f , if it is a fixed point of f^p , without being a fixed point of f^l , $1 \leq l < p$, (l and p integers).

Definition 1.1.8 :

A periodic orbit $(x_0, x_1, \dots, x_p)$ of a dynamical system governed by a map f is stable if every point x_i , $i = 0, 1, 2, \dots, p - 1$ is a stable fixed point of the dynamical system gouverned by the map f^p .

Definition 1.1.9 :

A periodic orbit $(x_0, x_1, \dots, x_{p-1})$ that is not stable is said to be unstable.

Limit points, limit sets, and aperiodic orbits

Let $\theta(x_0) = \{x_0, x_1, x_2, \dots, x_n, \dots\}$ be an orbit of a dynamical system

$$x_{n+1} = f(x_n), \quad (1.4)$$

where x_0 is a given initial condition.

Definition 1.1.10 :

A point z is called a limit point of the orbit $O(x_0)$ if there exists a subsequence $\{x_{n_k} : k = 0, 1, \dots\}$ of $\theta(x_0)$ such that.

$$\lim_{k \rightarrow +\infty} \|x_{n_k} - z\| = 0 \quad (1.5)$$

Remark 1.1.2 :

1. A stationary orbit has a single limit point (the fixed point).
2. A p -périodique orbit $O(x_0)$ has exactly p limit points: $x_0, x_1, x_2, \dots, x_{p-1}$.

Definition 1.1.11 :

The set $L(x_0)$ of all limit points of an orbit $O(x_0)$ is called limit set.

The fundamental relationship between $L(x_0)$ and its image under the map f is

$$f(L(x_0)) = L(x_0)$$

Definition 1.1.12 :

- 1- An orbit $O(x_0)$ is called asymptotically stationary if its limit set is a stationary point (fixed point), and asymptotically periodic if its limit set is a periodic orbit.
- 2- An orbit $O(x_0)$ is called eventually periodic with period p if it is not periodic itself but there exists an $m > 0$ such that $f^{p+k}(x) = f^k(x)$ for all $k \geq m$ (i.e. $f^k(x)$ is a periodic point for $k \geq m$). If $p = 1$, $O(x_0)$ is called eventually stationary.

Definition 1.1.13 :

An orbit $O(x_0)$ is called aperiodic if its limit set $L(x_0)$ contains infinitely many elements.

Theorem 1.1.1 : (Li-Yorke theorem [26])

Let I be an interval and $f : I \rightarrow I$ a continuous map, if f has a periodic orbit of period 3, then it has periodic orbits of period p for all $p \geq 1$ and there exists an infinite set $S \subset I$ such that every orbit starting from a point in S is aperiodic.

Topological Equivalence of Systems

Let D And E be two metric spaces, and $f : D \rightarrow D, E \rightarrow E$ two maps defining dynamical systems on D and E , respectively.

Definition 1.1.14 :

Two dynamical systems (D, f) and (E, g) are topologically conjugate if there exists a homeomorphism (continuous bijection) $h : D \rightarrow E$ such that $h \circ f = g \circ h$.

Theorem 1.1.2 :

Suppose (D, f) and (E, g) are topologically conjugate via a homeomorphism $h : D \rightarrow E$, then

- 1- The inverse map $h^{-1} : E \rightarrow D$ also satisfies the definition, ensuring topological equivalence.
- 2- $h \circ f^n = g^n \circ h$, for all $n \in \mathbb{N}$.
- 3- if $x^* \in D$ is a periodic point of f with fundamental period p , then $h(x^*) \in E$ is a periodic point of g with fundamental period p .

1.2 First-Order One-Dimensional Discrete Dynamical Systems

The discrete dynamical system:

$$x_{n+1} = f(x_n), \quad (1.6)$$

is one-dimensional if the state variable x is scalar, i.e., $x \in \mathbb{R}$, depends on one or more parameters, called bifurcation parameters, in this case, system (1.6) becomes

$$x_{n+1} = f(x_n, a), \quad (1.7)$$

where $a = (a_1, a_2, \dots, a_m) \in \mathbb{R}^m$.

1.2.1 Graphical Study of One-Dimensional Dynamical Systems

We now describe a graphical method for analyzing trajectories, called the Koenigs-Lemery method or Cobweb diagram.

Recall that the fixed points of a map f are the abscissas of the intersection points between the graph of f and the line $x = y$, suppose the parameter a in (1.7) is fixed.

The trajectory $O(x_0)$ can be visualized in the plane by drawing

- A vertical segment from $(x_0, 0)$ to $(x_0, f(x_0)) = (x_0, x_1)$ on the graphe of f .
- A horizontal segment to (x_1, x_1) on the line $x = y$.
- A vertical segment to $(x_1, f(x_1)) = (x_1, x_2)$ on the graph of f .

Continue this process to obtain sufficient information about the behavior of $O(x_0)$, the resulting plot is called a cobweb diagram.

1.2.2 Linear Systems

After a translation of the origin, a first-order one-dimensional linear dynamical system can always be written as:

$$x_{n+1} = Sx_n, \quad -\infty < x < +\infty. \quad (1.8)$$

The solution for an arbitrary initial condition x_0 is explicit,

$$x_n = S^n x_0. \quad (1.9)$$

The origin ($x = 0$) is the only fixed point, from (1.8) and (1.9), the nature of this point is determined by the real number S (the multiplier of the fixed point $x = 0$).

- 1- If $|S| < 1$, for any x_0 the orbit $O(x_0)$ of system (1.8) tends to the fixed point $x = 0$, this point is attractive or asymptotically stable.
- 2- If $|S| > 1$, for any x_0 the orbit $O(x_0)$ of system (1.8) moves away from 0 as n increases, this point is repulsive or unstable.

The nature of the fixed point also depends on the sign of S

- 1- If $S > 0$, the sequence x_n has the same sign as x_0 , here, 0 is called a type 1 point.
- 2- If $S < 0$, the sign of x_n alternates with n , and the sequence oscillates around 0, here, 0 is called a type 2 point.

1.2.3 One-Dimensional Nonlinear Dynamical Systems

Consider the nonlinear dynamical system defined by:

$$x_{n+1} = f(x_n, a) \quad (1.10)$$

where $a \in \mathbb{R}^m$ is the bifurcation parameter, and $f(x)$ is a class C^1 function of the real variable x , not reducible to the form (1.8) by a linear change of variables.

Existence of Fixed Points and Periodic Orbits

The fixed points of system (1.10) are solutions of

$$x = f(x) \quad (1.11)$$

similarly, periodic points of period p are solutions of:

$$x = f^p(x) \quad (1.12)$$

which are not solutions of:

$$x = f^m(x) \quad (1.13)$$

for any $m < p$.

Theorem 1.2.1 :

Suppose $h : [a, b] \rightarrow \mathbb{R}$ is continuous, then the image of h is a closed and bounded interval.

Lemma 1.2.1 :

Let I be an interval and $f : I \rightarrow \mathbb{R}$ a continuous map, at a point $x_0 \in I$, if $|f(x_0)| < 1$, then for any $k \in (|f(x_0)|, 1)$, there exists $r > 0$ such that

$$|f(x)| \leq k \quad \text{for all } x \in [x_0 - r, x_0 + r] \cap I$$

.

Theorem 1.2.2 : (Finite Increments Theorem)

For any real-valued function of a real variable $f : [a, b] \rightarrow \mathbb{R}$ assumed continuous on the closed interval $[a, b]$ and differentiable on the open interval $]a, b[$, there exists a real number $]a, b[$, such that:

$$\frac{f(b) - f(a)}{b - a} = f'(c). \quad (1.14)$$

Theorem 1.2.3 : (Intermediate Value Theorem)

For any continuous function $f : [a, b] \rightarrow \mathbb{R}$ and any real number $f(a)$ and $f(b)$, there exists at least one real $c \in (a, b)$ such that $f(c) = u$.

Theorem 1.2.4 :[3]

Let $I = [a, b]$, be an interval, and let $f : I \rightarrow I$ be a continuous map. Then f has at least one fixed point in I .

Theorem 1.2.5 :[3]

Let $f : I \rightarrow I$ and suppose $|f'(x)| < 1$ Then f has a unique fixed point in I , moreover:

$$|f(x) - f(y)| < |x - y|$$

for all $x, y \in I, x \neq y$.

Theorem 1.2.6 : (Sharkovsky Theorem)

Let I be an interval and $f : I \rightarrow I$ continuous, if f has a periodic point of period n , then for every m succeeding n in the Sharkovsky order, f has a periodic point of period m .

Attractors

Theorem 1.2.7 :[19]

Let I be an open interval, and x^* a fixed point of a continuous function $f : I \rightarrow I$, suppose there exists $r > 0$ such that f is differentiable on $]x^* - r, x^* + r[$ except possibly at x^* and $|f'(x)| \leq 1$, then x^* is stable.

Theorem 1.2.8 :[19]

Let I be an open interval, and $(x_0, x(1), \dots, x(p))$ a periodic orbit of period p of a continuous

function $f : I \rightarrow I$, suppose there exists $r > 0$ such that for each $j = 0, 1, \dots, p$, f is differentiable on $]x_j - r, x_j + r[$, except possibly at x_j and

$$|\frac{d}{dx}f^p(x)| \leq 1$$

then the periodic orbit $(x_0, x_1, \dots, x_p)$ is stable.

Definition 1.2.1 :

Let I be an open interval, and $f : I \rightarrow I$ continuous, a fixed point x^* is called attractive if there exists $\epsilon > 0$ such that:

$$\forall x_0 \in B_\epsilon(x^*), \quad x_n = f^n(x_0) \rightarrow x^* \quad n \rightarrow \infty.$$

A periodic orbit of period p $(x_0, x_1, \dots, x_p)$ is attractive if each of its points is an attractive fixed point of the map $f(p)$.

Theorem 1.2.9 :[19]

Let I be an interval and $f : I \rightarrow I$ continuous with a fixed point x^* , suppose there exists $\epsilon > 0$ such that f is differentiable on the neighborhood $]x^* - \epsilon, x^* + \epsilon[$, and the derivative of f , and the derivative of f is continuous at x^* , then $|\frac{d}{dx}f(x^*)| < 1$, implies x^* is attractive.

Theorem 1.2.10 :[3]

Let I be an open interval, and $(x_0, x(1), \dots, x(p))$ a periodic orbit of period p of a continuous function $f : I \rightarrow I$, suppose there exists $\epsilon > 0$ such that f is differentiable on the neighborhood $]x^* - \epsilon, x^* + \epsilon[$ of each point x_j and the derivative f' is continuous at x_j and the derivative is discontinuous at $j = 1, 2, \dots, p$, then

$$|\frac{d}{dx}f^p(x)| < 1 \tag{1.15}$$

implies that the periodic orbit $(x_0, x_1, \dots, x_p)$ is attractive.

Lemma 1.2.2 :

1- Every increasing and bounded sequence converges.

2- Every decreasing and bounded below sequence converges.

Theorem 1.2.11 :[3]

Let I be an interval and $f : I \rightarrow I$ continuous with a fixed point x^* , suppose there exists $\epsilon > 0$ such that f is differentiable on $]x^* - \epsilon, x^* + \epsilon[$ except possibly at x^* and $|f'(x)| < 1$, then x^* is attractive.

◆ Sources

Definition 1.2.2 :

Let I be an open interval and $f : I \rightarrow I$ continuous. A fixed point x^* of f is a source (or repulsive) if there exists $\epsilon > 0$ such that for all $x_0 \in I$, $0 < |x_0 - x^*| \leq \epsilon$, there exists $n_0 \in \mathbb{R}$ such that for all $n \geq n_0$

$$|x_n - x^*| > \epsilon.$$

Definition 1.2.3 :

A periodique orbites $O(x_0)$ of period p of f is a source (or repulsive) if every point of the orbit is a source for the dynamical system generated by f^p .

Theorem 1.2.12 :[3]

Let I be an open interval and x^* a fixed point of a continuous function $f : I \rightarrow I$, suppose there exists $\epsilon > 0$ such that f is differentiable on the neighborhood $]x^* - \epsilon, x^* + \epsilon[$ of point x^* and the derivative is continuous at x^* , then $|\frac{d}{dx}f(x^*)| > 1$, implies x^* is repulsive.

Theorem 1.2.13 :[3]

Let I be an open interval and $\theta(x_0) = x_0, x_1, \dots, x_p$ a periodic orbit of period p of a continuous function $f : I \rightarrow I$, suppose there exists $\epsilon > 0$ such that f is differentiable on the neighborhood $]x^* - \epsilon, x^* + \epsilon[$ of each point x_j and the derivative is continuous at x_j for all $j = 1, 2, \dots, p - 1$, then

$$|\frac{d}{dx}f^p(x_0)| > 1 \tag{1.16}$$

implies $O(x_0)$ is repulsive.

Theorem 1.2.14 :[3]

Let I be an interval and $f : I \rightarrow I$ continuous with a fixed point x^* , suppose there exists $\epsilon > 0$ such that f is differentiable on $]x^* - \epsilon, x^* + \epsilon[$ except possibly at x^* and $|f'(x)| > 1$, then x^* is repulsive.

Global Attractors

Lemma 1.2.3 :

Let I be an interval and $h : I \rightarrow \mathbb{R}$ continuous on I and differentiable at every point in the interior of I , suppose $h'(x) \geq 0$, then h is increasing on I , if additionally h is zero only at finitely many points on every bounded subinterval $J \subset I$, then h is strictly increasing on I .

Theorem 1.2.15 :[3]

Let I be an interval and $h : I \rightarrow \mathbb{R}$ continuous on I and differentiable at every point in the interior of I , suppose:

- 1- f has a fixed point x^* .
- 2- $|f'(x)| \leq 1$ for all $x \in I$.
- 3- There exists an open interval $J \subset I$ containing $x^* \in J$, such that $|f'(x)| < 1$, for all $x \in J$, $x \neq x^*$.

Then x^* is the unique fixed point of f , and every orbit $\Theta(x_0)$ converges to x^*

1.3 Stability of Two-Dimensional Linear Dynamical Systems

The characterization of trajectories in two-dimensional autonomous linear systems provides the conceptual foundation for generalizing the analysis to higher-dimensional, nonlinear, nonautonomous systems.

Consider the following linear discrete dynamical system:

$$\begin{cases} x_{1,n+1} = a_{1,1}x_{1,n} + a_{1,2}x_{2,n}, \\ x_{2,n+1} = a_{2,1}x_{1,n} + a_{2,2}x_{2,n}, \quad \infty < x_1, x_2 < +\infty, \end{cases} \quad (1.17)$$

with initial condition $(x_{1,0}, x_{2,0})$, where $a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}$ are real constants, system (1.3) can be written in matrix form as:

$$\begin{pmatrix} x_{1,n+1} \\ x_{2,n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{1,n} \\ x_{2,n} \end{pmatrix} = A \begin{pmatrix} x_{1,n} \\ x_{2,n} \end{pmatrix} \quad (1.18)$$

où $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, is the only fixed point of the map f (if $\det A \neq 0$) defined by

$$f(x_1, x_2) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} a_{1,1}x_{1,n} + a_{1,2}x_{2,n} \\ a_{2,1}x_{1,n} + a_{2,2}x_{2,n} \end{pmatrix}.$$

A solution of the linear system (1.3), is a trajectory $\{X_n\}_{n \geq 0}$ where $X_n = \begin{pmatrix} x_{1,n} \\ x_{2,n} \end{pmatrix}$, for all n , the value of x_n at steps $1, 2, 3, \dots, n$, is

$$\begin{aligned} X_1 &= AX_0, \\ X_2 &= AX_1 = A^2X_0, \\ &\vdots \\ X_n &= AX_{n-1} = A^nX_0. \end{aligned}$$

Jordan Matrix

This subsection summarizes fundamental propositions of linear algebra used in the derivation and qualitative analysis of two-dimensional discrete dynamical systems.

Lemma 1.3.1 :

Let $A = (a_{i,j})$ be a 2×2 , matrix with $a_{i,j} \in \mathbb{R}$

- 1- If A has two distinct real eigenvalues $\{\lambda_1, \lambda_2\}$, then there exists a nonsingular 2×2 matrix Q and a diagonal matrix D such that $A = QDQ^{-1}$

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

and Q is an invertible 2×2 matrix whose columns are the eigenvectors of A .

- 2- If A has a repeated real eigenvalue $\lambda_1 = \lambda_2 = \lambda$ then there exists a nonsingular 2×2 matrix Q such that $A = QDQ^{-1}$ where

$$D = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$$

- 3- If A has two complex conjugate eigenvalues, $\lambda_{1,2} = a \pm ib$ and $w = u \pm iv$ are the associated eigenvectors, then there exists a nonsingular 2×2 matrix Q such that $A = QDQ^{-1}$ where

$$D = \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

and Q is an invertible 2×2 matrix, moreover $Q = (v, u)$.

Characterization of Solutions of a Linear Dynamical System

To characterize solutions of the two-dimensional system:

$$X_{n+1} = AX_n, \tag{1.19}$$

with $X \in \mathbb{R}^2$ we first consider the system:

$$Y_{n+1} = DY_n, \tag{1.20}$$

where $X = QY$ and $D = Q^{-1}AQ$, and deduce the solutions of (1.19) using $X = QY$ and $Y = \begin{pmatrix} x \\ y \end{pmatrix}$.

Distinct Real Eigenvalues

Consider the dynamical system:

$$Y_{n+1} = DY_n. \quad (1.21)$$

avec $D = \begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{pmatrix}$, when $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, with $Y_n = D^n Y_0$ where $D^n = \begin{pmatrix} \lambda_1^n & 0 \\ 1 & \lambda_2^n \end{pmatrix}$ i.e

$$\begin{cases} x_n = (\lambda_1)^n x_0, \\ y_n = (\lambda_2)^n y_0, \end{cases} \quad (1.22)$$

Here $(x_0, y_0)^t$ $(x, y)^t$ is the initial condition.

If $\lambda_1 > 0$ and $\lambda_2 > 0$, eliminating n gives a family of invariant curves:

$$y = cx^{\ln(\lambda_1)/(\ln \lambda_2)}, \quad c = \text{constant fixed par } x_0, y_0. \quad (1.23)$$

These are invariant under the transformation T defined by (1.21).

- * If $0 < \lambda_2 < \lambda_1 < 1$ these curves are parabolic (stable node).
- * If $0 < \lambda_2 < 1 < \lambda_1$ these curves are hyperbolic (saddle).
- * If λ_1, λ_2 one multiplier is negative or both are negative, the curves (1.3) are invariant under T^2 which has two positive multipliers.

1- If $|\lambda_2| < |\lambda_1| < 1, 0$ is an attractive fixed point (stable node):

- (a) Type 1: $\lambda_1 > 0$ and $\lambda_2 > 0$.
- (b) Type 2: $\lambda_1 > 0$ and $\lambda_2 < 0$, or $\lambda_1 < 0$ and $\lambda_2 > 0$.
- (c) Type 3: $\lambda_1 < 0$ and $\lambda_2 < 0$.

2- If $|\lambda_2| > |\lambda_1| > 1, 0$ is an unstable node (types 1, 2, 3 as above).

3- If $|\lambda_2| < 1, |\lambda_1| > 1$, 0 is an unstable saddle (types 1, 2, 3).

4- If $|\lambda_1| = |\lambda_2|$, 0 is a star node (attractive or repulsive):

(a) Type 1: $\lambda_1 = \lambda_2 > 0$.

(b) Type 2: $\lambda_1 = -\lambda_2$.

Repeated Real Eigenvalues

Consider system (1.21) with $D = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$ and $\lambda \in \mathbb{R}$.

Then $Y_n = D^n Y_0$, where $D^n = \begin{pmatrix} \lambda^n & 0 \\ n\lambda^{n-1} & \lambda^n \end{pmatrix}$ i.e

$$\begin{cases} x_n = \lambda^n x_0, \\ y_n = n\lambda^{n-1} x_0 + \lambda^n y_0. \end{cases} \quad (1.24)$$

For $\lambda > 0$ and $\frac{x}{x_0} > 0$ trajectories lie on invariant curves:

$$y = \frac{y_0}{x_0} x + \frac{\ln \frac{x}{x_0}}{\lambda \ln(\lambda)} x. \quad (1.25)$$

1- If $\lambda > 0$, 0 is an attractive improper node..

2- If $\lambda < 0$, 0 is a repulsive improper node.

Complex Eigenvalues

Consider system (1.21) with $D = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, and $a, b \in \mathbb{R}$, geometrically, $\lambda_{1,2} = a \pm ib = r \exp^{\pm i\theta}$, where $r = \sqrt{a^2 + b^2}$, $a = r \cos \theta$ and $b = r \sin \theta$, $0 < \theta < \pi$, then

$$D = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = r \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad (1.26)$$

$$D^n = r^n \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}, \quad (1.27)$$

the vector Y_n is given by:

$$\begin{aligned} \begin{pmatrix} x_n \\ y_n \end{pmatrix} &= r^n \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ &= r^n \begin{pmatrix} x_0 \cos n\theta - y_0 \sin n\theta \\ x_0 \sin n\theta + y_0 \cos n\theta \end{pmatrix} \\ &= r^n \sqrt{x_0^2 + y_0^2} \begin{pmatrix} \frac{x_0}{\sqrt{x_0^2 + y_0^2}} \cos n\theta - \frac{y_0}{\sqrt{x_0^2 + y_0^2}} \sin n\theta \\ \frac{x_0}{\sqrt{x_0^2 + y_0^2}} \sin n\theta + \frac{y_0}{\sqrt{x_0^2 + y_0^2}} \cos n\theta \end{pmatrix} \end{aligned}$$

where

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = r^n \rho_0 \begin{pmatrix} \cos(\varphi_0 + n\theta) \\ \sin(\varphi_0 + n\theta) \end{pmatrix} \quad (1.28)$$

where $\rho_0 = \sqrt{x_0^2 + y_0^2}$, $x_0 = \rho_0 \cos(\varphi_0)$ and $y_0 = \rho_0 \sin(\varphi_0)$

1- If $r = 1$,

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = r^n \rho_0 \begin{pmatrix} \cos(\varphi_0 + n\theta) \\ \sin(\varphi_0 + n\theta) \end{pmatrix}$$

or equivalently:

$$x_0^2 + y_0^2 = \rho_0^2, \quad (1.29)$$

he circles centered at 0 are invariant curves, the fixed point 0 is a center (stable but not asymptotically stable).

2- If $r \neq 1$, using polar coordinates $x = \rho \cos \varphi$, $y = \rho \sin \varphi$, the system becomes

$$\begin{aligned} \begin{pmatrix} \rho_{n+1} \cos(\varphi_{n+1}) \\ \rho_{n+1} \sin(\varphi_{n+1}) \end{pmatrix} &= r \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix} \begin{pmatrix} \rho_n \cos(\varphi_n) \\ \rho_n \sin(\varphi_n) \end{pmatrix} \\ &= \rho_n r \begin{pmatrix} \cos(\varphi_0 + n\theta) \\ \sin(\varphi_0 + n\theta) \end{pmatrix}, \end{aligned}$$

So

$$\begin{cases} \rho_{n+1} &= r\rho_n, \\ \varphi_{n+1} &= \varphi_n + \theta, \end{cases}$$

implies:

$$\begin{cases} \rho_n &= r^n \rho_0, \\ \varphi_n &= \varphi_0 + n\theta, \end{cases} \quad (1.30)$$

thus:

$$\rho = r^{(\varphi - \varphi_0)/\theta} \rho_0$$

or in Cartesian coordinates:

$$x_0^2 + y_0^2 = r^{2(\varphi - \varphi_0)/\theta} \rho_0, \quad (1.31)$$

the logarithmic spirals (1.3) are invariant curves. Moreover:

- (a) If $|r| < 1$, 0 is a stable focus.
- (b) If $|r| > 1$, 0 is an unstable focus.

Characterization of Solutions via Trace and Determinant

Consider the linear system $X_{n+1} = AX_n$, qualitative properties can be classified using $tr(A)$, $det(A)$, the eigenvalues solve the characteristic equation:

$$P(\lambda) = \lambda^2 - tr(A)\lambda + det(A) = 0.$$

1- If $(\text{tr}(A))^2 > 4\det(A)$, eigenvalues are real.

2- If $(\text{tr}(A))^2 < 4\det(A)$, eigenvalues are complex.

Proposition 1.3.1 :

Consider $X_{n+1} = AX_n$ and let $\lambda_{1,2}$ be the eigenvalues of A

1- If $(\text{tr}(A))^2 < 4\det(A)$ then $\lambda_{1,2}$ are real ($\lambda_1 > \lambda_2$)

(a) 0 is a saddle iff

$$\begin{cases} P(1) < 0 & \text{and} & P(-1) > 0, \\ & \text{or} & \\ P(1) > 0 & \text{and} & P(-1) < 0, \end{cases}$$

(b) 0 is a stable node iff

$$P(1) > 0 \quad \text{and} \quad P(-1) > 0$$

(c) 0 is an unstable node iff

$$P(1) < 0 \quad \text{and} \quad P(-1) < 0$$

2- If $(\text{tr}(A))^2 < 4\det(A)$ eigenvalues are complex.

(a) 0 is a stable focus if $\det(A) < 1$.

(b) 0 is an unstable focus if $\det(A) > 1$.

1.4 Stability of Two-Dimensional Nonlinear Discrete Dynamical Systems

A two-dimensional nonlinear discrete system is given by:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = f \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} f_1 \begin{pmatrix} x_n \\ y_n \end{pmatrix} \\ f_2 \begin{pmatrix} x_n \\ y_n \end{pmatrix} \end{pmatrix}$$

or

$$\begin{cases} x_{n+1} = f_1(x_n, y_n), \\ y_{n+1} = f_2(x_n, y_n), \end{cases} \quad (1.32)$$

where $f_1(x_n, y_n)$, $f_2(x_n, y_n)$ are continuous, single-valued, nonlinear functions of real variables (x, y) .

Fixed point

Nonlinear systems allow for multiple fixed points.

Let X^* be a fixed point of (1.32), if $f_1(x_n, y_n)$, $f_2(x_n, y_n)$ are at least once differentiable (class C_1) at $X^* = (x^*, y^*)$ a Taylor expansion gives:

$$\begin{cases} x_{n+1} = f_1(X^*) + \left(\frac{\partial f_1}{\partial x}\right)(X^*)(x_n - x^*) + \left(\frac{\partial f_1}{\partial y}\right)(X^*)(y_n - y^*) + \bar{X} \\ y_{n+1} = f_2(X^*) + \left(\frac{\partial f_2}{\partial x}\right)(X^*)(x_n - x^*) + \left(\frac{\partial f_2}{\partial y}\right)(X^*)(y_n - y^*) + \bar{Y} \end{cases}$$

when $\bar{X}, \bar{Y}$ are nonlinear terms.

When X is close to X^* nonlinear terms become negligible, and the system is approximated by:

$$\begin{cases} x_{n+1} - x^* = \left(\frac{\partial f_1}{\partial x}\right)(X^*)(x_n - x^*) + \left(\frac{\partial f_1}{\partial y}\right)(X^*)(y_n - y^*) \\ y_{n+1} - y^* = \left(\frac{\partial f_2}{\partial x}\right)(X^*)(x_n - x^*) + \left(\frac{\partial f_2}{\partial y}\right)(X^*)(y_n - y^*) \end{cases}. \quad (1.33)$$

After a change of variables $X' = X - X^*$ the fixed point moves to 0:

$$\begin{cases} x'_{n+1} = \left(\frac{\partial f_1}{\partial x}\right)(X^*)(x'_n) + \left(\frac{\partial f_1}{\partial y}\right)(X^*)(y'_n) \\ y'_{n+1} = \left(\frac{\partial f_2}{\partial x}\right)(X^*)(x'_n) + \left(\frac{\partial f_2}{\partial y}\right)(X^*)(y'_n) \end{cases}.$$

i.e

$$X'_{n+1} = JX'_n.$$

where

$$J = \begin{pmatrix} \left(\frac{\partial f_1}{\partial x}\right)(X^*) & \left(\frac{\partial f_1}{\partial y}\right)(X^*) \\ \left(\frac{\partial f_2}{\partial x}\right)(X^*) & \left(\frac{\partial f_2}{\partial y}\right)(X^*) \end{pmatrix}.$$

Theorem 1.4.1 [3]

- (a) *If all eigenvalues of J have modulus < 1 , the fixed point X^* of system (1.33) is locally asymptotically stable.*
- (b) *If J has at least one eigenvalue with modulus > 1 , X^* is unstable.*
- (c) *If some eigenvalues lie on the unit circle and others inside, stability cannot be concluded X^* .*

m-cycle

An m-cycle is a fixed point of f^m that is not a fixed point of f^k , for $1 \leq k < m$.

It consists of m points in the plane (x, y) and has two multipliers λ_1, λ_2 .

1.5 Bifurcation and Chaos

Chaos theory studies dynamical systems highly sensitive to initial conditions. Bifurcation theory analyzes changes in system behavior as parameters vary.

1.5.1 Bifurcation Theory

A solution of a dynamical system undergoes a bifurcation under parameter variation when it forms a boundary between two qualitatively different solution behaviors. Specifically, consider:

$$x_{n+1} = f(x_n, \mu), \tag{1.34}$$

where μ , $f(x, \mu)$ is a real parameter. If there exists a $\epsilon > 0$, as small as desired, such that the structure of the singularity of (1.34) $\mu < \mu_0$, and $\mu > \mu_0$, then μ_0 is a bifurcation point.

Fold Bifurcation

A catastrophic bifurcation where two cycles collide at $\mu = \mu_0$ with multiplier $S = 1$.

Flip (Period-Doubling) Bifurcation:

Occurs when a stable K -cycle has a multiplier crossing $S = 1$, the cycle becomes unstable, giving birth to a stable $2K$ -cycle.

Neimark-Sacker Bifurcation

Defined for systems of dimension ≥ 2 , characterized by the birth of a closed invariant curve from a fixed point when it loses stability via a pair of complex eigenvalues crossing the unit circle. It can be supercritical (stable curve) or subcritical (unstable curve).

1.5.2 The Chaos

Chaotic Dynamical Systems

Definition 1.5.1 :

$f : J \rightarrow J$ is topologically transitive if for every pair of open sets $U, V \subset J$ there exists $K > 0$ such that $f^K(U) \cap V \neq \emptyset$.

Definition 1.5.2 :

$f : J \rightarrow J$ has sensitivity to initial conditions if there exists $\delta > 0$ such that for every $x \in J$ and every neighborhood N_x there exists $y \in N_x$ and $n > 0$ such that $|f^n(x) - f^n(y)| > \delta$.

Definition 1.5.3 :

A map $f : V \rightarrow V$ is chaotic on V if:

- 1- f has sensitivity to initial conditions.
- 2- f is topologically transitive.
- 3- Periodic points are dense in V .

Attractors, Basins, and Boundaries

Definition 1.5.4 :

Let $U \subset \mathbb{R}$ be open, and $f : U \rightarrow U$, a closed bounded set $A \subset U$ is an attractor if $f(A) = A$ and there exists $r > 0$ such that $d(x_0, A) < r$ implies $\lim_{n \rightarrow +\infty} d(x_n, A) = 0$.

Attractors can be classified as:

- 1- Fixed points (single point).
- 2- Periodic orbits (finite set of points).
- 3- Chaotic attractors (strange attractors): All other types.

Definition 1.5.5 :

The basin of attraction of an attractor A is the set of all initial conditions x_0 , such that $\lim_{n \rightarrow +\infty} d(x_n, A) = 0$.

Lyapunov Exponents

In the analysis of a dynamical system the Lyapunov exponent quantifies the stability or instability of its trajectories, a Lyapunov exponent is either a real number, either is worth $+\infty$ or $-\infty$.

1- One-dimensional systems:

Let f be a map from $\mathbb{R}$ to $\mathbb{R}$, choose two very close initial conditions, x_0 and $x_0 + \epsilon$, and observe the behavior of the resulting trajectories, suppose they diverge on average at an exponential rate, then we can find a real number μ such that after n iterations

$$|f^n(x_{0+\epsilon}) - f^n(x_0)| \simeq \epsilon \exp(n\mu)$$

where

$$\log \left| \frac{f^n(x_{0+\epsilon}) - f^n(x_0)}{\epsilon} \right| \simeq \mu n,$$

taking $\epsilon \rightarrow 0$ we find:

$$\mu \simeq \log \left| \frac{df^n(x_0)}{dx_0} \right|,$$

finally, letting $n \rightarrow \infty$ and using the chain rule:

$$\frac{d}{dx} f^n(x_0) = \prod_{i=0}^{n-1} f'(x_i)$$

we find:

$$\mu = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log|f'(x_i)| \quad (1.35)$$

μ is called the Lyapunov exponent, by definition the Lyapunov exponent characterizes the stability of an orbit $O(x_0)$, if $\mu > 0$ the orbit $O(x_0)$ is unstable, if $\mu < 0$, it is stable.

2- Two-dimension system:

Let us now present a generalization of the Lyapunov exponent concept to the multi-dimensional case, let $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ and

$$x_{n+1} = f(x_n) \quad (1.36)$$

be the dynamical system defined by f .

First, note that an m-dimensional system will have m Lyapunov exponents $\mu_i = 1, 2, \dots, m$, for each $n > 1$, consider the Jacobian matrix of the map f^n (denoted $J_n(x_0)$) evaluated at a point x_0 , similar to the 1-dimensional case, we have

$$J_n(x_0) = \prod_{i=1}^n J(x_i) \quad (1.37)$$

where $x_i = f^i(x_0)$ are points of the orbit $O(x_0)$ and $J(x)$ is the Jacobian matrix of the map f evaluated at point x , to compute the $\mu_i = 1, 2, \dots, m$, we consider:

$$f^n(x_0 + \epsilon) = f^n(x_0)$$

set $x' = x_0 + \epsilon$ and perform a first-order Taylor expansion of $f^n(x_0)$ arrond x'_0 :

$$x_n - x'_n = J_n(x_0)(x_0 - x'_0),$$

If $J_n(x_0)$ is diagonalizable, then there exists an invertible matrix P_t such that: $D_n(x_0) = P_t^{-1} J_n(x_0) P_t$, where $D_n(x_0)$ is a diagonal matrix containing the eigen $J_0(x_0)$, denote these eigenvalues by θ_i , $i = 1, \dots, M$, the m Lyapunov exponents are then defined as follows:

$$\mu_i = \lim_{n \rightarrow +\infty} \frac{1}{n} \ln(\theta_i^n), \quad i = 1, \dots, m.$$

Fractal Dimension

1- Hausdorff Dimension:

Let $A \subset \mathbb{R}^d$, we cover the set A with a countable union of subsets denoted A_i , each having a diameter less than r , for any non-negative real number s , we introduce the quantity:

$$H_r^s = \inf_{\text{diam}(A_i) < r} \left\{ \sum_{i=1}^{\infty} \text{diam}(A_i)^s / A \subset \bigcup_{i=1}^{\infty} A_i \right\}$$

the function H_r^s is decreasing which guarantees the existence of a limit as r tends to 0, hence the definition:

$$H^s(X) = \lim_{r \rightarrow 0} H_r^s$$

H^s is called the s dimensional Hausdorff measure, the Hausdorff dimension relies on the Hausdorff measure, the Hausdorff dimension of $A \subset \mathbb{R}$ is defined by:

$$D_H = \sup \{s, H^s(A) = +\infty\} = \inf \{s, H^s(A) = 0\}$$

where $H^s(A)$ is the Housdorff measure of the set A

2- Correlation Dimension:

The idea of the capacity dimension is as follows, suppose A is a bounded subset of $\mathbb{R}^n$, choose $\epsilon > 0$ and select hypercubes of side length ϵ , Let $N(\epsilon)$ be the minimum number of such cubes needed to cover A , defined

$$D_c = \lim_{\epsilon \rightarrow 0} \frac{\ln(N(\epsilon))}{\ln(\epsilon)}$$

D_c is called the capacity dimension of the set A

3- Carrelation Demension:

Let $O(x_1)$ be a trajectory of a dynamical system, the correlation dimension of the set $O(x_1)$ is calculated as follows:

Given a positive real number r , we form the correlation intefral given by:

$$C(r) = \lim_{n \rightarrow +\infty} \frac{1}{n^2 - n} \sum_{i \neq j}^n H(r||x_i - x_j||) \quad (1.38)$$

where

$$H(x) = \begin{cases} 0 & \text{si } x < 0 \\ 1 & \text{si } x \geq 0 \end{cases}, \quad (1.39)$$

is the Heaviside step function $\mathbb{R}$.

Definition 1.5.6 :

The corrlation dimension D_{Cor} of $O(x_1)$ is defined by:

$$D_{Cor} = \lim_{r \rightarrow 0} \frac{\ln(C(r))}{\ln(r)}. \quad (1.40)$$

4- Kaplan-Yorke Dimension:

Let $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$ be the n Lyapunov exponents of an attractor of a dynamica system, and let j be the largest natural integer such that:

$\mu_1 + \mu_2 + \dots \geq \mu_j$, then Karlan-York dimension is given by:

$$D_{KY} = \frac{\sum_{i=1}^j \mu_i}{|\mu_{j+1}|}$$

si aussi j existe alors $D_{KY} = 0$, et si l'attracteur est chaotique alors $D_{KY} > 0$ est un nombre fractionnaire.

Route to chaos

1- Period-Doubling Cascade:

This rout describes the transition of a dynamical system from a stat where it

reaches a equilibrium state to a chaotic region through a sequence of period-doubling bifurcations, as the bifurcation parameter μ increases, the stable fixed point is replaced at $\mu = \mu_0$ by a stable 2-cycle.

Then, at $\mu = \mu_1$, this cycle loses its stability and is replaced by a stable 4-cycle, and so on the periodic orbit of period 2^j observed for $\mu < \mu_j$ loses its stability at $\mu = \mu_j$ and is replaced by a stable periodic orbit of thed 2^{j+1} , the increasing sequences $(\mu_j)_{j \geq 0}$ of bifurcation values tends towards a limit μ_c , at $\mu = \mu_c$, a chaotic region appears, the bifurcation values $(\mu_j)_{j \geq 0}$ are specific to the system under consideration, as is their limit value μ_c , however, their accumulation near μ_c follows a geometric progression:

$$\lim_{j \rightarrow +\infty} \frac{\mu_{j+1} - \mu_j}{\mu_{j+2} - \mu_{j+1}} = \delta, \quad (1.41)$$

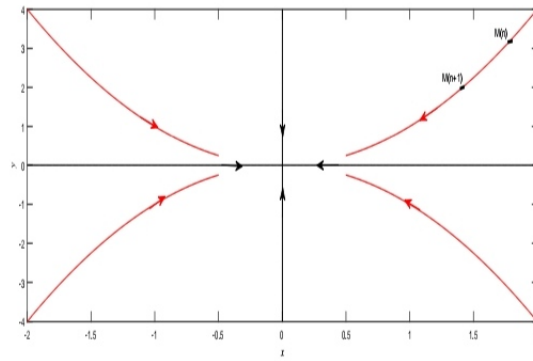
where δ is a universal number: $\delta = 466920\dots$ this means that the exponent δ is identical in all systems where such a sequence of periode doublings leading to chaos is observed, a qualitative similarity in asymptotic behaviors implies a quantitative identity.

2- Intermittency:

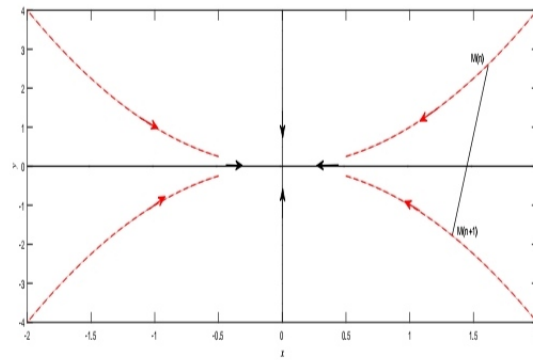
The route called intermittency desc ribes the persistence of regular and predictable phases within globally chaotic dynamics, the main idea is that after the disappearance of a stable fixed point x_μ^* via a saddle-node bifurcation at $\mu = 0$, the dynamics remain slow in the vicinity of x_μ^* as if sensing the ghost of the fixed point.

3- Ruelle-Takens Scenario:

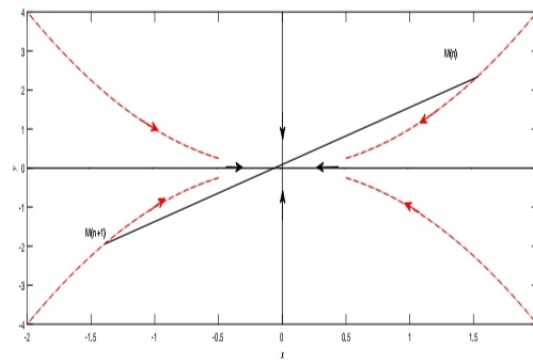
The exact statemebt of this route is quite technical, as it requires introducing a topology on the space of vector fields to define the proximity of two continuous dynamical systems, its approximate formulation is, a continuous dynamical systems undergoing three successive Hopf bifurcations generally possesses a strabger attractor, each Hopf bifurcation correponds to the appearance of an unstable mode, the above sattement can be rephrased.



(a)

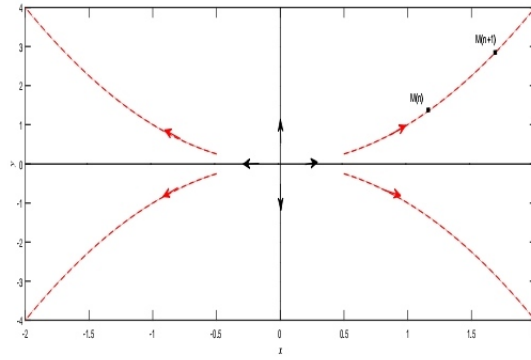


(b)

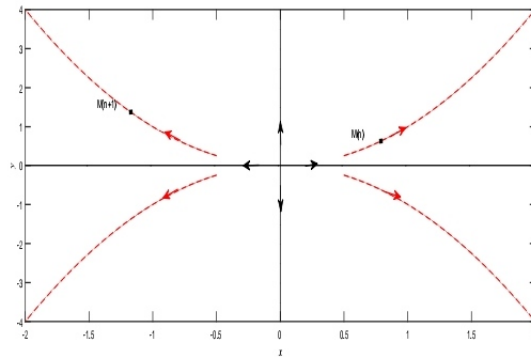


(c)

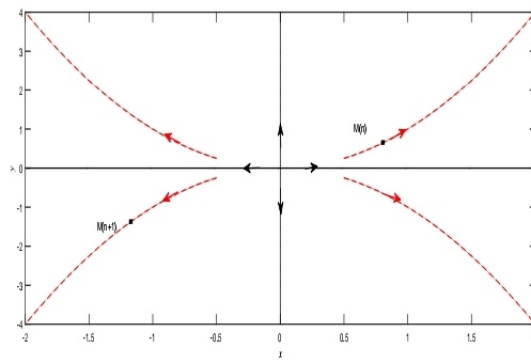
Figure 1.1: Phase portrait of (1.20) if $|\lambda_1| < |\lambda_2| < 1$ (a). Type 1 stable node $0 < \lambda_2 < \lambda_1$. (b) Type 2 stable node $0 < \lambda_1 < 0$ et $\lambda_2 < 0$. (c) Type 3 stable node $\lambda_1 < 0$ and $\lambda_2 < 0$.



(a)

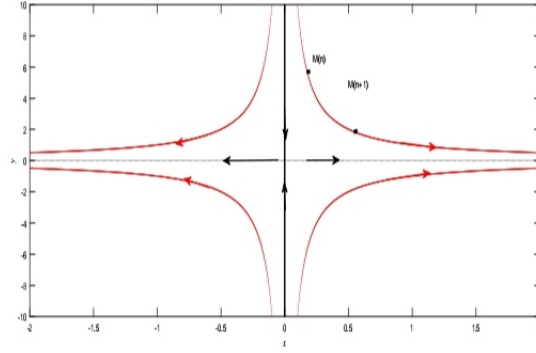


(b)

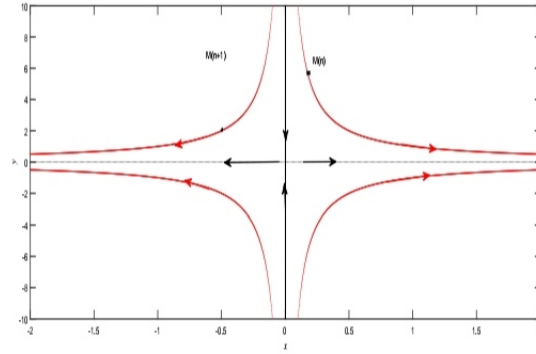


(c)

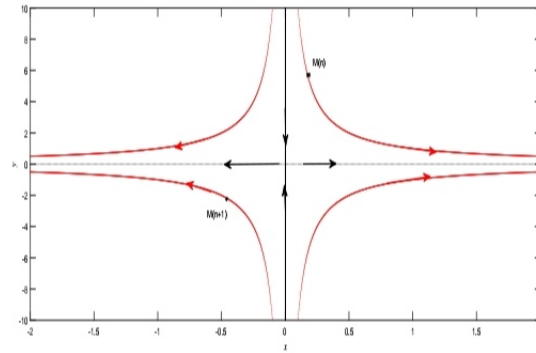
Figure 1.2: Phase portrait of (1.20) if $|\lambda_1| < |\lambda_2| < 1$ (a). Type 1 unstable node $0 < \lambda_2 < \lambda_1$. (b) Type 2 unstable node $0 < \lambda_1 < 0$ et $\lambda_2 < 0$. (c) Type 3 stable node $\lambda_1 < 0$ and $\lambda_2 < 0$.



(a)



(b)



(c)

Figure 1.3: Phase portrait of (1.20) if $|\lambda_1| < 1, |\lambda_2| > 1$ (a). Type 1 col $0 < \lambda_2, \lambda_1 > 0$. (b) Type 2 col $0 < \lambda_1 > 0$ and $\lambda_2 < 0$. (c) Type 3 col $\lambda_1 < 0$ and $\lambda_2 < 0$.

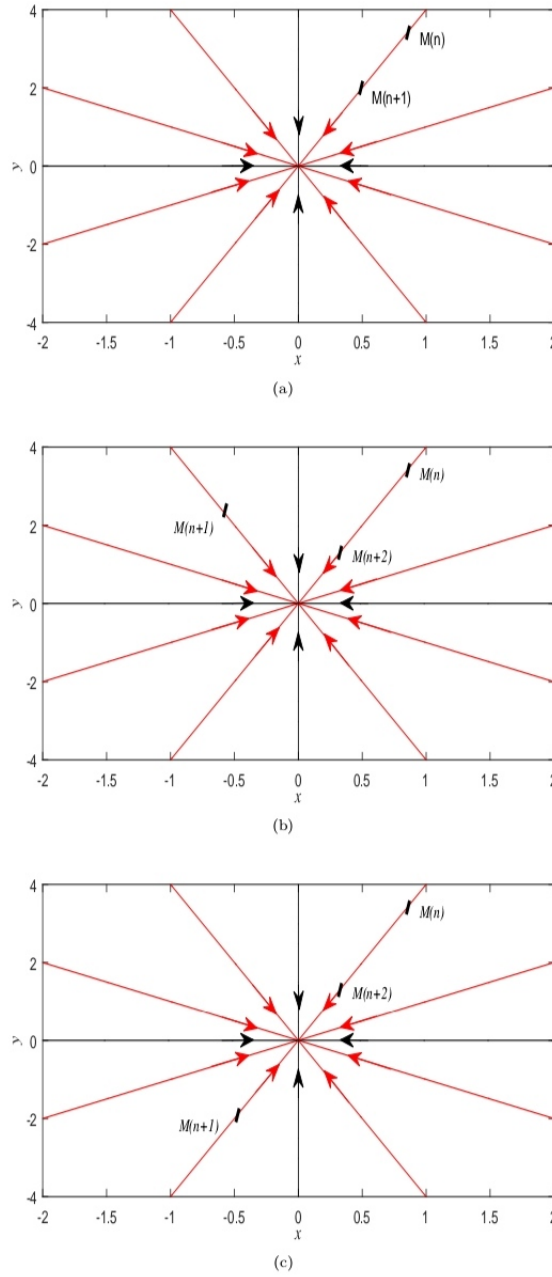
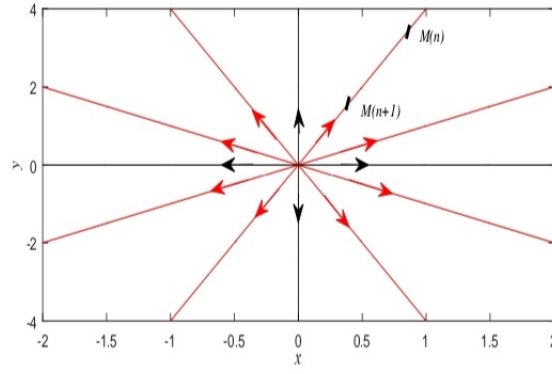
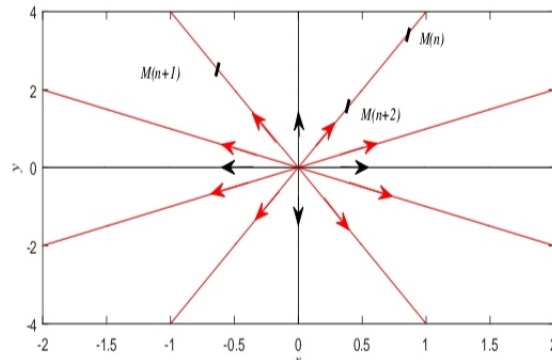


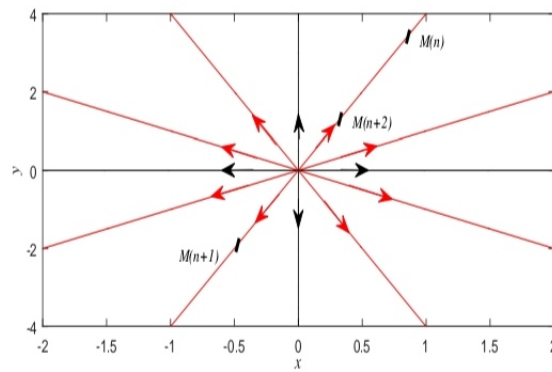
Figure 1.4: Phase portrait of (1.20) if $|\lambda_1| = |\lambda_2| < 1$ (a). Type 1 stable star node $0 < \lambda_2 = \lambda_1$. (b) Type 2 stable star node $\lambda_1 = \lambda_2$. (c) Type 3 stable star node $\lambda_1 = \lambda_2 < 0$.



(a)

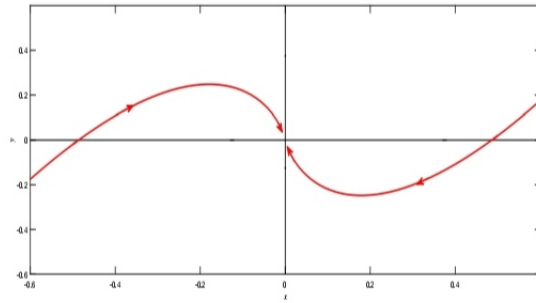


(b)

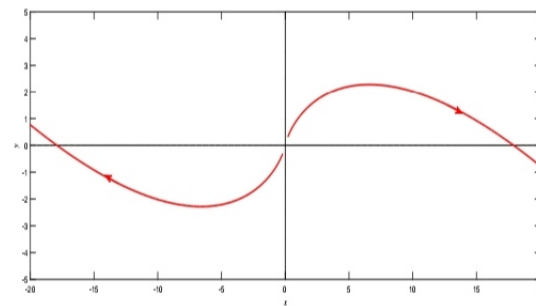


(c)

Figure 1.5: Phase portrait of (1.20) if $|\lambda_1| = |\lambda_2| < 1$ (a). Type 1 unstable star node $0 < \lambda_2 = \lambda_1$. (b) Type 2 unstable star node $\lambda_1 = \lambda_2$. (c) Type 1 unstable star node $\lambda_1 = \lambda_2 < 0$.



(a)



(b)

Figure 1.6: Phase portrait of (1.20) if $\lambda_1 = \lambda_2 = \lambda$ (a) Stable improper node $0 < \lambda < 1$. (b) Unstable improper node $1 < \lambda < \infty$.

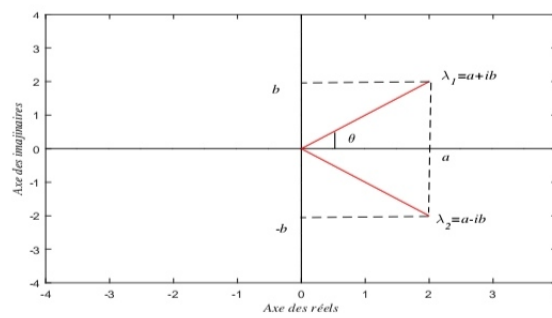
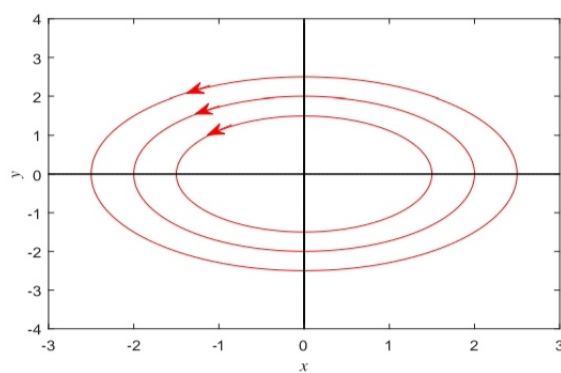
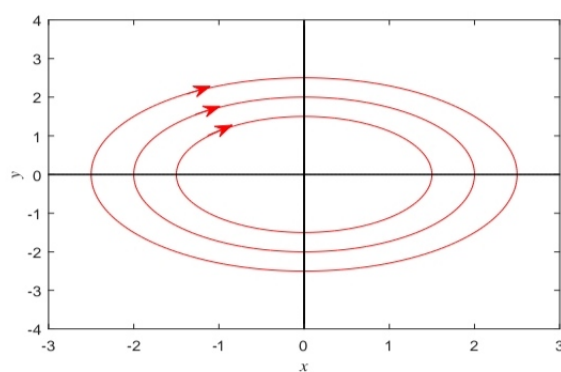


Figure 1.7: Geometric representation of the eigenvalues $\lambda_{1,2} = a \pm i = r \exp^{\pm i\theta}$.



(a)



(b)

Figure 1.8: Fixed point center $r = 1$, (a) $b > 0$, (b) $b < 0$.

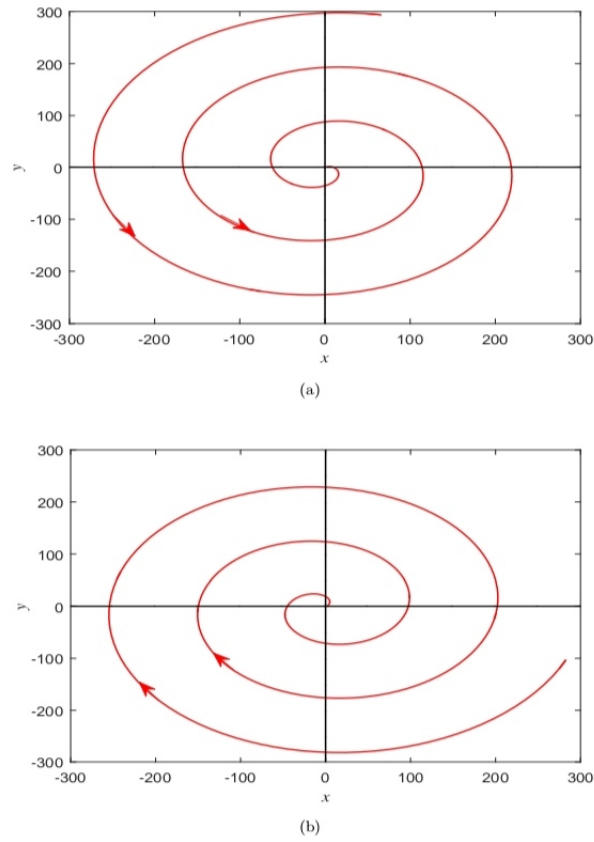


Figure 1.9: Stable focus, (a) $b > 0$, (b) $b < 0$.

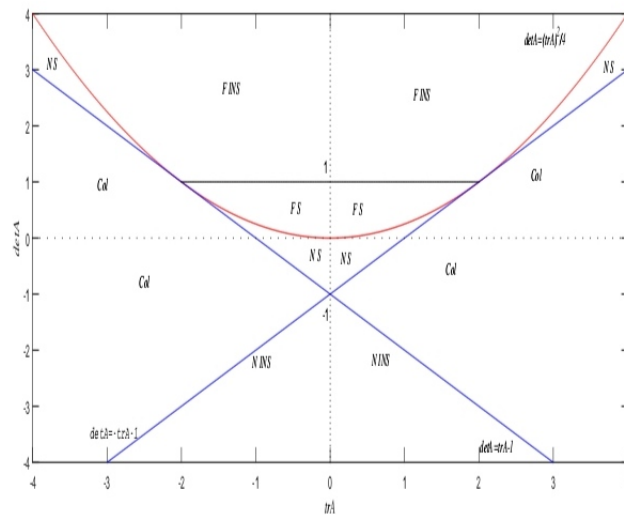


Figure 1.10: Characterization of solutions as a function of $tr(A)$ and $det(A)$.

CHAPTER 2

BIFURCATION ANALYSIS OF THREE DIMENSIONAL HENON MAP

The well known Henon map with the constant Jacobian determinant was proposed by Henon as a simplified model to the Poincare section of the Lorenz system [14]. This map is a two-dimensional discrete-time system with a single quadratic non-linearity and exhibits chaotic behavior for the parametric values $a = 1.4$ and $b = 0.3$, since the late 1970, the Hénon map has been extensively studied because of its generality and wide applications.

In this chapter, we study the dynamics of a generalized three-dimensional Hénon map, necessary and sufficient conditions for the existence and stability of the fixed points of this system are established, by applying the center manifold theorem and bifurcation theory, we show that the system exhibits Fold, Flip, and Neimark-Sacker bifurcations under certain conditions.

2.1 Three-dimensional Henon map

Consider the three-dimensional Henon map:

$$\begin{cases} x_{n+1} = 1 + y_n - ax_n^2 \\ y_{n+1} = bx_n + z_n \\ z_{n+1} = -bx_n \end{cases}, \quad (2.1)$$

where $(a, b \in \mathbb{R})$ and $(a, b \neq 0)$, let $(\bar{u} = ax)$, and $(\bar{v} = \frac{a}{b}y)$ et $(\bar{w} = \frac{a}{b}z)$, it reduces to :

$$\begin{cases} \bar{u}_{n+1} = a - \bar{u}_n^2 + b\bar{v}_n \\ \bar{v}_{n+1} = \bar{u}_n + \bar{w}_n \\ \bar{w}_{n+1} = -\bar{u}_n \end{cases},$$

for simplicity, we still use x, y and z instead of $\bar{u}, \bar{v}$ and $\bar{w}$, respectively (2.1) becomes:

$$\begin{cases} x_{n+1} = a + by_n - x_n^2 \\ y_{n+1} = x_n + z_n \\ z_{n+1} = -x_n \end{cases}, \quad (2.2)$$

it is easy to see that, when $a > \frac{1}{4}$, system(2.1) has two fixed points:

$$p_1(x_1, 0, -x_1) \quad \text{and} \quad p_2(x_2, 0, -x_2),$$

where $x_1 = \frac{-1 + \sqrt{1+4a}}{2a}$ and $x_2 = \frac{-1 - \sqrt{1+4a}}{2a}$, when a tends $-\frac{1}{4}$, these two fixed points approach to the point $p_0\left(-\frac{1}{2}, 0, \frac{1}{2}\right)$. The Jacobian matrix J_i of system (2.1) at the fixed point $p_i (i = 1, 2)$ is given by:

$$\begin{pmatrix} -2x_i & b & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

and the characteristic equation of the Jacobian matrix J_i is:

$$\lambda^3 + 2x_i\lambda^2 - b\lambda + b = 0. \quad (2.3)$$

Denote $\lambda = \frac{r+1}{r-1}$ The above equation (??) changes to:

$$c_0r^3 + c_{12}r^2 + c_2r + c_3 = 0, \quad (2.4)$$

where $c_0 = 1 + 2x_i$, $c_1 = 3 - 2b + 2x_i$, $c_2 = 3 + 4b - 2x_i$, $c_3 = 1 - 2b - 2x_i$, obviously, $|\lambda| < 1$ is equivalent to $(Re(r) < 0)$, let

$$\Delta_1 = c_1, \quad \Delta_2 = \begin{vmatrix} c_1 & c_0 \\ c_3 & c_2 \end{vmatrix} = c_1c_2 - c_0c_3 \quad \text{and} \quad \Delta_3 = \begin{vmatrix} c_1 & c_0 & 0 \\ c_3 & c_2 & c_1 \\ 0 & 0 & c_3 = c_3\Delta_2 \end{vmatrix} = c_1.$$

According to the Routh-hurwitz theorem $(Re(r) < 0)$ if and only if:

$$c_j > 0, (j = 0, 1, 2, 3) \quad \text{and} \quad \Delta_k > 0, (K = 1, 2, 3).$$

Which is equivalent to:

$$\begin{cases} x_i > -\frac{1}{2}, & x_i > b - \frac{3}{2}, & x_i < 2b + \frac{3}{2}, & x_i < \frac{1}{2} - b, \\ 2bx_i - b^2 + b + 1 > 0. \end{cases}$$

Since $\left(x_2 = \frac{-1 - \sqrt{1+4a}}{2} < \frac{1}{b}\right)$, the fixed point $p_2(x_2, 0, -x_2)$ is always unstable.

For la stability of the fixed point $p_1(x_1, 0, -x_1)$, substituting $\left(x_1 = \frac{-1 + \sqrt{1+4a}}{2}\right)$ into (2.1), we get:

$$\begin{cases} -1 < b < -\frac{1}{3}, \\ \sqrt{1+4a} < \frac{b^3-1}{b} \end{cases} \quad \text{or} \quad \begin{cases} -\frac{1}{3} \leq b < 1, \\ \sqrt{1+4a} < 2(1-b). \end{cases}$$

Notice that the existence of the fixed point p_1 is assured under the condition $a > -\frac{1}{4}$. The fixed point P_1 is stable if and only if the following conditions are satisfied:

$$\begin{cases} -1 < b < -\frac{1}{3}, \\ -\frac{1}{4} < a < \left(\frac{b^{2-1}}{2b}\right)^2 - \frac{1}{4}, \end{cases} \quad \text{or} \quad \begin{cases} -\frac{1}{3} \leq b < 1, \\ -\frac{1}{4} < a < (1-b)^2 - \frac{1}{4}. \end{cases}$$

Based on the above analysis, we have the following Lemma:

Lemma 2.1.1 :

The following statements are true:

- 1- The fixed point $p_2(x_2, 0, -x_2)$ is unstable.
- 2- The fixed point $p_1(x_1, 0, -x_1)$ is stable if and only if one of the following conditions holds:
 - i) either $\left(-1 < b < -\frac{1}{3}\right)$ and $(a_0 < a < a_h)$,
 - ii) $\left(-\frac{1}{3} \leq b < 1\right)$ and $(a_0 < a < a^*)$, where $(a_0 = -\frac{1}{4})$, $(a_h = \left(\frac{b^{2-1}}{2b}\right)^2 - \frac{1}{4})$, and $(a^* = (1-b)^2 - \frac{1}{4})$.

2.2 Codimension one bifurcations

In this section, we investigate the fold bifurcation, Flip bifurcation and Neimark-Sacker bifurcation at the fixed point of system (2.1), We choose the coefficient a as a bifurcation parameter for analyzing the fold bifurcation, Flip bifurcation and Neimark-Sacker bifurcation by means of the center Manifold Theorem and Bifurcation Theory.

2.2.1 Bifurcation de Pli

Consider the fold bifurcation at the unique fixed point (p_0) of system (2.1), i.e $(a = a_0 = -\frac{1}{4})$, the eigenvalues of the Jacobian matrix at p_0 are $(\lambda_1 = 1)$ and $(\lambda_{2,3} = \pm \sqrt{b})$, assume

($b \neq \pm 1$), gives $|\lambda_{2,3}| \neq 1$ let:

$$\bar{x} = x + \frac{1}{2}, \quad \bar{y} = y, \quad \bar{z} = z - \frac{1}{2} \quad \text{and} \quad \bar{a} = a + \frac{1}{4}.$$

We translate the fixed point p_0 to the origin and take a as a new dependent variable, le system (2.1) becomes

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \\ \bar{a} \end{pmatrix} \mapsto \begin{pmatrix} 1 & b & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \\ \bar{a} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} f_1(\bar{x}, \bar{y}, \bar{z}, \bar{a}) \quad (2.5)$$

where $f_1(\bar{x}, \bar{y}, \bar{z}, \bar{a}) = -\bar{x}^2$.

For $b > 0$, we find an invertible matrix by making use of corresponding eigenvectors of the 4 matrix in the map (2.5):

$$T_1 = \begin{pmatrix} -1 & -1 & \sqrt{b} & -\sqrt{b} \\ 0 & -1 & -1 - \frac{1}{\sqrt{b}} & -1 + \frac{1}{\sqrt{b}} \\ 1 & 0 & 1 & 0 \\ 0 & b-1 & 0 & 0 \end{pmatrix}.$$

Use the transformation of $(\bar{x}, \bar{y}, \bar{z}, \bar{a})^t = T_1(u, v, w, \mu)^t$, the map (2.5) can be rewritten as

$$\begin{pmatrix} u \\ v \\ w \\ \mu \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\sqrt{b} & 0 \\ 0 & 0 & 0 & \sqrt{b} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ \mu \end{pmatrix} + \begin{pmatrix} \frac{1}{v-1} \\ 0 \\ 1 \\ \frac{1}{2(1+\sqrt{b})} \\ \frac{1}{2(1-\sqrt{b})} \end{pmatrix} g_1(u, v, w, \mu), \quad (2.6)$$

where $g_1(u, v, w, \mu) = -(-u - v + \sqrt{bw} - \sqrt{b\mu})^2$.

We know that there exists a center manifold $W^c(0)$ around the origin, which can be

expressed as follows:

$$W^c(0) = (u, v, \omega, \mu) \mid w = h_1(u, v), \mu = h_2(u, v), h_i(0, 0) = 0, Dh_i(0, 0) = 0, i = 1, 2,$$

where

$$h_1(u, v) = \alpha_1 u^2 + \beta_1 uv + \gamma_1 v^2 + O((|u| + |v|)^3),$$

and

$$h_2(u, v) = \alpha_2 u^2 + \beta_2 uv + \gamma_2 v^2 + O((|u| + |v|)^3).$$

Since the center manifold must satisfy

$$\begin{cases} h_1\left(u + v + \frac{g_1(u, v, h_1(u, v), h_2(u, v))}{b-1}\right) + \sqrt{b}h_1(u, v) - \frac{g_1(u, v, h_1(u, v), h_2(u, v))}{2(1+\sqrt{b})} = 0, \\ h_2\left(u + v + \frac{g_1(u, v, h_1(u, v), h_2(u, v))}{b-1}\right) + \sqrt{b}h_2(u, v) - \frac{g_1(u, v, h_1(u, v), h_2(u, v))}{2(1+\sqrt{b})} = 0. \end{cases}$$

By comparing the corresponding coefficients of the above two equations, we have

$$\begin{cases} \alpha_1 = -\frac{1}{(1+\sqrt{b})^2}, & \beta_1 = -\frac{\sqrt{a}}{(1+\sqrt{b})^3}, & \gamma_1 = -\frac{b-\sqrt{b}}{2(1+\sqrt{b})^4}, \\ \alpha_2 = -\frac{1}{(\sqrt{b}-1)^2}, & \beta_2 = -\frac{\sqrt{a}}{(\sqrt{b}-1)^3}, & \gamma_2 = -\frac{b-\sqrt{b}}{2(\sqrt{b}-1)^4}. \end{cases}$$

Consider the map on the center manifold given by

$$F_1 : u \mapsto u + v + \frac{1}{1-b}(u^2 + 2uv + v^2) + O((|u| + |v|)^3). \quad (2.7)$$

It is easy to see that

$$F_1(0, 0) = 0, F_{1w}(0, 0) = 1, F_{1v}(0, 0) = 1, F_{1uv}(0, 0) = \frac{2}{1-b} \neq 0.$$

Hence, the fixed point $(u, v) = (0, 0)$ is a fold bifurcation point for the map (2.7).

for $b = 0$, let

$$T_1 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

By using the transformation of $(\bar{x}, \bar{y}, \bar{z}, \bar{a})^t = T_1(u, v, w, \mu)^t$, the map (2.5) becomes

$$\begin{pmatrix} u \\ v \\ w \\ \mu \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\sqrt{b} & 0 \\ 0 & 0 & 0 & \sqrt{b} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ \mu \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tilde{g}_1(u, v, w, \mu) \quad (2.8)$$

, where $\tilde{g}_1(u, v, w, \mu) = -(v - u)^2$.

To simplify the map (2.8) on the center manifold, we assume that the center manifold is of the form

$$\widetilde{W}^c(0) = (u, v, \omega, \mu) \mid w = \tilde{h}_1(u, v), \mu = \tilde{h}_2 = (u, v), \tilde{h}_i(0, 0) = 0, D\tilde{h}_i(0, 0) = 0, i = 1, 2,$$

where

$$\tilde{h}_1(u, v) = \tilde{\alpha}_1 u^2 + \tilde{\beta}_1 uv + \tilde{\gamma}_1 v^2 + O((|u| + |v|)^3),$$

and

$$\tilde{h}_2(u, v) = \tilde{\alpha}_2 u^2 + \tilde{\beta}_2 uv + \tilde{\gamma}_2 v^2 + O((|u| + |v|)^3).$$

Using an approximate computation on the center manifold and equating the corresponding coefficients, we get

$$\begin{cases} \tilde{\alpha}_1 = -1, & \tilde{\beta}_1 = -2, & \tilde{\gamma}_1 = -1, \\ \tilde{\alpha}_2 = -1, & \tilde{\beta}_2 = 0, & \tilde{\gamma}_2 = 0. \end{cases}$$

Thus, the map restricted to the center manifold is given by

$$\widetilde{F}_1 : u \mapsto u - v + (u - v)^2 + O((|u| + |v|)^3), \quad (2.9)$$

From the map(2.9), we find

$$\widetilde{F}_1(0,0) = 0, \widetilde{F}_{1w}(0,0) = 1, \widetilde{F}_{1v}(0,0) = -1, \widetilde{F}_{uv}(0,0) = -2 < 0,$$

This implies that the fixed point $(u, v) = (0,0)$ is a fold bifurcation point for the map (2.9).

Pour $b = -s^2 < 0$ (s is a non zero real number), let

$$T_1 = \begin{pmatrix} -1 & -1 & -s & 0 \\ 0 & -1 & -\frac{1}{s} & -1 \\ 1 & 0 & 0 & 1 \\ 0 & -(1+s^2) & 0 & 0 \end{pmatrix}.$$

By using the transformation of $(\bar{x}, \bar{y}, \bar{z}, \bar{a})^t = T_1(u, v, w, \mu)^t$, the map(2.5) becomes

$$\begin{pmatrix} u \\ v \\ w \\ \mu \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -s \\ 0 & 0 & s & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ \mu \end{pmatrix} + \frac{1}{1+s^2} \begin{pmatrix} -1 \\ 0 \\ -s \\ 1 \end{pmatrix} \widetilde{g}_1(u, v, w, \mu) \quad (2.10)$$

, where $\widetilde{g}_1(u, v, w, \mu) = -(-u - v - sw)^2$.

Applying the center manifold theorem again, we know that there exists a center manifold for (2.10) which can be written as

$$\widetilde{W}^c(0) = (u, v, w, \mu) \mid w = \widetilde{h}_1(u, v), \mu = \widetilde{h}_2 = (u, v), \widetilde{h}_i(0,0) = 0, D\widetilde{h}_i(0,0) = 0, i = 1, 2,$$

where

$$\widetilde{h}_1(u, v) = \widetilde{\alpha}_1 u^2 + \widetilde{\beta}_1 uv + \widetilde{\gamma}_1 v^2 + O((|u| + |v|)^3),$$

and

$$\widetilde{h}_2(u, v) = \widetilde{\alpha}_2 u^2 + \widetilde{\beta}_2 uv + \widetilde{\gamma}_2 v^2 + O((|u| + |v|)^3).$$

After substitution, by a straightforward computation on equating the corresponding coefficients, it gives

$$\begin{cases} \widetilde{\alpha}_1 = \frac{2s}{(1+s^2)^2}, & \widetilde{\beta}_1 = \frac{-2s(1-3s^2)}{(1+s^2)^3}, & \widetilde{\gamma}_1 = \frac{s-10s^2+5s^3}{s^2(1+s^2)^4}, \\ \widetilde{\alpha}_2 = \frac{-1+s^2}{(1+s^2)^2}, & \widetilde{\beta}_2 = \frac{-2s^2(3-s^2)}{(1+s^2)^3}, & \widetilde{\gamma}_2 = \frac{s^2(5-10s^2+s^4)}{(1+s^2)^4}. \end{cases}$$

Thus, the map restricted to the center manifold is given by

$$\widetilde{F}_1 : u \mapsto u + v + \frac{1}{1+s^2}(u-v)^2 + O((|u| + |v|)^3), \quad (2.11)$$

since

$$\widetilde{F}_1(0,0) = 0, \widetilde{F}_{1w}(0,0) = 1, \widetilde{F}_{1v}(0,0) = 1, \widetilde{F}_{uv}(0,0) = \frac{2}{1+s^2} > 0,$$

the fixed point $(u, v) = (0, 0)$ is a fold bifurcation point of the map (2.11).

Based on the above discussions, we obtain the following results immediately.

Theorem 2.2.1 :

System (2.1) undergoes a fold bifurcation at the fixed point P_0 if the conditions $b \neq \pm 1$ and $a = a_0$ hold, moreover, two fixed points bifurcate from p_0 .

Moreover, two fixed points bifurcate from P_0 for $a > a_0$, coalesce as the fixed point P_0 at $a = a_0$, and disappear for $a = a_0$.

2.2.2 Flip bifurcation

In this subsection, we consider the Flip bifurcation occurring at the fixed point P_1 of system (2.1), when $a = a^*$, the fixed point $P_1 = (\widehat{x}, \widehat{y}, \widehat{z})$, where $\widehat{x} = \frac{1}{2} - b$, $\widehat{y} = 0$, and $\widehat{z} = -\widehat{x}$, the eigenvalues of the associated Jacobian matrix at the fixed point P_1 sont $\lambda_1 = -1$ and $\lambda_{2,3} = b \pm c$, where $c = \sqrt{b^2 - b}$, the condition $|\lambda_{2,3}| \neq 1$ means $b \neq -\frac{1}{3}$, hence, we assure that $b \neq -\frac{1}{3}$ and $b \neq 1$ in the following discussion.

Let

$$\bar{x} = x - \widehat{x}, \quad \bar{y} = y - \widehat{y}, \quad \bar{z} = z - \widehat{z}, \quad \bar{a} = a - \widehat{a}.$$

We translate the fixed point P_1 into the origin and take $\bar{a}$ as a variable, then system (2.1) can be rewritten as

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \\ \bar{a} \end{pmatrix} \mapsto \begin{pmatrix} 2b-1 & b & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \\ \bar{a} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} f_2(\bar{x}, \bar{y}, \bar{z}, \bar{a}), \quad (2.12)$$

Where $f_2(\bar{x}, \bar{y}, \bar{z}, \bar{a}) = -\bar{x}^2$. In the case of $b \neq 0$, we get an invertible 4×4 matrix by making use of corresponding eigenvectors of the in the map (2.12):

$$\begin{pmatrix} 1 & -\frac{1}{2(b-1)} & \frac{c-b}{b} & \frac{-b-c}{b} \\ -2 & 0 & -\frac{(c-b)(c+b-1)}{b} & -\frac{(c+b)(c-b+1)}{b} \\ 1 & \frac{1}{2(b-1)} & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Use the transformation $(\bar{x}, \bar{y}, \bar{z}, \bar{a})^t = T_2(u, v, w, \mu)^t$, the map (2.12) becomes:

$$\begin{pmatrix} u \\ v \\ w \\ \mu \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b-c & 0 \\ 0 & 0 & 0 & b+c \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ \mu \end{pmatrix} + \frac{1}{2(1+3b)c} \begin{pmatrix} 2c \\ 0 \\ 2b-c \\ -2b-c \end{pmatrix} g_2(u, v, w, \mu), \quad (2.13)$$

where $g_2(u, v, w, \mu) = -\left[u + \frac{1}{2(1-b)v} + (c-b)w - (c+b)\mu \right]$.

Similarly, we reduce the map (2.13) on the center manifold, which can be represented as :

$$W^c(0) = (u, v, \omega, \mu) \mid w = h_3(u, v), \mu = h_4 = (u, v), h_i(0, 0) = 0, Dh_i(0, 0) = 0, i = 3, 4,$$

for u and v being sufficiently small, assume that the center manifold is of the form:

$$h_3(u, v) = \alpha_3 u^2 + \beta_3 uv + \gamma_3 v^2 + O((|u| + |v|)^3),$$

and

$$h_4(u, v) = \alpha_4 u^2 + \beta_4 uv + \gamma_4 v^2 + O((|u| + |v|)^3).$$

So the center manifold must satisfy

$$\begin{cases} h_3\left(-u + \frac{g_2(u, v, h_3(u, v), h_4(u, v))}{1 + 3b}, v\right) - (b - c)h_3(u, v) - \frac{(2b - c)g_2(u, v, h_3(u, v), h_4(u, v))}{2(1 + 3b)c} = 0 \\ h_4\left(-u + \frac{g_2(u, v, h_3(u, v), h_4(u, v))}{1 + 3b}, v\right) - (b - c)h_4(u, v) + \frac{(2b - c)g_2(u, v, h_3(u, v), h_4(u, v))}{2(1 + 3b)c} = 0 \end{cases}$$

By comparing the coefficients of u^2 , uv and v^2 in the above two equations, respectively, we get :

$$\begin{aligned} \alpha_3 &= \frac{b - c}{2(c - 1 + b)(2(b - c)c)}, & \beta_3 &= \frac{b - c}{2(c - 1 + b)(1 + b - c)c}, \\ \gamma_3 &= \frac{b - c}{8(c - 1 + b)(1 - b + c)(1 + b - c)c}, \end{aligned}$$

then, the map (2.13) restricted to the central variety can be expressed as:

$$u \mapsto F_2(u, v), \tag{2.14}$$

where

$$\begin{aligned} F_2(u, v) &= -u - \frac{1}{1 + 3b}u^2 - \frac{1}{(1 - b)(1 + 3b)}uv - \frac{1}{4(1 + 3b)(b - 1)^2}v^2 + \\ &\quad \frac{4b}{(1 - b)(1 + 3b)^2}u^3 + \frac{2b(-3 + 7b)}{(b - 1)^2(1 + 3b)^3}u^2v - \frac{b(-3 + 7b)}{(-1 - 2b + 3b^2)^3}uv^2 + \\ &\quad \frac{b}{2(b - 1)^4(1 + 3b)^2}v^3 - O((|u| + |v|)^4), \end{aligned}$$

in order for the map (2.14) to undergo a Flip bifurcation, we require that two discriminatory quantities k_1 and k_2 are not zero, where

$$k_1 = \left(\frac{\partial F_2}{\partial v} \cdot \frac{\partial^2 F_2}{\partial u^2} + 2 \frac{\partial^2 F_2}{\partial u \partial v} \right) \Big|_{(0,0)} = \frac{2}{(1-b)(1+3b)} \neq 0,$$

and

$$k_2 = \left(\frac{1}{2} \cdot \frac{\partial^2 F_2}{\partial u^2} + 2 \frac{\partial^2 F_2}{\partial u^3} \right) \Big|_{(0,0)} = \frac{2}{(1-b)(1+3b)} \neq 0.$$

in the cas of $b = 0$, we take the invertible matrix:

$$T_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

, use the transformation $(\bar{x}, \bar{y}, \bar{z}, \bar{a})^t = T_1(u, v, w, \mu)^t$, then the map (2.12) can be rewritten as:

$$\begin{pmatrix} u \\ v \\ w \\ \mu \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ \mu \end{pmatrix} + \frac{1}{2(1+3b)c} \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix} \tilde{g}_2(u, v, w, \mu), \quad (2.15)$$

where $\tilde{g}_2(u, v, w, \mu) = -(u+v)^2$.

We reduce the map (2.15) on a center manifold which can be expressed as

$$\widetilde{W}^c(0) = (u, v, \omega, \mu) \mid w = \tilde{h}_3(u, v), \mu = \tilde{h}_4 = (u, v), h_i(0, 0) = 0, Dh_i(0, 0) = 0, i = 3, 4,$$

Assume that $\tilde{h}_3(u, v), \tilde{h}_4 = (u, v)$ are of the following forms, respectively:

$$\tilde{h}_3(u, v) = \tilde{\alpha}_3 u^2 + \tilde{\beta}_3 uv + \tilde{\gamma}_3 v^2 + O((|u| + |v|)^3),$$

and

$$\tilde{h}_4(u, v) = \tilde{\alpha}_4 u^2 + \tilde{\beta}_4 uv + \tilde{\gamma}_4 v^2 + O((|u| + |v|)^3).$$

By a direct computation on the center manifold, we find

$$\begin{aligned}\widetilde{\alpha}_3 &= -1, & \widetilde{\beta}_3 &= 6, & \widetilde{\gamma}_3 &= -1 \\ \widetilde{\alpha}_4 &= 1, & \widetilde{\beta}_4 &= -2, & \widetilde{\gamma}_4 &= 1.\end{aligned}$$

Thus, the map (2.15) restricted to the center manifold is given by

$$\widetilde{F}_2 : (u, v) \mapsto -u - (u^2 + 2uv + v^2) + O((|u| + |v|)^2). \quad (2.16)$$

Note that

$$k_1 = \left(\frac{\partial \widetilde{F}_2}{\partial v} \cdot \frac{\partial^2 \widetilde{F}_2}{\partial u^2} + 2 \frac{\partial^2 \widetilde{F}_2}{\partial u \partial v} \right) \Big|_{(0,0)} = -4 \neq 0,$$

and

$$k_2 = \left(\frac{1}{2} \cdot \left(\frac{\partial^2 \widetilde{F}_2}{\partial u^2} \right)^2 + \frac{1}{3} \frac{\partial^3 \widetilde{F}_2}{\partial u^3} \right) \Big|_{(0,0)} = 2 \neq 0.$$

Thus, the fixed point $(u, v) = (0, 0)$ (2.16).

In addition, if $-\frac{1}{3} < b < 1$, then $|\lambda_{2,3}| < 1$ and $k_2 > 0$, we summarize our discussions into the following results.

Theorem 2.2.2 :

The system (2.1) undergoes a Flip bifurcation at the fixed point P_1 when $b \neq -\frac{1}{3}$ and $a = a^*$.

Moreover, if $-\frac{1}{3} < b < 1$ ($b < -\frac{1}{3}$ or $b > 1$) holds, then the period-2 orbits that bifurcate from the fixed point P_1 are stable (unstable).

2.2.3 Neimark-Sacker bifurcation

Now, we consider the Neimark-Sacker bifurcation of system (2.1) at the fixed point P_1 , let $a = a_h$ where a_h is given in Lemma (2.1.1).

If $\Delta = -(3b^2 - 2b - 1) < 0$, the characteristic equation (2.3) at the point P_1 has a pair of complex conjugate roots $\lambda_{1,2} = \frac{1 + b \pm i\sqrt{3b^2 - 2b - 1}}{2b} = \rho \pm i\omega$ and a real root $\lambda = -b$ with $|\lambda_{1,2}| = 1$ and $d = \frac{d|\lambda_{1,2}(a_h)|}{da} > 0$, the fixed point turns into $P_1 = (\widetilde{x}, \widetilde{y}, \widetilde{z})$, where $\widetilde{x} = \frac{b^2 - b - 1}{2b}$, $\widetilde{y} = 0$ et $\widetilde{z} = -\widetilde{x}$.

In addition, for a Neimark-Sacker bifurcation, it requires that

$$\lambda_{1,2}^n(a_h) \neq 1 \quad (n = 1, 2, 3, 4) \quad \text{and} \quad |\lambda_3| = |-b| \neq 1,$$

which implies that

$$b \neq -\frac{1}{3}, \quad -\frac{1}{2}, \quad -1, 1.$$

Note that $\Delta = -(3b^2 - 2b - 1) < 0$ is equivalent to $b < -\frac{1}{3}$ or $b > 1$. Hence, we consider the case of $b < -\frac{1}{3}$ ($b \neq -1, -\frac{1}{2}$) or $b > 1$ in this subsection.

Let $\bar{x} = x - \bar{x}$, $\bar{y} = y - \bar{y}$ and $\bar{z} = z - \bar{z}$, then the fixed point $(\bar{x}, \bar{y}, \bar{z})$ can be translated to the origin, and system (2.1) becomes

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} \mapsto \begin{pmatrix} \frac{-b^2 + b + 1}{b} & b & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} f_2(\bar{x}, \bar{y}, \bar{z}), \quad (2.17)$$

where $f_2(\bar{x}, \bar{y}, \bar{z}) = -\bar{x}^2$.

Using the corresponding eigenvectors of the 3 matrix in the map (2.17), we get an invertible matrix:

$$T = \begin{pmatrix} \frac{-\sqrt{3b^2 - 2b - 1}}{b} & \frac{-1 + b}{2b} & b \\ \frac{-\sqrt{3b^2 - 2b - 1}}{b} & \frac{1 - b}{2b} & \frac{b - 1}{2b} \\ 0 & 1 & 1 \end{pmatrix}.$$

By making use of the transformation $(\bar{x}, \bar{y}, \bar{z})^t = T_1(u, v, w)^t$, the map (2.17) becomes

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \mapsto \begin{pmatrix} \rho & -\omega & 0 \\ \omega & \rho & 0 \\ 0 & 0 & -b \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \frac{b}{2 + b + b^2} \begin{pmatrix} \frac{b}{\sqrt{2 + b + b^2}} \\ -1 \\ 1 \end{pmatrix} g(u, v, w), \quad (2.18)$$

where $g(u, v, w) = -(-wu + \rho v + bw)^2$.

We simplify the map (2.18) on the center manifold by assuming that the manifold is of

the form :

$$W^c(0) = (u, v, \omega) \parallel w = h(u, v), h(0, 0) = 0, Dh(0, 0) = 0$$

Let

$$h(u, v) = \alpha u^2 + \beta uv + \gamma v^2 + O((|u| + |v|)^3).$$

Substituting it into (2.18) yields

$$h \left(\rho - \omega v \frac{b(3+b)g(u, v, h(u, v))}{(2+b+b^2)(\sqrt{3b^2-2b-1})} \omega u + \rho v \frac{bg(u, v, h(u, v))}{(2+b+b^2)} + bh(u, v) - \frac{bg(u, v, h(u, v))}{2+b+b^2} = 0 \right).$$

Equating the corresponding coefficients to zero gives

$$\alpha = \frac{b(-1+b)^2(1+3b^3)}{4(2+9b+9b^2+6b^3+4b^4+b^5+b^6)}, \quad \beta = \frac{b(1+b)\sqrt{3b^2-2b-1}}{2(2+7b+2b^2+4b^3+b^5)},$$

$$\gamma = \frac{b(3+11b+b^2+b^3)}{4(2+9b+9b^2+6b^3+4b^4+b^5+b^6)}.$$

Thus, the map (2.18) restricted to the center manifold is given as follows

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \mapsto \begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} f^1(u, v) \\ g^1(u, v) \end{pmatrix}. \quad (2.19)$$

Where

$$\begin{aligned}
 f^1(u, v) = & \frac{(-1+b)(3+b)(1+3b)u^2}{4b(2+b+b^2)\sqrt{3b^2-2b-1}} + \frac{(3+b)(1+b)^2v^2}{4b(2+b+b^2)\sqrt{3b^2-2b-1}} \\
 & + \frac{(3+b)(1+b)uv}{2b(2+b+b^2)} + \frac{(-1+b)^2b^2(3+b)(1+3b)u^3}{4(2+b+b^2)^2(1+4b+2b^2+b^4)} \\
 & + \frac{b^2(-1+b)(3+b)(1+3b)^2u^2v}{4(2+b+b^2)^2(1+3b-b^2+b^3)\sqrt{3b^2-2b-2}} \\
 & + \frac{b^2(3+b)(3+11b+b^3+b^3)v^3}{4(2+b+b^2)^2(1+3b-b^2+b^3)\sqrt{3b^2-2b-2}} \\
 & + \frac{b^2(3+b)(1+3b)(5+2b+b^2)uv^2}{4(2+b+b^2)^2} \\
 & + O((|u|+|v|)^4),
 \end{aligned}$$

and

$$\begin{aligned}
 g^1(u, v) = & \frac{(-1+b)(1+3b)}{4b(2+b+b^2)u^2} + \frac{(1+b)\sqrt{3b^2-2b-2}}{2b(2+b+b^2)}uv + \frac{(1+b)^2}{4b(2+b+b^2)}v^2 \\
 & + \frac{(-1+b)^2b^2(1+3b)\sqrt{3b^2-2b-2}}{4(b+1)(2+b+b^2)^2(1+3b-b^2+b^3)}u^3 \\
 & + \frac{b^2(1+3b)(5+2b+b^2)\sqrt{3b^2-2b-2}}{4(b+1)(2+b+b^2)^2(1+3b-b^2+b^3)}uv^2 \\
 & + \frac{(-1+b)^2b^2(1+3b)^2}{4(2+b+b^2)^2(1+3b-b^2+b^3)}u^2v \\
 & + \frac{b^2(3+11b+b^2+b^3)}{4(2+b+b^2)^2(1+3b-b^2+b^3)}v^3 \\
 & + O((|u|+|v|)^4).
 \end{aligned}$$

In order for the map (2.19) to undergo a Neimark-Sacker bifurcation, we require that the following discriminatory quantity l_1 is not equal to zero

$$l_1 = -\operatorname{Re} \left(\frac{(1-2\lambda)\bar{\lambda}^2}{1-\lambda} \xi_{11}\xi_{20} \right) - \frac{1}{2}|\xi_{11}|^2 - |\xi_{02}|^2 + \operatorname{Re}(\bar{\lambda}\xi_{21}),$$

where the sign of l_1 determines the stability of the invariant circle, and

$$\begin{aligned}\xi_{20} &= \frac{1}{8}[(f_{uu}^1 - f_{vv}^1 + 2g_{uv}^1) + i(g_{uu}^1 - g_{vv}^1 + 2f_{uv}^1)], \\ \xi_{11} &= \frac{1}{4}[(f_{uu}^1 + f_{vv}^1) + i(g_{uu}^1 + g_{vv}^1)], \\ \xi_{02} &= \frac{1}{8}[(f_{uu}^1 - f_{vv}^1 - 2g_{uv}^1) + i(g_{uu}^1 - g_{vv}^1 - 2f_{uv}^1)], \\ \xi_{21} &= \frac{1}{16}[(f_{uuu}^1 - f_{uvv}^1 + g_{uuu}^1 - g_{vvv}^1)] + i(g_{uuu}^1 - g_{uvv}^1 + f_{uuu}^1 - f_{vvv}^1).\end{aligned}$$

By a direct calculation, we get

$$l_1 = \frac{b^2(1 + 2b - 2b^2 + b^3)}{2(b^2 - 1)(2 + b + b^2)^2(1 + 3b - b^2 + b^3)}.$$

Note that $-1 < b < 1$, then $|\lambda_3| < 1$, by the fact that $b < -\frac{1}{3}$ ($b \neq -1, -\frac{1}{2}$), it follows that $l_1 < 0$ for $-1 < b < -\frac{1}{3}$ and $b \neq -\frac{1}{2}$.

Consequently, we summarize our result as the following theorem. :

Theorem 2.2.3 :

Suppose that $b < -\frac{1}{3}$ ($b \neq -1, -\frac{1}{2}$) or ($b > 0$), then the system (2.1) undergoes a Neimark-Sacker bifurcation at the fixed point P_1 for $a = a_h$.

Moreover, if $-1 < b < -\frac{1}{3}$ and $b \neq -\frac{1}{2}$, then the bifurcation is supercritical and an attracting invariant closed curve bifurcates from the fixed point for $a > a_h$.

In Figure 1, it shows the stability region and bifurcation region of system (2.1) in the parametric space.

2.3 Numerical simulations

In this section, numerical simulations are provided, including bifurcation diagrams, maximum Lyapunov exponents and phase portraits, to illustrate consistency of our theoretical results, and to show the rich complex dynamics of system (2.1)

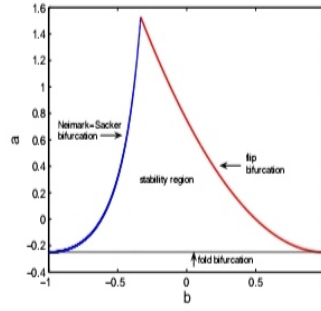


Figure 2.1: The stability region and bifurcation region of system (2.1) in the (b, a) -plane.

2.3.1 Numerical simulations of stability and codimension one bifurcations

Based on Lemma (2.1.1) and Theorem (2.2.1), we consider the bifurcation parameters in the following three cases.

cas 1: Let $b = -0.6$, and $a \in (-0.3, 0, 4)$ be the bifurcation parameter, then $a^* = 2.31$ and $a_h \sim 0.0344$, From Figure 2 (A), we observe that if $a < a_0$, there has no fixed point; if $a = a_0$, there is only one fixed point P_0 , if $a > a_0$, there are two fixed points $P_1(x_1, 0, -x_1)$ and $P_2(x_2, 0, -x_2)$ bifurcating from P_0 .

It shows that the number of the fixed points varies as the value of the parameter a changes. We also find that x_2 is unstable x_1 is stable for $a \in (a_0, a_h)$ in Figure 2 (A). This agrees well with Lemma (2.1.1) and Theorem (2.2.1). Furthermore, we see that the fixed point P_1 loses its stability at $a_h \sim 0.0344$ and an attracting invariant closed curve bifurcates from the fixed point P_1 for $a > a_h$, This is in good agreement with Theorem (2.2.3).

cas 2: Let $b = 0.4$, and $a \in (-0.4, 1)$ be the bifurcation parameter, then $a^* = 0.11$ and $P_1(0.1, 0, -0.1)$, by a straightforward computation, the eigenvalues of the corresponding Jacobian matrix at the fixed point P_1 are $\lambda_1 = -1$ and $\lambda_{2,3} = 0.4 \pm 0.4898979i$, with $k_1 = -1.5151515$ and $k_2 = 1.5151515$, On the basis of Lemma (2.1.1) and Theorem (2.2.2), we know that the fixed point P_1 is stable for $a \in (a_0, a^*)$ and loses its stability when $a > a^*$, the Flip bifurcation occurs at $a = a^*$, and the

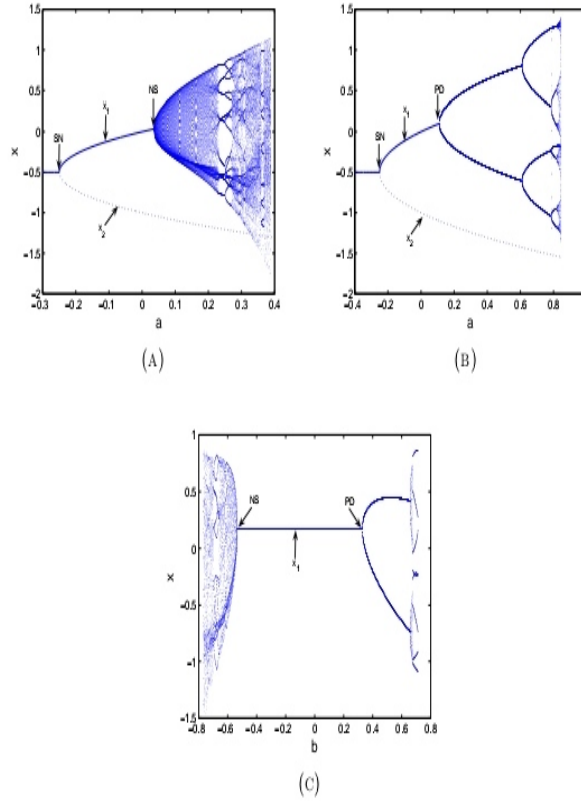


Figure 2.2: (A)-(B) bifurcation diagrams of system (2.1) in the plan (a, x) : (A) ($b = -0.6$), and (B) ($b = 0.4$); (C) bifurcation diagrams of system (2.1) in the plan (b, x) avec ($b \in (-0.8, 0.8)$) and ($a = 0.2$), here, the fold bifurcation, Flip bifurcation and Neimark-Sacker bifurcation are labeled as 'SN', 'PD' and 'NS', respectively

period-2 orbits that bifurcate from P_1 are stable, All these phenomena are clearly presented in Figure 2 (B).

cas 3: Let $a = 0.2$ and $b \in (-0.5333, 0.3292)$, be the bifurcation parameter. According to Lemma (2.1.1), is stable for $b \in (0.5333, 0.3292)$, from Theorem (2.2.2), the Flip bifurcation occurs at P_1 for $b \sim -0.5333$, from Theorem (2.2.3), the supercritical Neimark-Sacker bifurcation occurs at P_1 for $b \sim 0.5333$. All these phenomena are clearly presented in Figure 2 (C) which agree well with Lemma (2.1.1) and Theorem (2.2.1)-(2.2.3).

2.3.2 More numerical simulations of system (2.1)

In this subsection, we show that new complex dynamical behaviors change as the parameters of system (2.1) vary by using the bifurcation diagrams, maximum Lyapunov exponents and phase portraits.

The bifurcation diagrams in the three-dimensional (a, b, x) space are shown in Figure 3. For the bifurcation diagrams in the two-dimensional space, we consider the bifurcation parameters in the following three cases:

- (i) changing a in the range $0 \leq a \leq 0.4$, and fixing $b = -0.6$.
- (ii) changing a in the range $0.1 \leq a \leq$, and fixing $b = 0.4$.
- (iii) changing a in the range $-0.8 \leq a \leq 0.8$, and fixing $a = 0.23$.

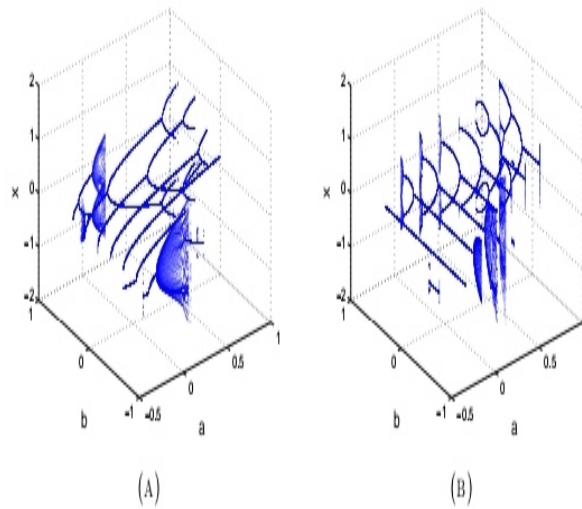


Figure 2.3: Bifurcation diagrams of system (2.1) in the threedimensional (a, b, x) space.

cas(i): The bifurcation diagram of system (2.1) in the plan (a, x) for $(b = -0.6)$ and $(- \leq a \leq 0.4)$ with the initial value $(0.03429, 0, -0.03429)$ is demonstrated in Figure 4 (A). The maximum Lyapunov exponents corresponding to Figure 4 (A) are shown in Figure 4 (B). The local amplified bifurcation diagram of Figure 4 (A) for $a \in (0.22, 0.32)$ is presented in Figure 4 (C), from Figures 4 (A)-(C), we observe

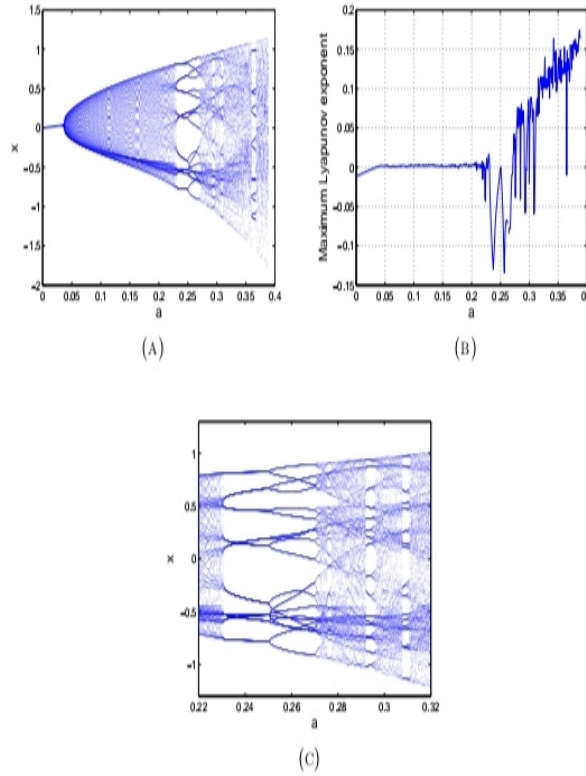


Figure 2.4: (A) Bifurcation diagrams of system (2.1) in the plan (a, x) ($a \in (0, 0.4)$) for $(b = -0.6)$; (B) maximum Lyapunov exponent corresponding to (A); (C) the local amplified bifurcation diagram of (A) for $(a \in (0.22, 0.32))$

that there is a stable fixed point for $(a \sim 0.0344)$, and the fixed point loses its stability at $a \sim 0.0344$.

The Neimark-Sacker bifurcation occurs at $a \sim 0.0344$ and an attracting invariant cycle appears for $a > 0.0344$. Then, we can see the process of the period-doubling bifurcation which is from the period-10 orbits to chaos and these chaotic regions are interspersed with several periodic windows, for example, we can observe orbits of period-17 for $a = 0.294$, period-34 for $a = 0.31$, and period-14 for $a = 0.361$, Phase portraits for various values of a are shown in Figure 5, which clearly depicts how a smooth invariant curve bifurcates from the stable fixed point and an invariant curve to chaotic attractors.

From Figure 5, we observe that there are an attracting invariant circle, quasi-

periodic orbits which are routines to chaos, period-20 orbits, ten-coexisting chaotic attractors, and chaotic sets. In particular, Figures 6 (A)-(C) show chaotic attractors for $(a = 0,385)$ dans les plans (x, y) , (x, z) and (y, z) respectivement.

Furthermore, from Figure 4 (B), one can see that the maximum Lyapunov exponents are negative for the parameter $a \in (0, 0,0344)$ when the fixed point is stable, for $a \in (0,0344, 0,23)$, the maximum Lyapunov exponents are in the neighborhood of zero, which correspond to quasi-period orbits or coexistence of chaos and quasiperiod orbits, for $a \in (0,23, 0,4)$, the majority of maximum Lyapunov exponents are positive with a few being negative, which shows that there exist stable fixed points or stable period windows in the chaotic region.

cas (ii): The bifurcation diagram of system (2.1) in the plan (a, x) for $(b = 0,4)$ and $(0, 1 \leq a' \leq 1)$ with the initial value $(0,101, 0, -0,101)$ is shown in Figure 7(A) and the maximum Lyapunov exponents corresponding to Figure 7 (A) are shown in Figure 7 (B).

Figure 7 (C) shows the local amplified bifurcation diagram and Figure 7 (D) shows the maximum Lyapunov exponents corresponding to Figure 7 (C), Figure 7 clearly depicts the period-doubling bifurcation being a routine to chaos; from Figures 7 (A) and (C), we can see that the fixed point is stable for $a < 0,11$, loses its stability at $a = 0,11$ and then a cascade of period 2, 4, 8, ... orbits emerge, from Figures 7 (B) and (D), we observe that there exist positive Lyapunov exponents when a exceeds 0.11, which indicates that the chaotic sets arise, this means that the system tends to a stable state at the very beginning, then circulates along period-doubling orbits, and finally follows irregular chaotic sets with some period-2 orbits, for example, the chaotic behavior suddenly disappears for $a \sim 0,8446$, and then the chaotic behavior suddenly appears for $a \sim 0,8447$ and converges to period-2 orbits for $a \sim 0,845$, Phase portraits of four-coexisting chaotic attractors for $a = 0.835$ and a chaotic attractor for $a = 0.8445$ are illustrated in Figures 7 (E) and (F), respectively.

cas (iii) The bifurcation diagram of system (2.1) in (b, x) plan for $(a = 0,23)$ is displayed in

Figure 8 (A). The maximum Lyapunov exponents corresponding to Figure 8 (A) are calculated and shown in Figure 8 (B), which indicates the existence of chaotic regions and periodic orbits as the parameter b varies, figures 8 (C) and (E) present the local amplications of Figure 8 (A) and the corresponding maximum Lyapunov exponents are shown in Figures 8 (D) and (F), respectively.

From Figures 8 (D) and (F), we can see when $(b < -0,59)$ or $(b > 0,67)$, some Lyapunov exponents are greater than 0 and some are smaller than 0, so there exist the stable fixed points or stable periodic windows in the chaotic region.

Figure 8 clearly depicts that there are period-14, -10, -13, -16 windows and invariant cycles in the chaotic regions, and two onsets of chaos at $(b \sim -0,74)$ and $(b \sim 0,67)$, respectively, from Figures 8 (C) and (E), the following phenomena are observed: the period-windows within the chaotic regions and the period-doubling bifurcations from period-2 orbits to chaos, Figures 9 (A) and (B) show two nice chaotic attractors at $(b = -0,72)$ and $(b = -0,6)$, respectively, Figure 9 (C) shows an invariant circle for $(b = -0,53)$, Furthermore, Figures 9 (D)-(F) exhibit the processes from period-doubling bifurcations to chaos corresponding to Figure 8 (E), in which two invariant circles at $(b = 0,66)$ are presented in Figure 9 (D), eight-coexisting chaotic attractors at $(b = 0,675)$ are demonstrated in Figure 9 (E), and a chaotic attractor at $(b = 0,691)$ is shown in Figure 9 (F).

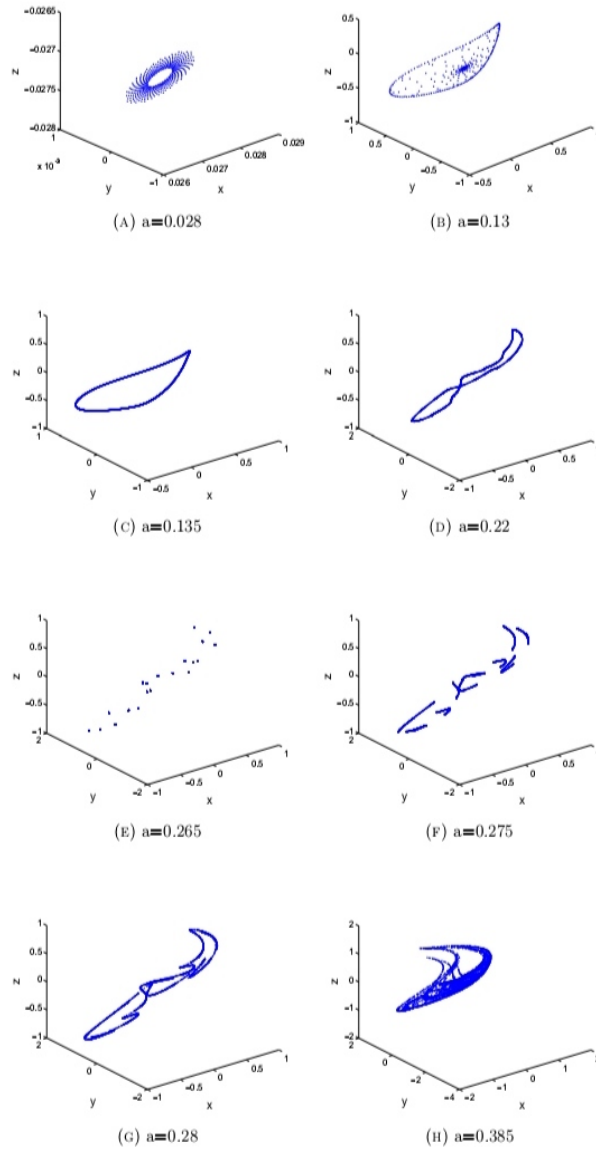


Figure 2.5: (A)-(H) phase portraits for various values of a corresponding to Figure 4 (A).

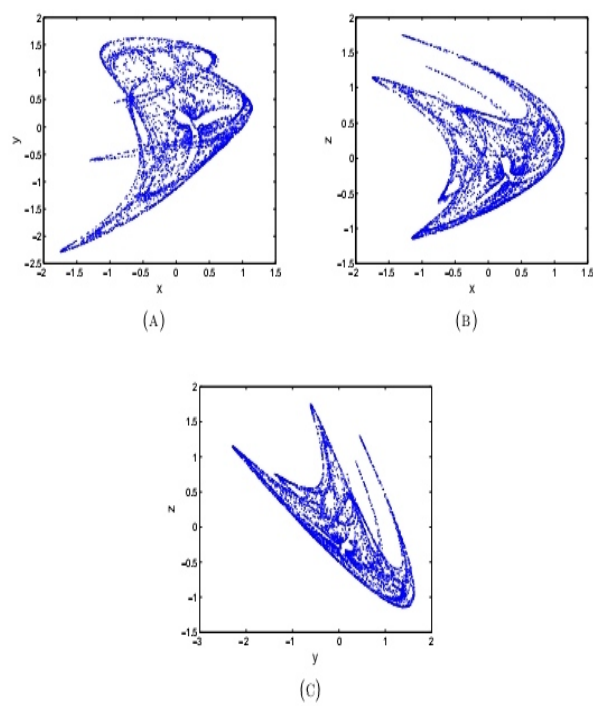


Figure 2.6: (A)-(C) phase portraits for ($a = 0.385$) in the (x, y) , the plan (y, z) , and the (x, z) .

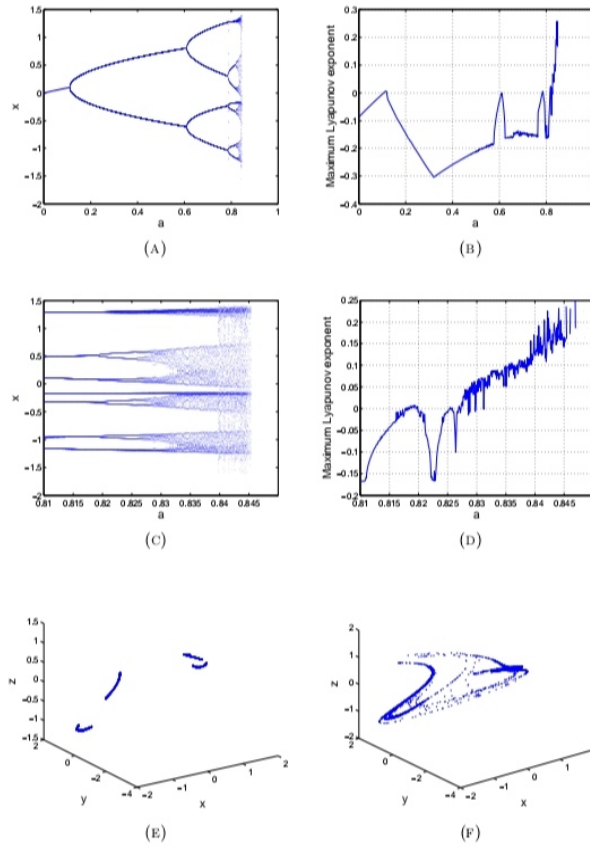


Figure 2.7: (A) Bifurcation diagram of system (2.1) in the plan (a, x) ($a \in (0, 1)$) for ($b = 0.4$); (B) maximum Lyapunov exponent corresponding to (A); (C) the local amplified bifurcation diagram of (A) for ($a \in (0.81, 0.85)$); (D) maximum Lyapunov exponent corresponding to (C); (E)-(F) chaotic attractors for ($a = 0.835$) and ($a = 0.8445$), respectively.

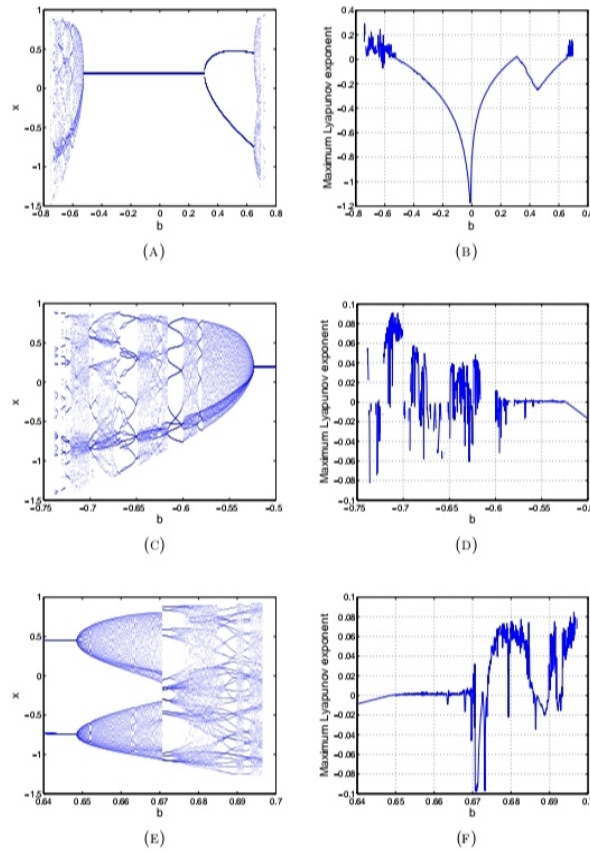


Figure 2.8: (A) bifurcation diagram of system (2.1) in the plan (x, y) with $(b \in (-0, 8, , 0, 8))$ and $(a = 0, 23)$; maximum Lyapunov exponent corresponding to (A); (C) the local amplified bifurcation diagram of (A) for $(b \in (-0, 75, -0, 5))$; (D) maximum Lyapunov exponent corresponding to (C); (E) the local amplified bifurcation diagram of (A) for $(b \in (0, 64, 0, 7))$; (F) maximum Lyapunov exponent corresponding to (E).

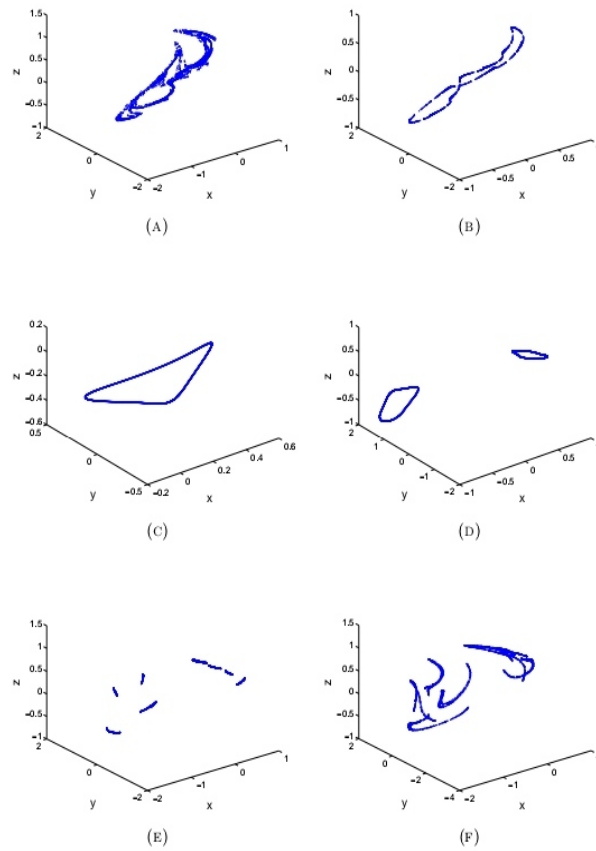


Figure 2.9: In (A)-(C), phase portraits corresponding to Figure 8 (C): (A) $b = -0,72$, (B) $b = -0,6$, and (C) $b = -0,53$, In (D)-(F), phase portraits corresponding to Figure 8 (E): (D) $b = 0,66$, (E) $b = 0,675$, and (F) $b = 0,691$.

CONCLUSION

This thesis demonstrates that the three-dimensional Hénon map exhibits a remarkable variety of dynamics, including complex bifurcations and chaotic attractors, the results open avenues for:

- . The study of higher-dimensional systems.
- . Applications in cryptography or the modeling of physical systems.
- . The exploration of synergies between theory and simulations for more complex nonlinear systems.

In summary, this work provides an in-depth analysis of a rich dynamical system, serving as a bridge between theoretical foundations and practical applications.

BIBLIOGRAPHY

- [1] A. C. J. Luo et Y. Guo, *Dynamical Systems: Discontinuity, Stochasticity and Time-Delay*, Springer-Verlag, New York, 2010.
- [2] A. S. Gonchenko et S. V. Gonchenko, *Variety of strange pseudohyperbolic attractors in three-dimensional generalized Hénon maps*, Phys. D, 337 (2016), 43-57.
- [3] B. Rabeh, *Systèmes dynamique chaotique: Application à l'optimisation*, University frères Mentouri-Constantine, departement of Mathematics, (2018-2019).
- [4] C. Robinson, *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos*, CRC Press, New York, 1999.
- [5] C. Mira, *Chaotic Dynamics*, World Scientific, Singapore, 1987.
- [6] D. L. Hitzl et F. Zele, *An exploration of Hénon quadratique map*, Physics D, 14(1985), 305-326.
- [7] E. N. Lorenz, *Compound windows of the Hénon map*, Physics D, 237 (2008), 1689-1704.
- [8] E. Ott, *Chaos in dynamical Systems*, 2nd edition, Cambridge University Press, Cambridge, 2002.
- [9] F. R. Marotto, *Chaotic behavior in the Hénon mapping*, Communications Maths Physics, 68 (1979), 184-194.

- [10] G. Baier and M. Klein, *Maximum hyperchaos in generalized Hénon maps*, Physics Letters A, 151(1990), 281-284.
- [11] H. L. An and Y. Chen, *The function cascade synchronization scheme for discrete-time hyper-chaotic*, Communications Nonlinear Science Numerical Simulation, 14 (2009), 1494-1501.
- [12] H. Richter, *The generalized hénon maps: Examples for higher dimensional chaos*, International Journal of Bifurcation Chaos, 15 (2002), 1371-1384.
- [13] H. K. Lam, *Synchronization of generalized Hénon map using bpolynomial controller*, Physics Letters A, 374 (2010), 552-556.
- [14] H. R. Dullin et J. D. Meiss, *Generalized Hénon maps: The cubic diffeomorphisms of the plane*, Physics D, 142 (2000), 262-289.
- [15] J. Carr, *Applications of Center Manifold Theory*, Springer-Verlag, New York, 1981.
- [16] J. H. Curry, *On the henon transformation*, Communications Maths Physics, 68 (1975), 129-140.
- [17] J. Guckenheimer et P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Springer-Verlag, New York, 1990.
- [18] M. Hénon, *A two-dimensional mapping with a strange attractor*, communications maths physics, 50 (1976), 69-77.
- [19] M. Martelli, *Discrete Dynamical Systems*, (2007).
- [20] R. L. Filal, S. Hammad, M. Benrejeb et P. Borne, *on synchronization, anti-synchronization and hybrid synchronization of 3D discrete generalized Hénon map*, Nonlinear Dynamics and systems Theory, 12 (2012), 81-95.
- [21] S. V. Geonchenko, I. I. Ovsyannikov, C. Simo et D. Turaev, *Three-dimensional Hénon-Like maps and wild lorenz-like attractors*, Int J Bifurcat Chaos, 15 (2005), 2493-3508.

- [22] S. Micheal, *Once more on Hénon map: Analysis of bifurcations*, *Chaos, Solitons and fractals*, 7 (1996), 2215-2234.
- [23] S. Smale, *Differentiable dynamical systems*, *Bulletin of the American Mathematical Society*, 73 (1967), 747-817.
- [24] S. Winggins, *Introduction to Applied Nonlinear Dynamical System and Chaos*, Springer-Verlag, New York, 1990.
- [25] Y. J. Xue et S. Y. Yang, *Synchronization of generalized Hénon map by using adaptive fuzzy controller*, *Chaos, Solitons and Fractals*, 17 (2003), 717-722.
- [26] Y. A. Kuznetsov, *Elements of Applied Bifurcation Théory*, 2nd edition, Springer-Verlag, New York, 1998.
- [27] Z. Y. Yan, *Q-S synchronization in 3D Hénon-like map and generalized Hénon map via a scalar controller*, *Physics Letters A*, 342 (2005), 309-317.