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University Center Abdelhafid Boussouf
Institute of Mathematics and Computer Sciences
Department of Mathematics

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Bifurcation and Chaos Analysis of a Two-Dimensional Discrete-Time Predator–Prey Model

Presented by: OUAIL SEKHRI ZEGGAR

Supervised by: Rabah Bouden

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DEDICATION

I dedicate this work to my dear parents, whose endless love, support, and prayers have been my constant source of strength.

To my brothers and sisters, and to all my family members.

To all my professors and friends who have accompanied me throughout my academic journey.

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Abstract

This work is devoted to the study of the dynamical behavior of a discrete predator-prey system with a nonmonotonic functional response. We investigate the local asymptotic stability of the positive equilibrium of the system by examining the characteristic equation of the linearized system corresponding to the model. By choosing the growth rate as a bifurcation parameter, the existence of Neimark-Sacker and period-doubling (flip) bifurcations at the positive equilibrium is established. Furthermore, the effects of perturbations on the system dynamics are investigated, and numerical simulations are presented to illustrate our main analytical results.

Keywords: Discrete predator-prey model, Neimark-Sacker bifurcation, flip bifurcation, center manifold theorem.

Résumé

Ce travail est consacré à l'étude du comportement dynamique d'un système prédateur-proie discret avec une réponse fonctionnelle non monotone. Nous étudions la stabilité asymptotique locale de l'équilibre positif du système en examinant l'équation caractéristique du système linéarisé correspondant au modèle. En choisissant le taux de croissance comme paramètre de bifurcation, l'existence de bifurcations de Neimark-Sacker et de doublement de période (flip) à l'équilibre positif est établie. De plus, les effets des perturbations sur la dynamique du système sont étudiés, et des simulations numériques sont présentées pour illustrer nos principaux résultats analytiques.

Mots clés : Modèle prédateur-proie discret, bifurcation de Neimark-Sacker, bifurcation flip, théorème de la variété centrale.

ملخص

يتمحور هذا العمل حول دراسة السلوك الديناميكي لنظام متقطع مفترس-فريسة مع استجابة وظيفية غير رتيبة. حيث قمنا بالبحث في الاستقرار المقارب المحلي للتوازن الإيجابي للنظام من خلال فحص المعادلة المميزة للنظام الخطي المقابل للنموذج. وباختيار معدل النمو كمعلمة تفرع، تم إثبات وجود تفرعات نيمارك-ساكر وتفرع مضاعفة الفترة (فليب) عند التوازن الإيجابي. علاوة على ذلك، تم تقصي تأثيرات الاضطرابات على ديناميكيات النظام، مع تقديم محاكاة عددية لتوضيح نتائجنا التحليلية الرئيسية.

الكلمات المفتاحية: نموذج المفترس والفريسة المتقطع، تفرع نيمارك-ساكر، تفرع فليب، مبرهنة المنوع المركزي.

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General Introduction

The study of dynamical systems constitutes one of the fundamental pillars of contemporary applied mathematics, providing a powerful mathematical framework for modeling and analyzing the time-dependent evolution of complex phenomena across various fields, including physics, economics, engineering, and population biology [1]. These systems are generally classified into continuous-time systems, governed by differential equations, and discrete-time systems, governed by difference equations and recurrence relations [3]. In recent decades, discrete-time dynamical systems have received increasing attention from researchers. This growing interest arises not only because they provide more accurate modeling for biological or ecological phenomena that are inherently discrete (such as non-overlapping generations), but also because they naturally emerge when numerical discretization techniques—such as the Euler method or Runge–Kutta methods—are applied to continuous-time systems [1, 4]. Moreover, discrete systems are remarkably capable of exhibiting highly complex dynamical behaviors and rich geometric structures even in lower dimensions compared to their continuous counterparts [5].

The core objective in studying any discrete-time dynamical system lies in understanding the long-term behavior of its orbits and trajectories. This analysis typically begins by identifying the equilibrium points (or fixed points) and investigating their local stability using the Jacobian matrix and linearization techniques around these points [1, 6]. However, the nonlinear nature of real-world systems makes the stability of these equilibria highly sensitive to variations in the system's parameters [7]. This is where *Bifurcation Theory* plays a vital role as an analytical tool, studying the sudden structural and qualitative transformations in the system's behavior when its parameters cross certain critical thresholds [2, 8]. These transformations include the appearance or disappearance of new equilibrium points, the exchange of stability between them, or the birth of new periodic orbits, as observed in period-doubling (flip) bifurcations and Neimark–Sacker bifurcations, which lead to the emergence of invariant closed curves and

quasi-periodic motions in two-dimensional spaces [2, 9].

As parameters continue to vary through successive cascades of bifurcations (such as Feigenbaum cascades), the system often follows a path that leads to deterministic *Chaos* [5, 10]. Chaotic behavior is defined as a pseudorandom motion governed by purely deterministic, non-stochastic equations [1]. It is primarily characterized by a sensitive dependence on initial conditions—commonly known as the *Butterfly Effect*—where minuscule differences in the initial state lead to exponential divergence of trajectories over time, making long-term predictions practically impossible [11]. This exponential divergence is quantified by *Lyapunov Exponents*, where the presence of at least one positive Lyapunov exponent is considered a strong indicator of chaotic dynamics and the existence of strange attractors with fractional (fractal) dimensions [12].

These abstract mathematical concepts find important applications in mathematical ecology, particularly in modeling the interactions between living organisms, such as predator–prey systems [2, 13]. Understanding ecological balance and predicting population dynamics represent significant challenges because the functional response of predation is often nonlinear and nonmonotonic due to complex environmental factors, including group defense mechanisms, toxicity, and herd behavior [2, 14]. Recent studies have demonstrated that incorporating a nonmonotonic functional response into predator–prey models can generate complex periodic oscillations and multistability phenomena that do not arise in classical monotonic models [15].

This master’s thesis, entitled “**Bifurcation and Chaos Analysis of a Two-Dimensional Discrete-Time Predator–Prey Model**”, aims to investigate the dynamical behavior of a two-dimensional discrete-time predator–prey model with a nonmonotonic functional response [2]. The significance of this study lies in combining rigorous theoretical analysis with comprehensive numerical simulations to explore the rich dynamical characteristics of the model and identify the parameter regions in which ecological equilibrium loses stability, giving rise to periodic oscillations or chaotic dynamics.

To achieve the objectives of this study, the thesis is organized into three main chapters, progressing from the fundamental theoretical concepts to the analysis of the proposed mathematical model:

- **Chapter 1** introduces the fundamental concepts of discrete dynamical systems. It presents the discretization of continuous-time models using numerical methods, with particular emphasis on the Euler

method. In addition, it discusses one-dimensional discrete dynamical systems, including fixed points, periodic orbits, and local stability analysis. The chapter also introduces graphical techniques, such as cobweb plots, together with the concept of topological conjugacy [1].

- **Chapter 2** extends the discussion to two-dimensional and higher-dimensional discrete dynamical systems. It develops the theoretical foundations of bifurcation theory by examining several fundamental bifurcations, including saddle-node, transcritical, and pitchfork bifurcations. The chapter concludes with an introduction to chaos theory and the computation and interpretation of Lyapunov exponents [1,5].
- **Chapter 3** constitutes the main contribution of this thesis. It formulates and investigates a two-dimensional discrete-time predator–prey model with a nonmonotonic functional response [2]. The existence of equilibrium points is established, and their local stability is analyzed using the Jacobian matrix. Taking the intrinsic growth rate as the principal bifurcation parameter, the occurrence of flip (period-doubling) and Neimark–Sacker bifurcations is established analytically. Finally, extensive numerical simulations, including bifurcation diagrams and phase portraits, are presented to validate the theoretical results and illustrate the transition from regular dynamics to chaotic behavior [2].

Chapter 1

Fundamentals of Discrete Dynamical Systems

Definition 1 *Discrete-time dynamical systems arise naturally in mathematics, physics, biology, economics, and engineering whenever processes evolve in distinct time steps. Unlike continuous-time systems described by differential equations, discrete-time systems are governed by recurrence relations (difference equations).*

The general form of a first-order discrete dynamical system is:

$$x_{n+1} = f(x_n)$$

where x_n represents the state of the system at time n , $f : D \subset \mathbb{R}^m \rightarrow D$ is a given function, and $n \in \mathbb{Z}_+$.

The central objective is to understand the behavior of the sequence $\{x_n\}$ as $n \rightarrow \infty$, depending on the structure of f and the initial condition x_0 . [1]

1.1 From Continuous Time to Discrete Time

Discrete systems may arise either because the process is inherently discrete or by discretizing a continuous system.

1.1.1 Euler Discretization

Consider the continuous system:

$$\dot{x} = f(x).$$

Using Euler's method with step size Δt , we approximate:

$$\dot{x}(t_n) \approx \frac{x(t_{n+1}) - x(t_n)}{\Delta t}.$$

This leads to the discrete system:

$$x_{n+1} = x_n + \Delta t f(x_n).$$

Thus, differential equations can be transformed into difference equations.

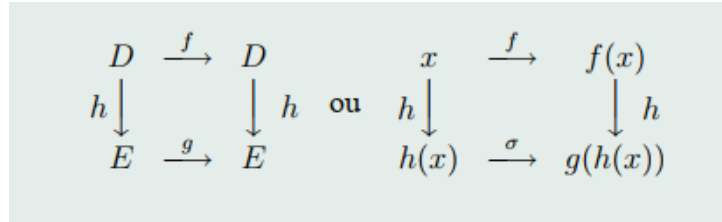


Figure 1.1: Euler discretization scheme

[1]

1.2 Definition of a Discrete Dynamical System

1.2.1 First-Order Systems in Dimension m

Let $D \subset \mathbb{R}^m$ and $f : D \rightarrow D$ be continuous and differentiable. A first-order discrete dynamical system is defined by:

$$x(0) = x_0 \in D, \quad x(n+1) = f(x(n)), \quad n \geq 0.$$

The pair (f, D) defines the system. [1]

1.2.2 Vector Formulation

For $m \geq 2$, the state vector is:

$$\mathbf{x}(n) = \begin{pmatrix} x_1(n) \\ x_2(n) \\ \vdots \\ x_m(n) \end{pmatrix}.$$

The system becomes:

$$\mathbf{x}(n+1) = \mathbf{f}(\mathbf{x}(n)).$$

The space $\mathbb{R}^m$ is called the phase space.

1.2.3 Autonomous and Non-Autonomous Systems

An autonomous system has the form:

$$x_{n+1} = f(x_n),$$

while a non-autonomous system depends explicitly on time:

$$x_{n+1} = f(n, x_n).$$

1.2.4 Higher-Order Systems

A higher-order difference equation

$$x(n+r) = f(x(n), \dots, x(n+r-1))$$

can be transformed into a first-order system by defining:

$$y(n) = \begin{pmatrix} x(n) \\ x(n+1) \\ \vdots \\ x(n+r-1) \end{pmatrix}.$$

1.2.5 Orbits (Trajectories)

Given an initial condition x_0 , the orbit of the system is:

$$O(x_0) = \{x_0, f(x_0), f^2(x_0), \dots\}.$$

The behavior of all orbits determines the global behavior of the system.

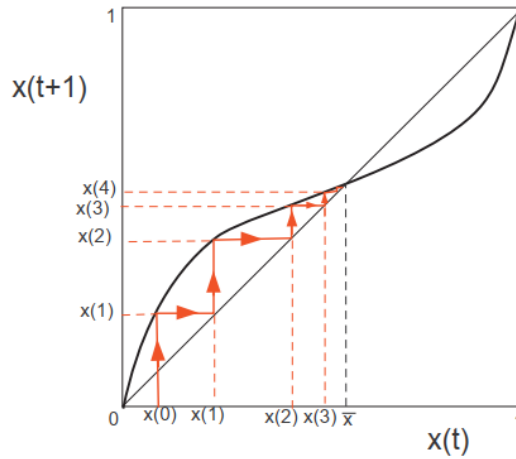


Figure 1.2: Example of an orbit

Example: $f(x) = x^2$

If $0 < x_0 < 1$, then $x_n \rightarrow 0$. If $x_0 = 1$, the orbit is constant. If $x_0 > 1$, then $x_n \rightarrow \infty$.

[1]

1.2.6 Fixed (Stationary) Points

A point x^* is called a fixed point if:

$$f(x^*) = x^*.$$

Graphically, fixed points are the intersections of the curves $y = f(x)$ and $y = x$.

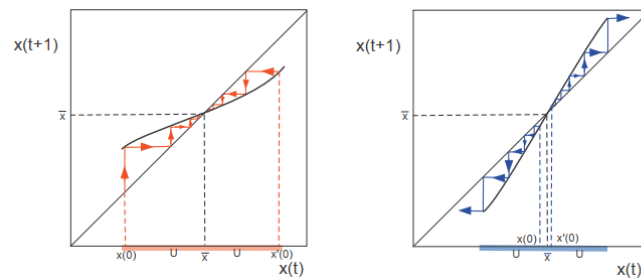


Figure 1.3: Graphical representation of fixed points

[3]

Example: Logistic Map

Consider:

$$f(x) = ax(1 - x).$$

Fixed points satisfy:

$$x = ax(1 - x).$$

The solutions are:

$$x_1^* = 0, \quad x_2^* = 1 - \frac{1}{a}.$$

[5]

1.2.7 Periodic Orbits

A point p is periodic of period k if:

$$f^k(p) = p,$$

and k is the smallest such positive integer.

A fixed point is a periodic point of period 1.

The orbit of a periodic point contains exactly k distinct points.

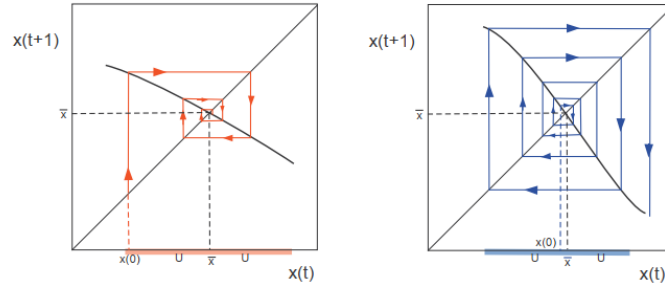


Figure 1.4: Periodic orbit example

[5]

1.3 Topological Equivalence (Conjugacy)

Two systems

$$x_{n+1} = f(x_n), \quad y_{n+1} = g(y_n)$$

are topologically conjugate if there exists a homeomorphism h such that:

$$h \circ f = g \circ h.$$

This means:

$$h(f(x)) = g(h(x)).$$

Topological conjugacy preserves:

- periodic points,
- stability properties,
- qualitative behavior.

[?]

1.4 Graphical Study of One-Dimensional Systems

The cobweb diagram is used to visualize the iteration process.

Procedure:

1. Plot $y = f(x)$ and $y = x$.

2. Start from x_0 .
3. Move vertically to the curve $y = f(x)$.
4. Move horizontally to the line $y = x$.
5. Repeat.

This method reveals convergence, divergence, periodic cycles, or complex behavior.

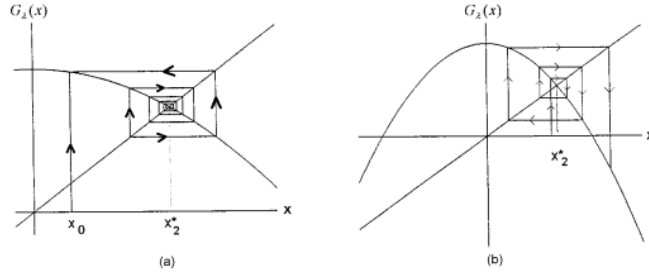


Figure 1.5: Cobweb diagram

1.4.1 Two-Dimensional Systems and Phase Portraits

A two-dimensional system has the form:

$$\mathbf{x}_{n+1} = f(\mathbf{x}_n).$$

Phase portraits are constructed by plotting the direction vector:

$$f(x) - x.$$

Fixed points satisfy:

$$f(x) = x.$$

The phase portrait provides qualitative information about attraction, repulsion, and orbit structure.

[4]

Conclusion

Discrete-time dynamical systems transform simple recurrence relations into rich mathematical structures. Even simple nonlinear maps can generate convergence, oscillation, divergence, or complex behavior.

This chapter establishes the fundamental concepts of orbits, fixed points, periodic points, graphical analysis, and topological equivalence, which form the theoretical foundation for stability analysis, bifurcation theory, and chaos studied in subsequent chapters.

1.5 Discrete One-Dimensional Dynamical Systems

In this section , we study discrete dynamical systems defined on subsets of $\mathbb{R}$, that is, one-dimensional systems of the form:

$$x_{n+1} = f(x_n),$$

where $f : I \subset \mathbb{R} \rightarrow I$ is a continuous and differentiable function.

The main objective is to analyze the long-term behavior of solutions using analytical tools. In particular, we focus on:

- Stability of fixed points,
- Local and global behavior,
- Linearization,
- Attractors and repellers.

1.5.1 Fixed Points and Stability

A point x^* is called a fixed point if:

$$f(x^*) = x^*.$$

The stability of a fixed point determines whether nearby trajectories approach or move away from it.

* Stability :

Let x^* be a fixed point.

- x^* is **stable** if for every $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$|x_0 - x^*| < \delta \Rightarrow |x_n - x^*| < \varepsilon \quad \forall n \geq 0.$$

- x^* is **asymptotically stable** if it is stable and:

$$\lim_{n \rightarrow \infty} x_n = x^*.$$

- x^* is **unstable** if it is not stable.

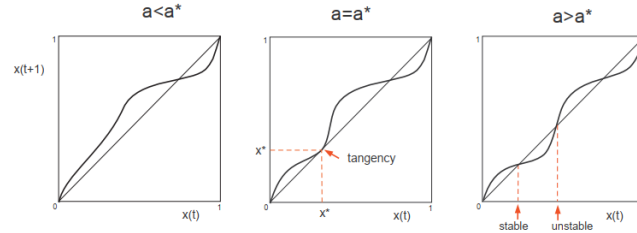


Figure 1.6: Stable and unstable fixed points

[5]

1.6 Linearization and Derivative Criterion

To determine stability, we study the derivative of f at the fixed point.

Assume f is differentiable at x^* .

Using Taylor expansion near x^* :

$$f(x) = f(x^*) + f'(x^*)(x - x^*) + o(|x - x^*|).$$

Since $f(x^*) = x^*$, we obtain:

$$x_{n+1} - x^* \approx f'(x^*)(x_n - x^*).$$

Thus, near x^* , the system behaves approximately like:

$$y_{n+1} = f'(x^*)y_n.$$

[5]

1.6.1 Stability Theorem

Let x^* be a fixed point.

- If $|f'(x^*)| < 1$, then x^* is asymptotically stable.
- If $|f'(x^*)| > 1$, then x^* is unstable.
- If $|f'(x^*)| = 1$, the linear test is inconclusive.

This is the fundamental stability criterion for one-dimensional discrete systems.

[5]

1.6.2 Geometric Interpretation

The derivative $f'(x^*)$ represents the slope of the curve $y = f(x)$ at the intersection with $y = x$.

- If $|f'(x^*)| < 1$, the graph crosses the diagonal with slope less than 1 in magnitude $\rightarrow$ trajectories contract.
- If $|f'(x^*)| > 1$, the slope is steep $\rightarrow$ trajectories expand.

If $f'(x^*)$ is negative, oscillations occur around the fixed point.

Example 1 Consider the logistic map:

$$f(x) = ax(1 - x), \quad a > 0.$$

Fixed points satisfy:

$$x = ax(1 - x).$$

Solutions:

$$x_1^* = 0, \quad x_2^* = 1 - \frac{1}{a}.$$

Derivative:

$$f'(x) = a(1 - 2x).$$

At $x_1^* = 0$:

$$f'(0) = a.$$

Thus:

- Stable if $0 < a < 1$,
- Unstable if $a > 1$.

At $x_2^* = 1 - \frac{1}{a}$:

$$f'(x_2^*) = 2 - a.$$

Thus:

- Stable if $1 < a < 3$,
- Unstable if $a > 3$.

This shows how parameter variation changes stability.

[3]

1.7 Attractors and Repellers

1.7.1 Attractor

A fixed point x^* is called an attractor if there exists a neighborhood U such that:

$$\lim_{n \rightarrow \infty} f^n(x) = x^* \quad \forall x \in U.$$

1.7.2 Repeller

A fixed point is a repeller if nearby points move away from it.

The derivative condition provides a simple test:

$$|f'(x^*)| < 1 \Rightarrow \text{attractor}$$

$$|f'(x^*)| > 1 \Rightarrow \text{repeller}$$

[8]

1.8 Monotonic Maps

If $f'(x) > 0$ for all x , then f is increasing.

If $f'(x) < 0$, then f is decreasing.

Increasing maps preserve order:

$$x_0 < y_0 \Rightarrow f^n(x_0) < f^n(y_0).$$

Decreasing maps may generate oscillatory behavior.

[14]

1.9 Global Stability

A fixed point x^* is globally asymptotically stable if:

$$\lim_{n \rightarrow \infty} f^n(x_0) = x^* \quad \forall x_0 \in I.$$

Sufficient conditions may include:

- f is continuous,
- I is invariant under f ,
- $|f'(x)| < 1$ for all $x \in I$.

[10]

1.10 Periodic Points

A point p is periodic of period k if:

$$f^k(p) = p.$$

Stability of a periodic orbit is determined by:

$$|(f^k)'(p)|.$$

- If $|(f^k)'(p)| < 1$, the cycle is stable.
- If $|(f^k)'(p)| > 1$, it is unstable.

For period 2:

$$(f^2)'(p) = f'(f(p)) \cdot f'(p).$$

[11]

1.11 Basin of Attraction

The basin of attraction of a fixed point x^* is defined as:

$$B(x^*) = \{x_0 \in I : \lim_{n \rightarrow \infty} f^n(x_0) = x^*\}.$$

Basins may be:

- Intervals,
- Disconnected sets,
- Fractal in complex systems.

[8]

Conclusion

One-dimensional discrete dynamical systems provide a complete framework for understanding nonlinear iteration processes. The derivative test offers a powerful tool for determining local stability.

Through fixed points, periodic orbits, and parameter dependence, we observe how simple nonlinear maps can exhibit rich and complex behavior.

This chapter establishes the analytical foundation necessary for the study of higher-dimensional systems and bifurcation theory in the following chapters.

Chapter 2

Higher-Dimensional Systems, Bifurcation and Chaos

Two-Dimensional Discrete Dynamical Systems

Definition 2 : *In this section , we extend the study of discrete dynamical systems to two-dimensional phase space. A two-dimensional discrete system is defined by:*

$$\mathbf{x}_{n+1} = F(\mathbf{x}_n),$$

where $\mathbf{x}_n = (x_n, y_n)^T \in \mathbb{R}^2$ and $F : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a differentiable map.

Two-dimensional systems are significantly richer than one-dimensional systems. They allow:

- *Spiral behavior,*
- *Rotations,*
- *Saddle points,*
- *Complex geometric structures.*

The analysis relies heavily on linear algebra, particularly eigenvalues and eigenvectors.

[4]

2.0.1 Fixed Points

A point $\mathbf{x}^*$ is a fixed point if:

$$F(\mathbf{x}^*) = \mathbf{x}^*.$$

To determine stability, we study the Jacobian matrix of F . [7]

2.0.2 Linearization

Let $\mathbf{x}^*$ be a fixed point. The Jacobian matrix is:

$$JF(\mathbf{x}) = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix}.$$

Evaluated at the fixed point:

$$A = JF(\mathbf{x}^*).$$

Near $\mathbf{x}^*$, the system behaves approximately like the linear system:

$$\mathbf{u}_{n+1} = A\mathbf{u}_n,$$

where $\mathbf{u}_n = \mathbf{x}_n - \mathbf{x}^*$. [7]

2.0.3 Linear Systems in $\mathbb{R}^2$

Consider the linear system:

$$\mathbf{x}_{n+1} = A\mathbf{x}_n,$$

where A is a 2×2 matrix.

The general solution depends on the eigenvalues of A . [7]

Eigenvalues

Eigenvalues satisfy:

$$\det(A - \lambda I) = 0.$$

For a 2×2 matrix:

$$\lambda^2 - (\text{tr}A)\lambda + \det A = 0.$$

Thus:

$$\lambda_{1,2} = \frac{\text{tr}A \pm \sqrt{(\text{tr}A)^2 - 4 \det A}}{2}.$$

The behavior of the system depends entirely on these eigenvalues. [7]

2.1 Stability Classification

Let λ_1 and λ_2 be the eigenvalues.

2.1.1 1. Sink (Asymptotically Stable)

If:

$$|\lambda_1| < 1, \quad |\lambda_2| < 1,$$

then the fixed point is asymptotically stable.

2.1.2 2. Source (Unstable)

If:

$$|\lambda_1| > 1, \quad |\lambda_2| > 1,$$

then the fixed point is unstable.

2.1.3 3. Saddle Point

If one eigenvalue satisfies:

$$|\lambda_1| < 1$$

and the other:

$$|\lambda_2| > 1,$$

then the fixed point is a saddle and is unstable.

2.1.4 4. Non-Hyperbolic Case

If:

$$|\lambda_i| = 1$$

for some i , linearization is inconclusive.

[8]

2.2 Geometric Interpretation

2.2.1 Real Eigenvalues

If eigenvalues are real and distinct:

- Solutions move along eigenvector directions.
- Contraction occurs if $|\lambda| < 1$.
- Expansion occurs if $|\lambda| > 1$.

2.2.2 Complex Eigenvalues

If eigenvalues are complex:

$$\lambda = re^{i\theta},$$

then:

- $r < 1 \rightarrow$ spiral sink,
- $r > 1 \rightarrow$ spiral source,
- $r = 1 \rightarrow$ pure rotation (neutral stability).

[8]

2.3 Local Stability Analysis of Two-Dimensional Discrete Systems

2.3.1 Trace-Determinant Plane

The eigenvalues depend on:

$$\tau = \text{tr} A, \quad \Delta = \det A.$$

The characteristic equation becomes:

$$\lambda^2 - \tau\lambda + \Delta = 0.$$

Stability conditions:

- Stable if $|\lambda_1| < 1$ and $|\lambda_2| < 1$.
- Equivalent to:

$$1 + \tau + \Delta > 0,$$

$$1 - \tau + \Delta > 0,$$

$$\Delta < 1.$$

The trace-determinant diagram provides a complete classification of linear fixed points.

2.3.2 Nonlinear Systems

For nonlinear systems:

$$\mathbf{x}_{n+1} = F(\mathbf{x}_n),$$

if all eigenvalues of $JF(\mathbf{x}^*)$ satisfy:

$$|\lambda_i| < 1,$$

then $\mathbf{x}^*$ is locally asymptotically stable.

If at least one eigenvalue satisfies:

$$|\lambda_i| > 1,$$

then the fixed point is unstable.

Such points are called hyperbolic when no eigenvalue satisfies $|\lambda| = 1$.

2.3.3 Invariant Manifolds

For a saddle point:

- The stable manifold consists of points approaching the fixed point.
- The unstable manifold consists of points moving away.

These manifolds are tangent to eigenvectors corresponding to stable and unstable eigenvalues.

Example 2 : *Consider:*

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}.$$

Eigenvalues are a and b .

- *If $|a| < 1$ and $|b| < 1$, origin is stable.*
- *If one is less than 1 and the other greater than 1, origin is saddle.*
- *If both exceed 1 in magnitude, origin is unstable.*

2.3.4 Phase Portraits

The phase portrait shows:

- Direction of trajectories,
- Convergence or divergence,
- Spiral motion,
- Saddle structure.

Graphical analysis complements algebraic classification.

[7]

Conclusion

Two-dimensional discrete dynamical systems exhibit richer dynamics than one-dimensional systems due to interactions between variables.

Linearization using the Jacobian matrix reduces local analysis to eigenvalue study. The classification via eigenvalues, trace, and determinant provides a complete description of local behavior near hyperbolic fixed points.

This chapter prepares the foundation for bifurcation theory, where qualitative changes occur as parameters vary.

2.4 Bifurcation Theory

Definition 3 : *In the previous section, we studied discrete dynamical systems for fixed parameter values. However, many real-world systems depend on one or more parameters. When parameters vary, the qualitative behavior of the system may change drastically.*

*Such qualitative changes are called **bifurcations**.*

Consider a parameter-dependent discrete system:

$$x_{n+1} = f(x_n, \mu),$$

where $\mu \in \mathbb{R}$ is a parameter.

A bifurcation occurs at $\mu = \mu_0$ if the qualitative structure of the system changes as μ passes through μ_0 .

[8]

2.4.1 Equilibrium Points and Parameter Dependence

Fixed points satisfy:

$$f(x^*, \mu) = x^*.$$

As μ varies:

- The number of fixed points may change.
- Their stability may change.
- New periodic orbits may appear.

Bifurcation theory studies these structural transitions. [8]

2.5 Bifurcation condition

From Chapter 2, a fixed point x^* is stable if:

$$|f_x(x^*, \mu)| < 1.$$

A bifurcation may occur when:

$$|f_x(x^*, \mu)| = 1.$$

There are two important cases:

- $f_x(x^*, \mu) = 1$
- $f_x(x^*, \mu) = -1$

These lead to different types of bifurcations.

[8]

2.5.1 Saddle-Node (Fold) Bifurcation

A saddle-node bifurcation occurs when two fixed points collide and annihilate each other.

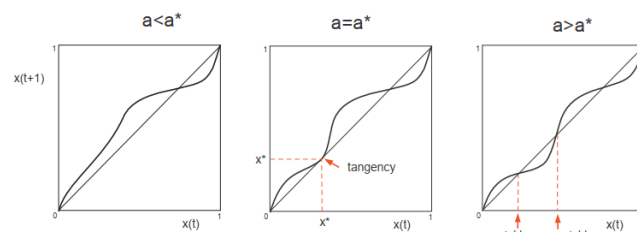


Figure 2.1: Sink (asymptotically stable)

The bifurcation diagram is the graph of a multivalued function, showing for every value of the parameter a in a neighbourhood of the bifurcation value a^* the fixed points of f_a in a neighbourhood of x^* . Stable fixed points are marked by a continuous line, unstable points by a dotted line.

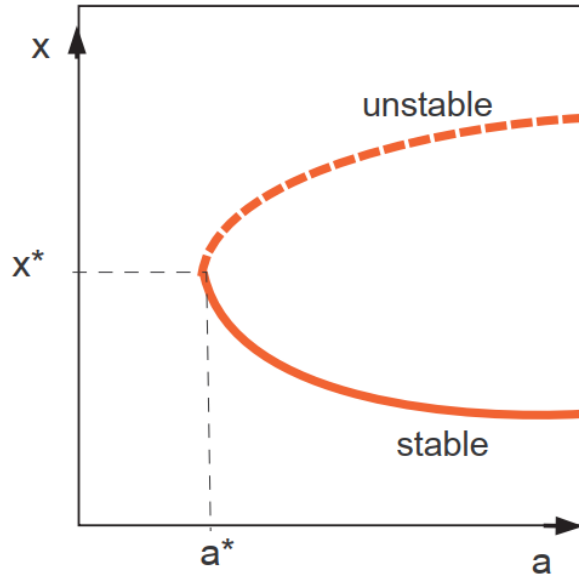


Figure 2.2: Geometric interpretation using eigenvectors

Normal Form

A typical normal form is:

$$x_{n+1} = x_n + \mu - x_n^2.$$

Fixed points satisfy:

$$x^2 = \mu.$$

- If $\mu > 0$, there are two fixed points.
- If $\mu = 0$, there is one (degenerate) fixed point.
- If $\mu < 0$, no fixed points exist.

Thus, at $\mu = 0$, a qualitative change occurs. [8]

2.5.2 Transcritical Bifurcation

In a transcritical bifurcation, two fixed points exchange stability.

Normal Form

$$x_{n+1} = x_n + \mu x_n - x_n^2.$$

Fixed points:

$$x_1^* = 0, \quad x_2^* = \mu.$$

As μ changes sign, stability is exchanged between the two equilibria. [8]

2.5.3 Pitchfork Bifurcation

Pitchfork bifurcations occur in symmetric systems.

Normal Form

$$x_{n+1} = x_n + \mu x_n - x_n^3.$$

Fixed points satisfy:

$$x(\mu - x^2) = 0.$$

- If $\mu < 0$, only $x = 0$ exists.
- If $\mu > 0$, three fixed points exist.

There are two types:

- Supercritical pitchfork (stable branches emerge),
- Subcritical pitchfork (unstable branches emerge).

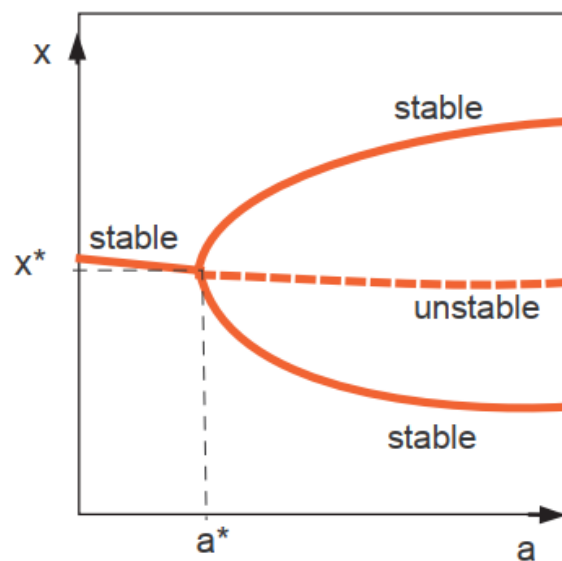


Figure 2.3: Bifurcation diagram of a pitch-fork bifurcation..)

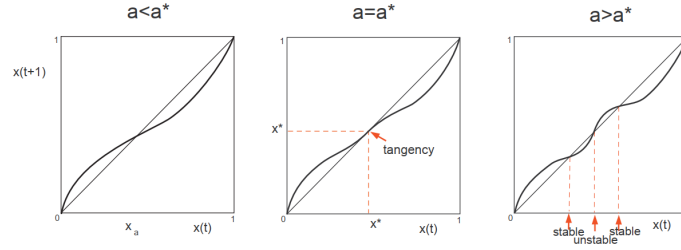


Figure 2.4: Bifurcation diagram of a saddle-node bifurcation.)

[8]

2.5.4 Period-Doubling (Flip) Bifurcation

A period-doubling bifurcation occurs when a fixed point loses stability as:

$$f_x(x^*, \mu) = -1.$$

A stable period-2 orbit is created.

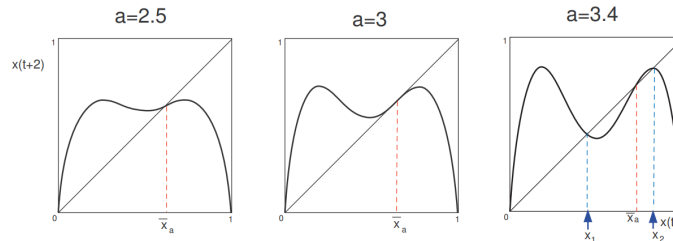


Figure 2.5: The pitchfork bifurcation exemple

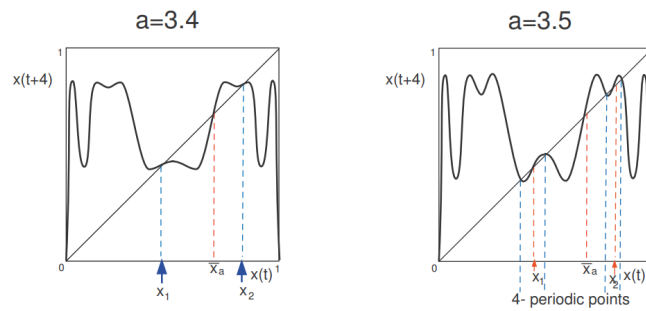


Figure 2.6: The loss of stability of x_1 and x_2 and birth of 4 stable 4-periodic points

Example 3 Consider the *logistic map*:

$$f(x) = ax(1 - x).$$

We previously determined:

- *Stable fixed point for $1 < a < 3$.*
- *At $a = 3$, $f'(x^*) = -1$.*

Thus:

- *For $a > 3$, the fixed point becomes unstable.*
- *A stable period-2 orbit appears.*

As a increases further, additional period-doubling bifurcations occur, leading to a cascade.

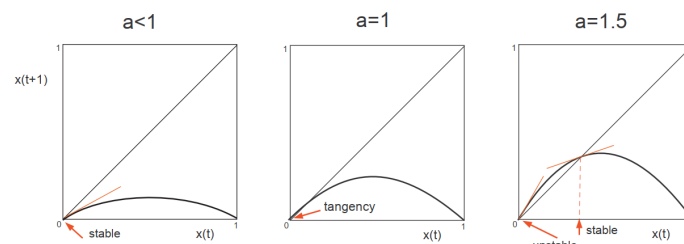


Figure 2.7: The logistic map for $a < 3$

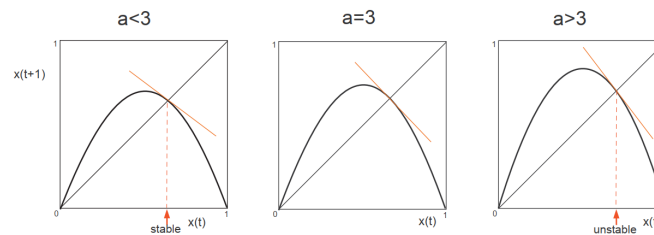


Figure 2.8: The logistic map for $a = 3$

[?]

2.6 Routes to Chaos and Higher-Dimensional Bifurcations

[9]

2.6.1 Bifurcation Diagram

A bifurcation diagram plots:

$$\mu \quad \text{vs} \quad x_n \text{ (long-term behavior).}$$

For the logistic map:

- Small $a \rightarrow$ single stable fixed point.
- Moderate $a \rightarrow$ period-2 cycle.
- Larger $a \rightarrow$ period-4, 8, 16, ...
- Eventually $\rightarrow$ chaotic regime.

The diagram visually demonstrates how complexity arises from simple nonlinear maps.

2.6.2 Feigenbaum Cascade

In many one-dimensional maps, period-doubling occurs in a universal pattern.

Let a_n be parameter values where period doubles.

Then:

$$\frac{a_{n+1} - a_n}{a_{n+2} - a_{n+1}} \rightarrow \delta,$$

where $\delta \approx 4.669$ is the Feigenbaum constant.

This constant is universal for a wide class of maps.

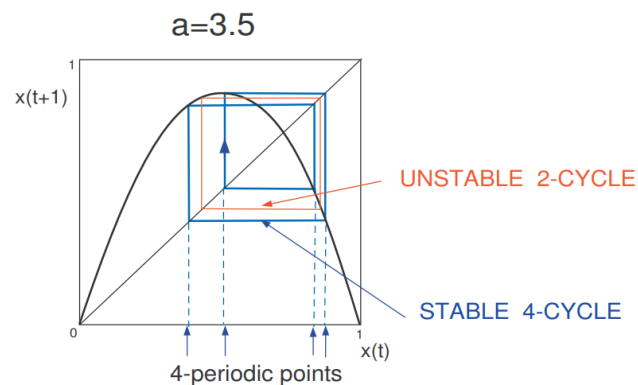


Figure 2.9: The stable 4-cycle and the unstable 2-cycle.

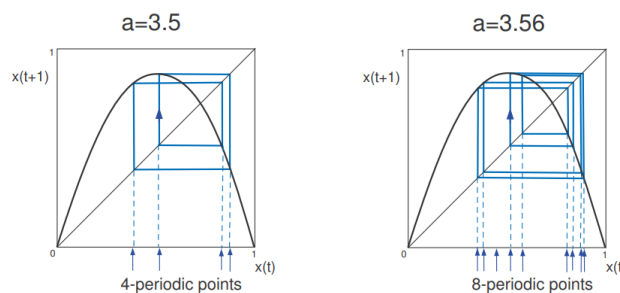


Figure 2.10: By a period doubling bifurcation a 4-cycle loses stability and appears a stable 8-cycle.

[8]

2.6.3 Two-Dimensional Bifurcations

In two-dimensional systems, bifurcations occur when eigenvalues of the Jacobian satisfy:

- $\lambda = 1 \rightarrow$ saddle-node type.
- $\lambda = -1 \rightarrow$ flip bifurcation.
- $|\lambda| = 1$ (complex eigenvalues) $\rightarrow$ Neimark-Sacker bifurcation.

[8]

Neimark-Sacker Bifurcation

This occurs when complex eigenvalues cross the unit circle:

$$\lambda = e^{i\theta}.$$

A closed invariant curve may appear, leading to quasi-periodic behavior.

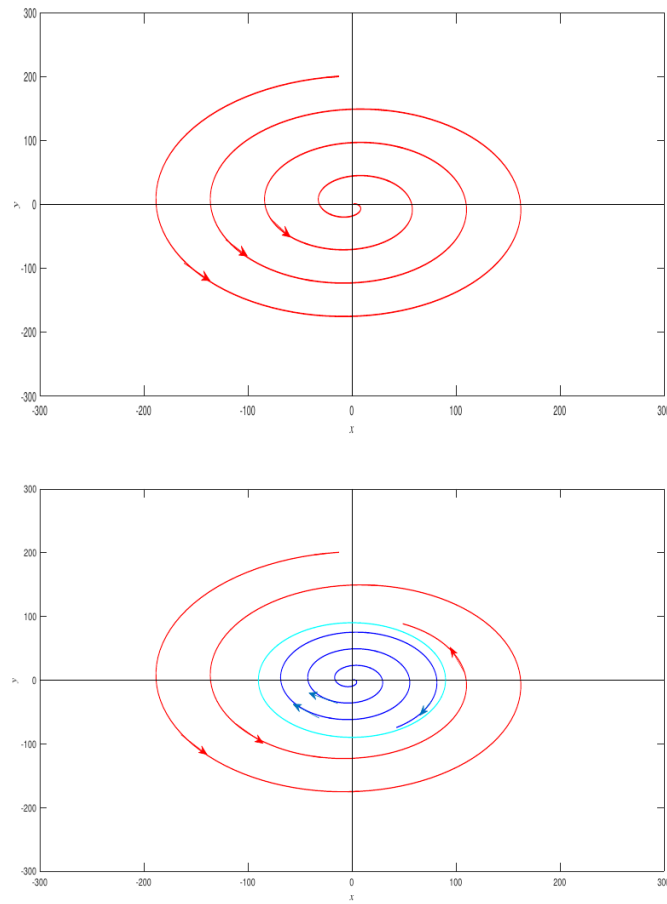


Figure 2.11: Supercritical Neimark–Sacker bifurcation. (a) Stable focus for uu_0 (b) Stable closed invariant curve for $\alpha > 0$.

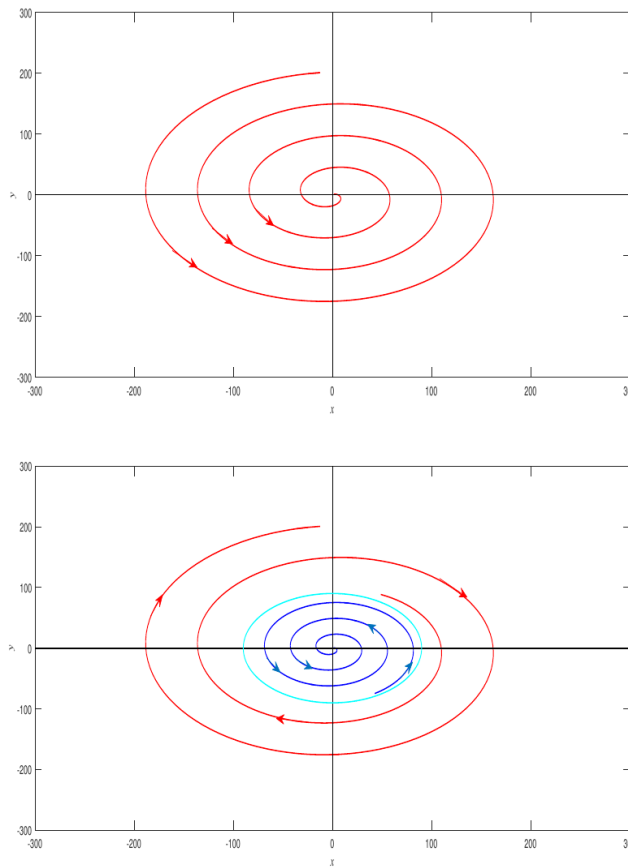


Figure 2.12: Subcritical Neimark-Sacker bifurcation. (a) Stable focus for (b) Unstable closed invariant curve for

[8]

2.6.4 Structural Stability

A system is structurally stable if small parameter changes do not alter its qualitative behavior.

At bifurcation points, structural stability is lost.

[9]

Conclusion

Bifurcation theory explains how qualitative changes arise in parameter-dependent discrete systems. Simple variations in parameters can lead to:

- Creation or destruction of equilibria,
- Stability exchange,

- Emergence of periodic orbits,
- Cascades toward chaos.

The period-doubling cascade, particularly in the logistic map, provides a fundamental route to chaos.

This chapter prepares the theoretical foundation for the final chapter, where chaotic dynamics are studied in detail.

2.7 Introduction to Chaos Theory

Definition 4 *Chaos theory studies deterministic systems that exhibit unpredictable and highly complex behavior. Although the governing equations are completely deterministic, the long-term behavior may appear random.*

In discrete dynamical systems, chaos arises naturally in nonlinear maps when parameters reach certain critical values. The logistic map is one of the most famous examples demonstrating chaotic dynamics.

In this chapter, we introduce the fundamental concepts of chaos in discrete systems.

2.7.1 Deterministic Chaos

A dynamical system is called chaotic if it satisfies three main properties:

- Sensitive dependence on initial conditions,
- Topological transitivity,
- Density of periodic orbits.

These properties define chaos in a rigorous mathematical sense. [9]

2.7.2 Sensitive Dependence on Initial Conditions

Sensitive dependence means that arbitrarily close initial conditions eventually separate exponentially.

Formally, the system exhibits sensitivity if there exists $\delta > 0$ such that for any x_0 and any neighborhood of x_0 , there exists y_0 arbitrarily close to x_0 and some n such that:

$$|f^n(x_0) - f^n(y_0)| > \delta.$$

Thus, small differences in initial conditions grow significantly over time.

This phenomenon explains why chaotic systems are practically unpredictable.

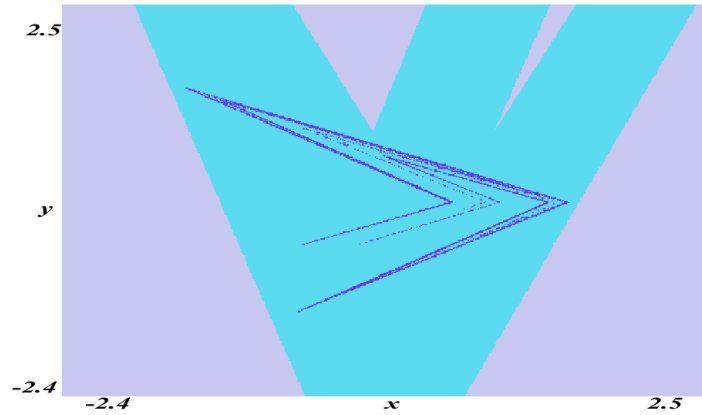


Figure 2.13: Lozi attractor obtained for $x = 1.7$ and $y = 0.5$.

[?]

2.7.3 Lyapunov Exponent

The Lyapunov exponent measures the average exponential rate of separation of nearby trajectories.

For a one-dimensional map:

$$x_{n+1} = f(x_n),$$

the Lyapunov exponent is defined by:

$$\lambda = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |f'(x_k)|.$$

- If $\lambda < 0$, trajectories converge.
- If $\lambda = 0$, neutral behavior.
- If $\lambda > 0$, trajectories diverge exponentially $\rightarrow$ chaos.

A positive Lyapunov exponent is a strong indicator of chaotic dynamics. Lyapunov exponent of the logistic map as a function of the parameter a .

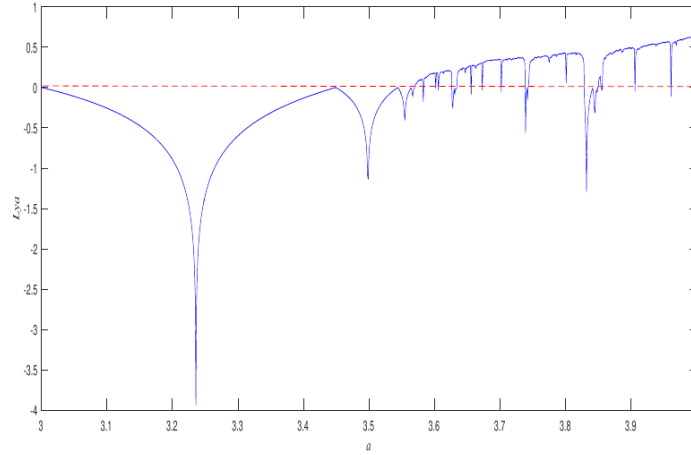


Figure 2.14: Lyapunov exponent of the logistic map as a function of the parameter a .

Figure (2.14) shows the two Lyapunov exponents of the Hénon map defined by:

$$\begin{cases} x_{n+1} = 1 - ax_n^2 + by_n \\ y_{n+1} = x_n \end{cases}$$

for $b = 0.3$ and $0.2 < a \leq 1.5$.

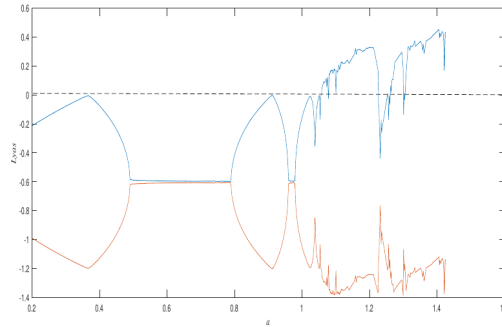


Figure 2.15: Lyapunov exponents of the Hénon map for $b = 0.3$ and $0.2 < a < 1.5$.

[12]

2.7.4 Topological Transitivity

A system is topologically transitive if, for any two open sets U and V , there exists n such that:

$$f^n(U) \cap V \neq \emptyset.$$

This means that trajectories eventually move throughout the phase space.

Topological transitivity implies strong mixing behavior.

[10]

2.7.5 Density of Periodic Points

Periodic points are dense if every neighborhood contains a periodic point.

This property implies that chaotic systems contain infinitely many unstable periodic orbits embedded within chaotic regions.

[11]

2.7.6 Chaos in the Logistic Map

Consider the logistic map:

$$f(x) = ax(1 - x).$$

For:

- $0 < a < 3 \rightarrow$ stable fixed point.
- $3 < a < 3.5699 \dots \rightarrow$ period-doubling cascade.
- $a > 3.5699 \dots \rightarrow$ chaotic regime (with periodic windows).

When $a = 4$, the map becomes fully chaotic on $(0, 1)$.

The bifurcation diagram shows a transition from order to chaos through period-doubling.

[10]

2.7.7 Invariant Sets

An invariant set S satisfies:

$$f(S) = S.$$

Chaotic attractors are invariant sets with complicated internal structure. [?]

2.7.8 Strange Attractors

In higher-dimensional systems, chaotic attractors may have fractal structure. These are called strange attractors.

They have the following properties:

- Fractal geometry,
- Sensitive dependence,

- Non-periodic motion.

Strange attractors combine geometric complexity with dynamical instability.

[8]

2.7.9 Symbolic Dynamics

Chaotic maps can often be described using symbolic sequences.

For example, the logistic map at $a = 4$ is conjugate to the shift map on binary sequences.

Each orbit can be encoded as a sequence of symbols, providing a combinatorial description of chaos.

[8]

2.7.10 Mixing Property

A system is mixing if for any two sets U and V :

$$\lim_{n \rightarrow \infty} \mu(f^n(U) \cap V) = \mu(U)\mu(V).$$

Mixing systems exhibit strong statistical randomness.

[10]

2.7.11 Fractal Dimension

Chaotic attractors often have non-integer dimension.

One measure is the Hausdorff dimension.

Fractal geometry explains the intricate structure of chaotic sets.

[?]

2.7.12 Route to Chaos

Discrete systems may reach chaos through:

- Period-doubling cascade,
- Quasi-periodicity,
- Intermittency.

The period-doubling route is the most common in one-dimensional maps.

[11]

2.7.13 Importance of Chaos Theory

Chaos theory has applications in:

- Physics,
- Biology,
- Economics,
- Engineering,
- Climate modeling.

It explains irregular behavior in deterministic systems and highlights limitations of long-term prediction.

[10]

Conclusion

Chaos theory reveals that simple nonlinear discrete systems can generate extremely complex behavior. Despite deterministic equations, chaotic systems exhibit unpredictability due to sensitive dependence on initial conditions.

The study of Lyapunov exponents, invariant sets, bifurcations, and symbolic dynamics provides a rigorous framework for understanding chaotic phenomena.

With this chapter, the theory of discrete dynamical systems progresses from fixed points and stability to bifurcations and ultimately to chaos, completing the theoretical structure of the course.

Chapter 3

Bifurcation and Chaos Analysis of a Two-Dimensional Discrete-Time Predator–Prey Model

3.1 Introduction

In the previous chapters, the theoretical foundations of discrete dynamical systems, bifurcation theory, and chaos were presented. These concepts provide the mathematical framework required to analyze nonlinear discrete-time models arising in biological systems.

In this chapter, we study a two-dimensional discrete-time predator–prey model with a nonmonotonic functional response. First, the mathematical model is derived from biological assumptions. Then, the existence and local stability of its equilibrium points are investigated by analyzing the Jacobian matrix and the associated characteristic equation.

Furthermore, the conditions for the occurrence of flip and Neimark–Sacker bifurcations are established using local bifurcation theory. Finally, numerical simulations are presented to illustrate the analytical results and to demonstrate the rich dynamical behaviors of the model, including periodic oscillations, invariant closed curves, and chaotic dynamics.

3.2 Derivation of the Model

We consider a predator–prey model consisting of two interacting populations.

Let

$$x(t)$$

denote the density of the prey population and

$$y(t)$$

denote the density of the predator population at time t .

Assumption 1

When predators are absent, the prey population grows according to the logistic map

$$x(t+1) = rx(t)(1-x(t)) \quad (3.1)$$

[13]

Assumption 2

When predators are present, prey decreases due to predation. The interaction follows a Holling type III functional response.

Thus equation (1) becomes

$$x(t+1) = rx(t)(1-x(t)) - \beta \frac{x(t)^2 y(t)}{x(t)^2 + 1} \quad (3.2)$$

[13]

Assumption 3

The predator population grows depending on prey consumption.

Hence

$$y(t+1) = \beta \frac{x(t)^2 y(t)}{x(t)^2 + 1} \quad (3.3)$$

Therefore the predator-prey system becomes

$$\begin{cases} x(t+1) = rx(t)(1-x(t)) - \beta \frac{x(t)^2 y(t)}{x(t)^2 + 1} \\ y(t+1) = \beta \frac{x(t)^2 y(t)}{x(t)^2 + 1} \end{cases} \quad (3.4)$$

with positive initial conditions

$$x(0) > 0, \quad y(0) > 0$$

where

- r is the intrinsic growth rate of the prey
- β is the maximum predation rate

The system can be written in the map form

$$F(x, y) = \begin{pmatrix} rx(1-x) - \beta \frac{x^2 y}{x^2 + 1} \\ \beta \frac{x^2 y}{x^2 + 1} \end{pmatrix} \quad (3.5)$$

[13]

3.3 Fixed Points: Existence and Stability

In this section, we look at the existence and stability of the system's fixed points in $\mathbb{R}_+^2$. Results about the existence of fixed points are summarized as follows.

Proposition 3.1. For model (3.4), we can have at most three fixed points:

1. The trivial fixed point

$$E_0(0, 0)$$

always exists.

2. The predator-free fixed point

$$E_1\left(\frac{r-1}{r}, 0\right)$$

exists if $r > 1$.

3. The interior fixed point

$$E_2\left(\frac{1}{\sqrt{\beta-1}}, \frac{(\beta-1)(r-1) - r\sqrt{\beta-1}}{(\beta-1)^{3/2}}\right)$$

exists if $\beta > 2$ and

$$r > \frac{1 - \beta\sqrt{\beta-1}}{\beta + 1 - \beta}.$$

Proof. By solving the following algebraic system, the fixed points correspond to the steady states of the model population (3.4):

$$rx(1-x) - \frac{\beta x^2 y}{x^2 + 1} - x = 0, \quad (3.6)$$

$$\frac{\beta x^2 y}{x^2 + 1} - y = 0. \quad (3.7)$$

Then all non-negative fixed points of system (3.4) are

$$E_0(0, 0), \quad E_1\left(\frac{r-1}{r}, 0\right),$$

which is positive if $r > 1$, and

$$E_2\left(\frac{1}{\sqrt{\beta-1}}, \frac{(\beta-1)(r-1) - r\sqrt{\beta-1}}{(\beta-1)^{3/2}}\right),$$

which is positive if $\beta > 2$ and the above condition on r holds. :contentReference[oaicite:1]index=1
[13]

3.3.1 Dynamic Behavior of the Model

Now, we investigate how the solutions of model (3.4) behave around E_0 , E_1 , and E_2 . Computing the variation matrix corresponding to each fixed point can be used to study the local stability of the model. :contentReference[oaicite:2]index=2

The Jacobian matrix of model (3.4) about the fixed point (x, y) is

$$J(x, y) = \begin{pmatrix} r(1-x) - rx - \frac{2\beta xy}{x^2 + 1} + \frac{2\beta x^3 y}{(x^2 + 1)^2} & -\frac{\beta x^2}{x^2 + 1} \\ \frac{2\beta xy}{(x^2 + 1)^2} & \frac{\beta x^2}{x^2 + 1} \end{pmatrix}.$$

Its characteristic equation is

$$\lambda^2 - p(x, y)\lambda + q(x, y) = 0, \quad (3.1)$$

where

$$p(x, y) = \frac{-2rx^5 + (r + \beta)x^4 - 4rx^3 + (2r + \beta)x^2 - 2(\beta y + r)x + r}{(x^2 + 1)^2},$$

and

$$q(x, y) = \frac{\beta r x^2 (1 - 2x)}{x^2 + 1}.$$

We recall the following definitions. :contentReference[oaicite:3]index=3

Definition 3.1. Let $E(x, y)$ be a fixed point of system (3.4) and let μ_1, μ_2 be the eigenvalues of (3.1).

1. E is called a **sink** (locally asymptotically stable) if

$$|\mu_1| < 1, \quad |\mu_2| < 1.$$

2. E is called a **source** if

$$|\mu_1| > 1, \quad |\mu_2| > 1.$$

3. E is called a **saddle** if

$$|\mu_1| < 1, |\mu_2| > 1 \quad \text{or} \quad |\mu_1| > 1, |\mu_2| < 1.$$

4. E is called **non-hyperbolic** if

$$|\mu_1| = 1 \quad \text{or} \quad |\mu_2| = 1.$$

[14]

3.4 Stability and Bifurcation Analysis

3.4.1 A Useful Lemma

Lemma 3.1. Let $R(\mu) = \mu^2 + A\mu + B$ and suppose $R(1) > 0$. Let μ_1, μ_2 be the roots of $R(\mu) = 0$. Then:

- (i) $|\mu_1| < 1, |\mu_2| < 1$ iff $R(-1) > 0$ and $B < 1$;
- (ii) one root inside and one outside the unit circle iff $R(-1) < 0$;
- (iii) $|\mu_1| > 1, |\mu_2| > 1$ iff $R(-1) > 0$ and $B > 1$;
- (iv) $\mu_1 = -1$ iff $R(-1) = 0$ and $B \neq 0, 2$;
- (v) complex conjugate roots with $|\mu_1| = |\mu_2| = 1$ iff $A^2 - 4B < 0$ and $B = 1$.

[14]

3.4.2 Stability of Fixed Points

Proposition 3.2. For the fixed point E_0 :

- (i) E_0 is a sink if $0 < r < 1$;
- (ii) E_0 is never a source;
- (iii) E_0 is a saddle if $r > 1$;
- (iv) E_0 is non-hyperbolic if $r = 1$.

Proposition 3.3. The fixed point E_1 is asymptotically stable iff:

$$1 < r < 3 \quad \text{and} \quad \beta < \frac{2r^2 - 2r + 1}{(r - 1)^2}.$$

Moreover, E_1 loses stability via:

- (i) a branch bifurcation at $r = 1$ (towards E_0);
- (ii) a branch bifurcation at $\beta = \frac{2r^2 - 2r + 1}{(r - 1)^2}$ (towards E_2);
- (iii) a supercritical flip bifurcation at $r = 3$ when $\beta < \frac{13}{4}$.

[14]

3.4.3 Stability of the Interior Fixed Point E_2

Proposition 3.4. The positive equilibrium E_2 is locally asymptotically stable iff:

- (i) $r < \frac{(2\beta - 1)\sqrt{\beta - 1}}{-\sqrt{\beta - 1} + \beta + 1},$
- (ii) $r > \frac{\sqrt{\beta - 1}}{\sqrt{\beta - 1} - 1},$
- (iii) $r(\sqrt{\beta - 1} - 2) < \sqrt{\beta - 1}.$

These conditions are obtained using Lemma 3.1 applied to the characteristic equation:

$$\lambda^2 - p\lambda + q = 0.$$

[14]

3.4.4 Bifurcation of E_2

Proposition 3.5. The fixed point E_2 loses stability via:

- (i) a flip bifurcation if $\frac{13}{4} < \beta < 6.8926$ and

$$r = \frac{(2\beta - 1)\sqrt{\beta - 1}}{-\sqrt{\beta - 1} + \beta + 1};$$

- (ii) a Neimark–Sacker bifurcation if $\beta > 6.8926$ and

$$r = \frac{\sqrt{\beta - 1}}{\sqrt{\beta - 1} - 2};$$

- (iii) a branch bifurcation if $\beta > \frac{13}{4}$ and

$$r = \frac{\sqrt{\beta - 1}}{\sqrt{\beta - 1} - 1},$$

where E_2 bifurcates to E_1 ;

- (iv) a branch-flip (BPPD) point at

$$D_1 : \left(r = 3, \beta = \frac{13}{4} \right).$$

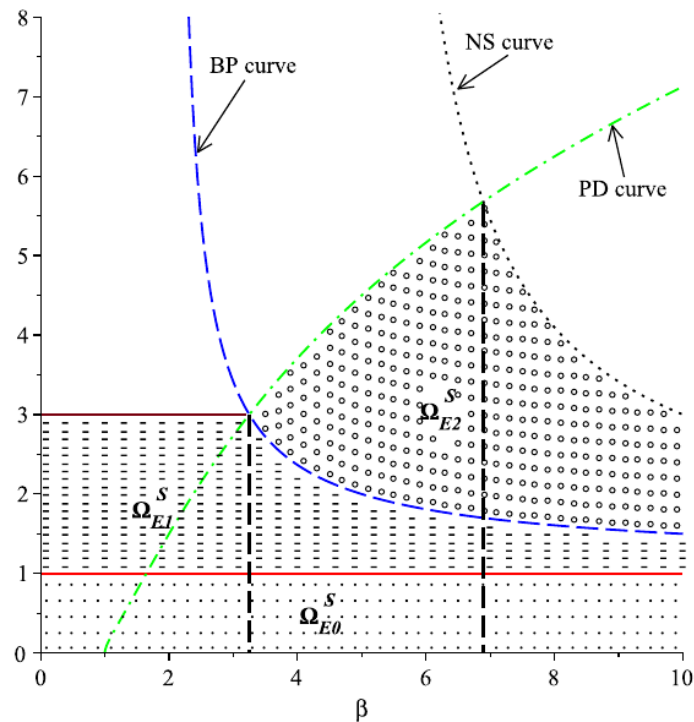


Figure 3.1: Stability regions in (β, r) -space.

3.5 Numerical Simulations

In this section, we present comprehensive numerical simulation results to verify and illustrate the theoretical analysis obtained for our main discrete-time predator-prey system. By selecting the intrinsic growth rate r and the predation rate β as control parameters, we investigate the diverse dynamical behaviors of the system, including stable states, periodic oscillations, and the transition into chaotic regimes via period-doubling cascades.

To deeply investigate the dynamic behaviors of the system, we consider the following numerical configurations and scenarios:

3.5.1 Case 1: Stability of the Trivial Equilibrium E_0

We choose the parameter values $r = 0.5$ and $\beta = 2$ with the initial conditions $(x_0, y_0) = (0.1, 0.1)$. Based on our previous analytical criteria, the eigenvalues of the Jacobian matrix evaluated at the origin fall strictly inside the unit circle, confirming that the trivial equilibrium point $E_0(0, 0)$ acts as a stable sink.

As shown in the numerical trajectory, both prey and predator populations decrease rapidly and converge directly to the origin. Furthermore, as the continuation curve indicates, when r increases and crosses the critical threshold $r = 1$, E_0 loses its stability and undergoes a branch point (BP) bifurcation.

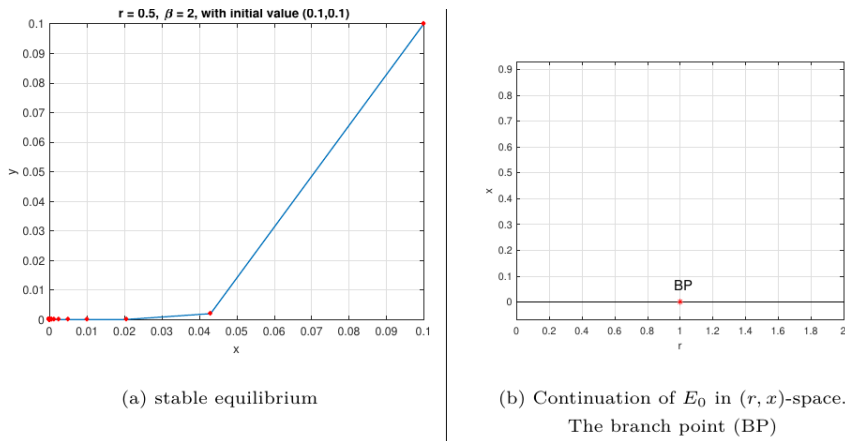


Figure 3.2: Phase portraits and bifurcation tracking of the trivial equilibrium point $E_0(0, 0)$ under parameter values $r = 0.5$ and $\beta = 2$: (a) Stable equilibrium trajectory directly converging to the origin with initial values $(0.1, 0.1)$, (b) Numerical continuation of E_0 in the (r, x) -space highlighting the precise boundary location of the Branch Point (BP) threshold.

3.5.2 Case 2: Bifurcations and Chaos of the Boundary Scenario E_1

Next, we adjust the parameter values to $r = 2.9$ and $\beta = 1$ with the initial conditions $(x_0, y_0) = (0.6, 0.4)$ to evaluate the behavior of the boundary equilibrium E_1 . In this scenario, the system displays sharp oscillatory behaviors that pulse near the state boundaries as shown in the trajectory plot.

The continuation curve in (r, x) -space demonstrates that after emerging from the Branch Point (BP), the boundary equilibrium curve propagates until it encounters a Period-Doubling (PD) bifurcation point, where E_1 loses its local stability and triggers a cascade toward complex orbits.

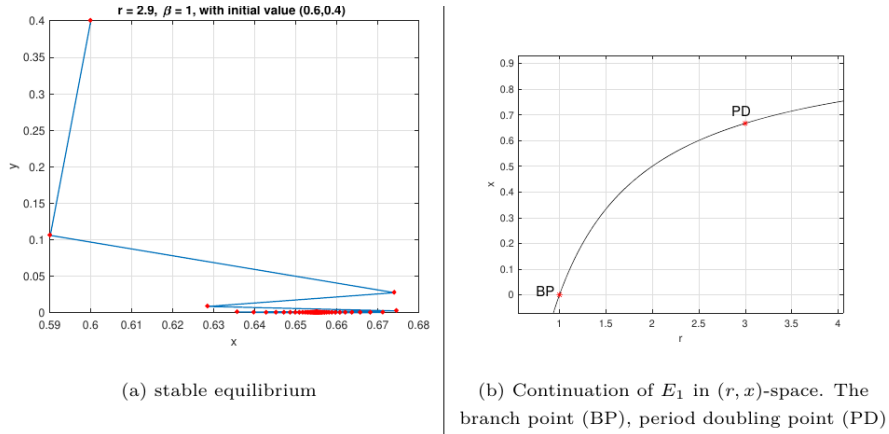


Figure 3.3: Phase portraits and bifurcation tracking of the boundary equilibrium point E_1 under parameter values $r = 2.9$ and $\beta = 1$: (a) Dynamic trajectory exhibiting stable oscillatory behavior under initial values $(0.6, 0.4)$, (b) Numerical continuation curve of E_1 in the (r, x) -space mapping the sequence from the Branch Point (BP) up to the onset of the Period-Doubling (PD) threshold.

3.5.3 Chaotic Regime and Strange Attractors

Further increase and variation of the system parameter space leads to a cascade of period-doubling bifurcations that completely break down periodic orbits, eventually evolving into chaotic dynamics. In another parameter region, the system undergoes a Neimark-Sacker bifurcation where an attractive closed invariant curve appears around the positive interior equilibrium state, confirming the emergence of highly complex strange attractors.

[15]

Conclusion

In this paper we studied a two-dimensional discrete-time predator–prey model with a nonmonotonic functional response.

The existence of equilibrium points was determined and the local stability of the positive equilibrium was analyzed using the Jacobian matrix.

By choosing the intrinsic growth rate as a bifurcation parameter, we showed that the system can undergo flip bifurcation and Neimark–Sacker bifurcation.

Numerical simulations were performed to illustrate the analytical results and showed that the system can exhibit complex dynamical behaviors including periodic oscillations and chaotic dynamics.

These results contribute to the understanding of the dynamical properties of discrete predator–prey systems.

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