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Dedication

To my dear mother and my dear father

*To my husband **Rabie***

*To my daughters **Lina and Naila***

To my brothers and sisters

To all my family and in-laws

To all my friends.

Abstract

The study of chaos control and synchronization has made significant progress over the past few decades. It is due to their use in various sectors such as communication protection, biological systems, and, more recently, in the epidemiological field that their scope is so broad. Thus, we have presented various control techniques as well as different forms of synchronization.

In this work, we aim To attain complete synchronization and anti-synchronization of the SIR model, various control methods will be implemented, including active control, adaptive control, backstepping, and sliding mode. Next, we conduct a comparative analysis of these methods to identify the most effective one among them.

Key words:Chaos, adaptive control, active control, synchronization, anti-synchronization.

Résumé

Le contrôle et la synchronisation du chaos a considérablement avancé au cours des dernières décennies. C'est en raison de leur emploi dans divers domaines, tels que la sécurisation des communications, les systèmes biologiques et, plus récemment, dans le domaine épidémiologique que leur portée est si large. Ainsi, nous avons exposé diverses techniques de contrôle ainsi que différentes formes de synchronisation.

Dans ce travail, nous visons à effectuer la synchronisation complète et l'anti-synchronisation du modèle SIR en mettant en œuvre diverses méthodes de contrôle comme le contrôle actif, adaptatif, backstepping et le mode glissant. Ensuite, nous procédons à une analyse comparative de ces méthodes pour identifier la plus efficace parmi elles.

Mots clés: Le chaos, contrôle adaptatif, contrôle actif, synchronisation, anti-synchronisation.

ملخص

دراسة التحكم و التزامن الفوضى عرفت تقدما كثيرا خلال العقود الأخيرة. يرجع ذلك إلى استخدامها في قطاعات مختلفة مثل حماية الإتصالات، الأنظمة البيولوجية ومؤخرا في المجال الوبائي، مما يجعل تطبيقها واسعا جدا. وعلى هذا الأساس قمنا بعرض تقنيات تحكم مختلفة وكذلك أشكال التزامن متنوعة. في هذا العمل نهدف إلى تحقيق التزامن الكامل والتزامن العكسي لنموذج SIR من خلال تنفيذ طرق تحكم متنوعة مثل التحكم النشط، التكيفي، التراجعي، والوضع الانزلاق ثم قمنا بإجراء تحليل مقارنة لهذه الطرق لتحديد الأكثر فعالية بينها.

الكلمات المفتاحية: الفوضى، التحكم التكيفي، التحكم النشط، التزامن والتزامن العكسي.

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Introduction

Scientists started examining dynamical systems defined by differential equations in the 19th century. It was assumed that if we knew enough about the initial condition of these systems, they would behave as predicted. But Henri Poincaré's work altered this appearance.

In the late nineteenth century, Henri Poincaré investigated the three-body problem, which is a celestial mechanics problem dealing with the movement of three bodies caused by gravity [58]. The objective was to figure out how to make these bodies move. However, Poincaré discovered that their actions could be more complicated than he had thought. He showed that even with little adjustment to the initial conditions, the trajectories of these entities displayed significant variations.

This finding was the first mathematical demonstration of deterministic instability, which states that some systems eventually lose their deterministic properties despite being subjected to precise equations. However, insufficient mathematical tools were available at the time to thoroughly examine this phenomenon, which was not known then as chaos. Although Poincaré's comments about chaos were not taken seriously until the mid-1900s. It was found in another dimension.

The American meteorologist Edward Lorenz was developing mathematical models using computers to predict the weather in the first part of the 1960s [6]. In one of his experiments, he discovered that changing the model's initial values causes significantly different outcomes. This impact, known as the butterfly effect, predicts that modest changes in beginning conditions result in dramatically different outcomes. In other words, in chaotic systems, even modest calculation errors make long-term prediction impossible.

Three non-linear equations that are differential are proposed by Lorenz as a basic mathematical model. In simulation, we see that it exhibits complicated behavior; it is not periodic but rather follows a special geometric route known as the Lorenz attractor, which is becoming an indication of chaos.

In the 1970s and 1980s, scientists realized that chaos is not a unique phenomenon in weather prediction but exists across physics, biology, and economics. This resulted in the development of chaos theory as a distinct discipline of mathematics. Among these researchers are David Ruelle and Floris Takens [63], who introduce the concept of strange attractors, which represent the geometric shape of chaotic models in phase space. In 1975, James Yorke brought the notion of chaos in literature [41].

He demonstrates that period-doubling bifurcation can cause some nonlinear systems to transition from, regular to chaotic states. During that time, various chaotic models emerge, including the logistic map and the Henon attractor. These were used to better understand how dynamic systems change from ordered to chaotic behavior.

By the 1990s, chaos was no longer only a theoretical idea; it had become of considerable practical relevance in several sectors, like as: Electronics: Designing electronic circuits that generate chaotic signals for use in secure communications [81]. Biological systems: that describe the interaction between various biological species [74]. Robotics and Control: Using chaos control to improve the stability of robots and nonlinear dynamic systems [82]. Economics and Finance: Modeling financial markets and analyzing their volatility using chaotic models [38]. Astrophysics: The study of the irregular motions of celestial bodies and complex gravitational effects [13].

The wider study of chaos presents a new challenge: controlling chaotic systems rather than rejecting them. The "OGY method" was developed in the 1990s by Edward Ott and James Yorke as a way to manage chaos [54]. It relies on posing little challenges to one of the system's parameters to force it toward stabilization. In addition to this fundamental approach, several techniques have been developed, we mentioned Adaptive control works well with complex systems because it dynamically modifies parameters to accommodate

unforeseen events or changes in the system [19, 21]. Active control, which stabilizes vibrations in mechanical systems, depends on introducing external control signals to compel the system to follow a predetermined course [39, 20]. Backstepping control is appropriate for multi-stage nonlinear systems because it produces stability by successively building sub-controllers [71, 78].

Slide Mode Control: It steers the system along a predetermined "sliding path" and is highly resistant to external disruptions. Along with regulating the chaos, other issues arose connected to changing the character of the system itself [73, 15]. Chaos Suppression: The transformation of a chaotic system into periodic or stable behavior using techniques such as damping or parametric modulation, which is critical in electrical and biological engineering applications [83]. Chaotification: Reversing the course by introducing chaos into stable systems, which is useful in domains such as data encryption (where chaos mixes signals) and sensor efficiency [50].

During the same period, chaotic synchronization becomes a significant topic of research. Carroll and Pecora [56] establish chaotic synchronization by connecting nonlinear electrical circuits. They define the master-slave system (drive-response), in which one "driver" system imposes its dynamics on another. It was developed and extensively researched in recent years due to its applicability across various engineering and information science fields, particularly in secure communication and cryptology. Synchronization is defined as a phenomenon where a slave system is engineered to replicate the behavior of a master system. The literature examines various forms of synchronization, including total synchronization [45], anti-synchronization [67], generalized synchronization [40], projective synchronization [24], delayed synchronization [79], $R - S$ synchronization [28], and so on.

This thesis aims to investigate the control and synchronization of chaotic dynamical systems, with a particular emphasis on their application to epidemic models like the SIR model. The work is structured into four chapters, each focusing on a distinct aspect of the problem to achieve the stated goal.

In chapter 1, we introduce the fundamental concepts of dynamical systems, including

equilibrium points, attractors, and stability analysis. Key tools such as Lyapunov stability and bifurcation theory are discussed, along with their implications for understanding system behavior. The chapter also covers the Poincaré section, a method for making the study of continuous systems easier by turning them into discrete maps. These foundational concepts provide the necessary background for studying more complex chaotic systems.

Chapter 2 explores the theoretical underpinnings of chaotic systems, emphasizing their defining traits, including sensitivity to beginning conditions and the presence of unusual attractors, and bifurcation diagrams. Mathematical tools like Lyapunov exponents and fractal dimensions are introduced to quantify and analyze chaotic behavior. This chapter sets the stage for understanding the difficulties of controlling and synchronizing such systems.

Chapter 3 explores various methodologies to control chaotic systems. Techniques such as the OGY method, Pyragas method, active control, adaptive control, sliding mode control, and backstepping are discussed in detail. These methods are applied to the Lorenz system, a classic example of a chaotic system, to demonstrate their effectiveness in stabilizing chaotic behavior and guiding systems toward desired states or trajectories.

Finally, Chapter 4 focuses on the synchronization of chaotic systems, with a particular emphasis on the SIR model. Different types of synchronization, including complete synchronization, anti-synchronization, delayed synchronization, projective synchronization, generalized synchronization, and Q-S synchronization, are examined. The chapter also investigates the application of control techniques to achieve synchronization in the SIR model, comparing their performance using metrics such as control energy and integral dynamic error. This analysis highlights the importance of synchronization in understanding and controlling the spread of infectious diseases.

This thesis adds to the expanding body of knowledge about chaotic systems and how to manage them by merging theoretical ideas with real-world examples. It also underscores the relevance of synchronization in epidemiology, offering new perspectives on how to model and manage disease dynamics. The findings of this work have the potential to inform public health strategies, improve epidemic forecasting, and inspire more research in the areas of

dynamical systems and the theory of control.

Chapter 1

Overview on dynamical systems

The theory of dynamical systems provides a framework for modeling and analyzing phenomena that evolve over time and space. A dynamical system consists of three main components: first, a space where the dynamic behavior occurs, such as a torus, topological space, manifold, or locally compact metric space; second, a mathematical rule that defines how the system evolves over time; and third, an initial condition representing the system's state at the starting point. Since its inception, the theory has focused on qualitatively studying the properties of dynamical systems. The evolution of a system can be described as either a continuous-time process or a discrete-time process.

1.1 Dynamical systems and global existence theorems

Definition 1. A dynamical system on F is a C^1 - map:

$$g : \mathbb{R}^+ \times F \longrightarrow F,$$

where F is an open subset of $\mathbb{R}^n$ and if $g_t(x) = g(t, x)$, then g_t satisfies:

- (i) $g(0, x) = x$ for all $x \in F$.
- (ii) $(g_t \circ g_s)(x) = g_{t+s}(x)$ for $t, s \in \mathbb{R}^+$, and $x \in F$.

Definition 2. If B is an $n \times n$ matrix then the function $g(t, x) = e^{Bt}x$ describes a dynamical system on $\mathbb{R}^n$ and also, for each $x_0 \in \mathbb{R}^n$, $g(t, x_0)$ is the solution of the initial value problem

$$\begin{cases} \dot{x}(t) &= Bx, \\ x(0) &= x_0. \end{cases} \quad (1.1)$$

1.2 Continuous and discrete systems

Definition 3. (Continuous Dynamical Systems) A continuous dynamical system can be defined by a system of differential equations of the form $\dot{x} = g(x, t)$ where $x \in \mathbb{R}^n$ and $g : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$

Definition 4. (Discrete Dynamical Systems) A discrete dynamical system consists of a sequence of iterations for a function $g : F \rightarrow F$ where $F \subset \mathbb{R}^n$, and the beginning point is x_0 . The sequence of iterates is formed as $x_0, g(x_0), g^2(x_0), g^3(x_0), \dots$

1.3 Autonomous and non autonomous systems

A differential equation of the form $\dot{x} = g(x, t)$ is termed non-autonomous since g explicitly depends on t , rather than relying entirely on $x(t)$. Any non-autonomous system can be transformed into autonomous by introducing t as an additional variable. We define the ordered pair $\tilde{x} = (x, t)$ and reformulate the differential equation as follows:

$$\begin{cases} \dot{x} = g(x, t), \\ \dot{t} = 1. \end{cases} \Leftrightarrow \dot{\tilde{x}} = \tilde{g}(\tilde{x}),$$

where $\tilde{g} : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ is defined by : $\tilde{x} \rightarrow \begin{pmatrix} g(x, t) \\ 1 \end{pmatrix}$.

1.4 Flows

The flow of the system

$$\begin{cases} \dot{x} = g(t, x), \\ x(0) = x_0, \end{cases} \quad (1.2)$$

is defined by $\chi_t(x) : W \rightarrow \mathbb{R}^n$ where $\chi_t(x) = \chi(t, x)$ is a smooth vector function of $x \in W \subset \mathbb{R}^n$ and $t \in I \subset \mathbb{R}^+$, satisfying the equation

$$\frac{d}{dt}\chi_t(x) = g(\chi_t(x)), \quad (1.3)$$

for all t and $\chi(0, x) = x$. The flow $\chi_t(x)$ satisfies the following properties:

- (a) $\chi_0 = I_W$.
- (b) $\chi_{t+s} = \chi_t \circ \chi_s$.

The function $\chi(t, x_0)$ is called the **trajectory** for a given initial state x_0 ; and the set $\{\chi(t, x_0), t \in I \subset \mathbb{R}^+\}$ is the **orbit** of $x_0 \in \mathbb{R}^n$: The set of all qualitative trajectories of the system is called **phase space**.

Example 5. We consider the Lorenz system given by:

$$\begin{cases} \dot{x}_1 = \sigma(x_2 - x_1), \\ \dot{x}_2 = \rho x_1 - x_2 - x_1 x_3, \\ \dot{x}_3 = x_1 x_2 - \beta x_3, \end{cases} \quad (1.4)$$

where x_1, x_2 , and x_3 represents the variable states of the system, and σ, ρ , and β denote the parameters of the system, respectively. Lorenz used the values $\sigma = 10$, $\rho = 28$, and $\beta = \frac{8}{3}$. Figure (1.1) gives the phase space trajectories of system (1.4).

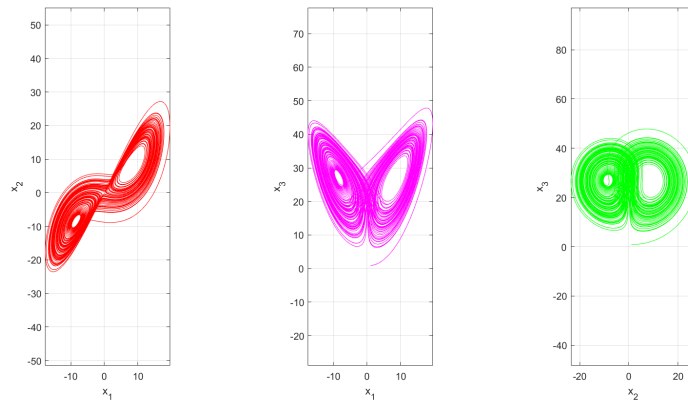


Figure 1.1: Phase space trajectories of system (1.4).

1.5 Conservative systems and dissipative systems

Conservative and dissipative systems are described by the divergence of their respective vector fields, indicating volume or area conservation in their state space or phase plane. A system is considered conservative if the divergence of its vector field is zero. Dissipative means the vector field has a negative divergence. In a conservative system, the phase volume remains constant during flow. In a dissipative system, the phase volume declines with time and eventually reaches zero as $t \rightarrow +\infty$. Positive vector field divergence leads to a progressive expansion of phase volume.

Lemma 6. We consider the system (1.1) which generates a flow χ_t . V a volume of phase space at time $t = 0$, $V(t) = \chi_t(V)$ the image of V by χ_t , we have:

$$\frac{dV}{dt}\bigg|_{t=0} = \int_V \text{div} g dx_1 \dots dx_n \text{ and } , \text{div} g = \sum_{i=1}^n \frac{\partial g_i}{\partial x_i}.$$

1.6 Some definitions

This section provides essential preliminary definitions concerning the flow of a system .

Definition 7. (Invariant set)[8]

A set $B \subset \mathbb{R}^n$ is considered an invariant set under the flow χ_t if for every point $q \in B$, $\chi_t(q) \in B$ for all $t \in \mathbb{R}$.

The asymptotic behavior of a trajectory may be associated with limit points, sets, or cycles.

Definition 8. (Limit points : μ - and σ - limit points)[57]

A point $q \in \mathbb{R}^n$ is called an μ - (resp σ -) limit point if there exists a sequence $\{t_j\}$ with $t_j \rightarrow +\infty$ (resp $t_j \rightarrow -\infty$) such that $\chi(t_j, x) \rightarrow q$ as $j \rightarrow +\infty$. The μ - limit set(cycle) is denoted by $\Delta(x)$ and is defined as:

$$\Delta(x) = \{x \in \mathbb{R}^n \mid \text{there exist } \{t_j\} \text{ with } t_j \rightarrow +\infty \text{ and } \chi(t_j, x) \rightarrow q \text{ as } j \rightarrow +\infty\}.$$

Similarly, the σ - limit set (cycle), $\Theta(x)$ is defined as:

$$\Theta(x) = \{x \in \mathbb{R}^n \mid \text{there exist } \{t_j\} \text{ with } t_j \rightarrow -\infty \text{ and } \chi(t_j, x) \rightarrow q \text{ as } j \rightarrow +\infty\}.$$

Definition 9. (Attracting set)[57]

A closed invariant set $B \subset \mathbb{R}^n$ for a flow χ_t is classified as an attractive set if there exists some neighborhood V within B such that:

$$\forall t \geq 0, \chi(t, V) \subset V \text{ and } \bigcap_{t \geq 0} \chi(t, V) = B$$

Definition 10. (Basin of attraction)[57]

The domain (called as basin of attraction) of an attracting set B is defined as $\bigcup_{t \leq 0} \varphi(t, V)$ where V is any open set in $B \subset \mathbb{R}^n$.

Definition 11. (Attractor)[48]

The set A is an attractor if:

- For any neighborhood W of A there is a neighbourhood U of A such that any solution $x(t, x_0) = \chi_t(x_0)$ will remain in W if $x_0 \in U$.
- $\chi_t(U) = A, \quad \forall t \geq 0$.
- There is a dense orbit in A .

Categories of attractors There exist two categories of attractors: Regular attractors and unusual (or chaotic) attractors.

(i) **Regular attractors**

There are three types of regular attractors that define the evolution of non-chaotic systems:

- **Fixed point:** is the simplest and most common attraction, where the system reaches a resting state.
- **Limit cycle:** all nearby trajectories are attracted to this closed trajectory.
- **Torus:** denotes the motions arising from two or more separate oscillations, sometimes referred to as quasi-periodic movements.

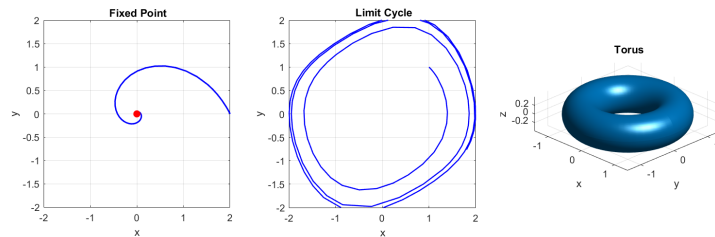


Figure 1.2: Regular attractors.

(ii) **Chaotic attractors**

Whether A is a set of $\mathbb{R}^n$, then A is termed a strange attractor if it is chaotic.

1.7 Poincaré maps

Consider the following ordinary differential equation:

$$\dot{x} = g(x), \quad x \in \mathbb{R}^n, \quad (1.5)$$

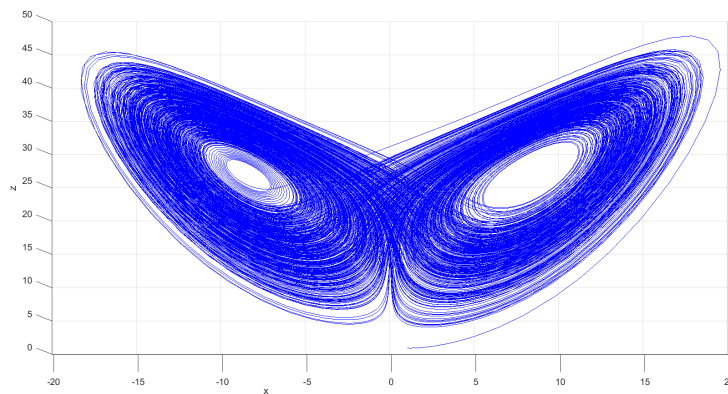


Figure 1.3: Chaotic attractor of Lorenz.

Where $g : D \rightarrow \mathbb{R}^n$ is $\mathcal{C}^r$ on some open set $D \subset \mathbb{R}^n$. Let $\chi(t, \cdot)$ denote the flow generated by (1.5). Suppose that (1.5) has a periodic solution of period T which we denote by $\chi(t, x_0)$ where $x_0 \in \mathbb{R}^n$ is any point through which this periodic solution transmits (i.e., $\chi(t+T, x_0) = \chi(t, x_0)$). Let Π be an $n-1$ dimensional surface transverse to the vector field at x_0 (note: "transverse" means that $g(x) \cdot n(x) \neq 0$ where $\cdot$ denotes the vector dot product and $n(x)$ is the normal to Π at x); we refer to Π as a cross-section to the vector field (1.5). We know that $\chi(t, x)$ is $\mathcal{C}^r$ if $g(x)$ is $\mathcal{C}^r$; thus, we can find an open set $U \subset \Pi$ such that the trajectories starting in U return to Π in a time close to T . The map that associates points in U with their points of first return to Σ is referred to as the Poincaré map, which we denote by P , such that

$$P : U \rightarrow \Pi,$$

$$x \rightarrow \chi(\tau(x), x),$$

where $\tau(x)$ is the time of first return of the point x to Π . Note that, by construction, we have $\tau(x_0) = T$ and $P(x_0) = x_0$. Therefore, a fixed point of P coincides. to a periodic orbit of (1.5), and a period k point of P (i.e., a point $x \in U$ such that $P^k(x) = x$ provided $P^i(x) \in U$, $i = 1, \dots, k$) coincides. to a periodic orbit of (1.5) that pierces Π k times.

1.8 System equilibrium points

Definition 12. A point x^* serves as an equilibrium point of system

$$\dot{x} = g(x), x \in \mathbb{R}^n, \quad (1.6)$$

if and only if $g(x^*) = 0$ equivalently $\chi(t, x^*) = x^*$, $\forall t \in \mathbb{R}$, where χ is the flow generated by the system (1.6).

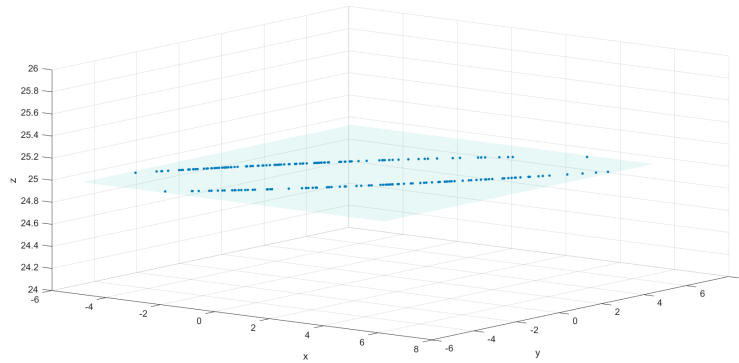


Figure 1.4: Poincaré maps of Lorenz system.

- Remark 13.**
1. The equilibrium point is often referred to as a crucial point, fixed point, or stationary point.
 2. A system can have an unlimited number of equilibrium points, a finite number of equilibrium points, one equilibrium point, or no equilibrium points at all. For example, $\dot{x} = 1$ (no equilibrium points), $\dot{x} = 5x - 1$ (only one equilibrium point), $\dot{x} = x^2 - 2$ (two equilibrium points), and $\dot{x} = \cos x$ (infinite number of equilibrium points).

1.9 Stability theory

Stability is crucial for understanding linear and nonlinear dynamic systems. Stability analysis examines a system's behavior when it deviates from its equilibrium state. Analyze the system's trajectory when it is close to equilibrium. Equilibrium points are essential for studying dynamic systems. Henri Poincaré (1854–1912) demonstrated that knowing the equilibrium points and stability of a dynamic system with several variables is enough to define it, rather than calculating exact solutions. This significant finding streamlines research on nonlinear systems around these spots. To assess equilibrium point stability, investigate the behavior of solutions in a limited area around it. Multiple definitions of stability have been presented. This list includes the Lyapunov notion of stability.

1.10 Stability according to Lyapunov

We consider the nonlinear dynamical system:

$$\begin{cases} \dot{x} = g(t, x), \\ x(0) = x_0, \end{cases} \quad (1.7)$$

where $x \in \mathbb{R}^n$ and $g : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, a continuous function.

Definition 14. In the sense of Lyapunov, the equilibrium point x^* of system (1.7) is referred to as stable.

$$\forall \epsilon > 0, \exists \lambda_\epsilon > 0; \|x^* - x_0\| < \lambda_\epsilon \Rightarrow \|x(t) - x^*\| < \epsilon,$$

in the opposite case, we said instable.

Thus, Lyapunov stability means that the trajectory $x(t)$, with an initial condition x_0 , must remain close to the equilibrium point x^* . For this, the solutions $x(t)$ must remain within the region bounded by $\|x(t) - x^*\| < \epsilon$.

Definition 15. The equilibrium point x^* of system (1.7) is referred to as asymptotically stable in Lyapunov sense if

$$\forall \epsilon > 0, \exists \lambda_\epsilon > 0; \|x^* - x_0\| < \lambda_\epsilon \Rightarrow \lim_{t \rightarrow +\infty} \|x(t) - x^*\| = 0.$$

Asymptotic stability means that not only is the equilibrium point stable, but that we are also able to determine a domain close to the equilibrium point such that any trajectory, originating from an initial state x_0 belonging to this domain, tends towards x^* when t approaches $+\infty$. Asymptotic stability is the property generally sought in practice. It is necessary however, note that the definition above does not provide any information about speed to which the trajectory $x(t)$ converges towards equilibrium. That's why we introduce the the notion of exponential stability that allows characterizing this speed.

Definition 16. The equilibrium point x^* of system (1.7) is called exponentially stable in the sense of Lyapunov if there exist $\delta, c, \alpha \in \mathbb{R}_+^*$ such that:

$$\|x^* - x_0\| < \delta \Rightarrow \|x(t) - x^*\| < c\|x_0\|e^{-\alpha t}, \quad \forall t > t_0.$$

This stability means that the state vector, for an initial condition x_0 , converges to an equilibrium point x^* faster than an exponential function; α is called the rate of convergence.

Moreover, exponential stability implies asymptotic stability, which in turn implies the stability of an equilibrium point. In each of the previous definitions, stability is defined locally.

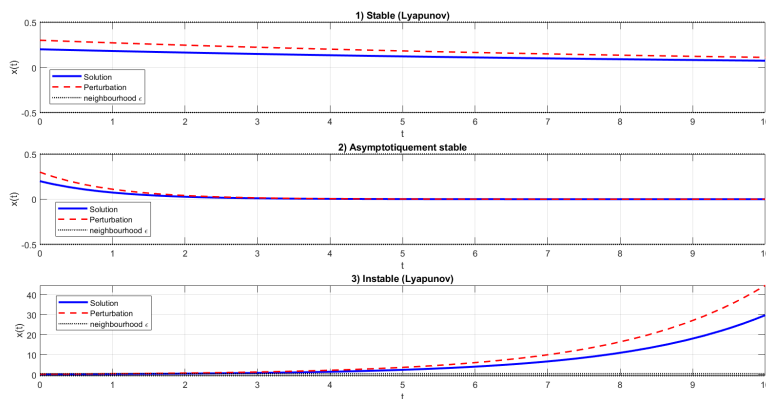


Figure 1.5: Stability in Lyapunov sense.

Remark 17. If the equilibrium point is stable, asymptotically stable (exponentially stable) for any initial state vector x_0 in $\mathbb{R}^n$ then the equilibrium point is globally stable, globally asymptotically stable (globally exponentially stable).

1.11 Techniques for analyzing stability in the context of Lyapunov

1.11.1 Direct method

Lyapunov introduced a technique known as the direct Lyapunov approach, which relies on identifying a certain function, denoted $P(x)$ and called the Lyapunov function. If such a function exists, then the system is stable. This method is difficult to implement, but it has a much broader scope. Observe that Lyapunov's direct method provides a sufficient requirement for stability, meaning that the system can be stable even in the absence of a Lyapunov function, as there is no general rule for finding such a function.

Definition 18. The function $P : E \rightarrow \mathbb{R}$, $E \subset \mathbb{R}^n$ is said to be positively (resp. negatively) definite if $P(x) > 0$ (resp $P(x) < 0$), for all $x \in E$ and $x \neq 0$. Similarly, P is called positively (resp negatively) semi-definite in D if $P(x) \geq 0$ (resp $P(x) \leq 0$), for all $x \in E$ and $x \neq 0$.

Definition 19. We consider the system $\dot{x} = g(x)$. The total derivative or orbital derivative of P in the direction of the vector field g is defined as:

$$\frac{dP}{dt} = \frac{\partial x}{\partial t} \frac{\partial P}{\partial x} = g(x) \frac{\partial P}{\partial x}$$

Definition 20. Let $E \subset \mathbb{R}$ be an open neighborhood of the equilibrium point x^* . Then the function $P : E \rightarrow \mathbb{R}$, satisfying the following properties:

- (i) P is continuously differentiable,
- (ii) $P > 0$ for all $x \in E - \{x^*\}$ and $P(x^*) = 0$

is called a Lyapunov function.

Theorem 21. (Lyapunov theorem)[46]

Assume that the origin constitutes an equilibrium point of $\dot{x} = g(x)$ with $x \in \mathbb{R}^n$ and consider P a Lyapunov function in a vicinity E of the origin. If

- (i) If $\dot{P}$ is negative semi-definite in E , the origin is stable.
- (ii) If $\dot{P}$ is negative definite in E , then the origin is asymptotically stable.
- (iii) If $\dot{P}$ is positive definite in E , the origin is unstable.

1.11.2 Indirect method

Lyapunov and others have noted through the study of trajectories of integral curves in the neighborhood of equilibrium that, in the majority of cases, the equilibrium points of nonlinear systems can be simplified to correspond with the equilibrium points of linear systems. Therefore, the study of a linear system is easy since it is resolved using a purely algebraic criterion. Consequently, the most traditional approach for ascertaining the nonlinear stability of the equilibrium point reduces to the linearization of the system (1.7) at this point (equilibrium point).

Dynamic system linearization

Examine the nonlinear system (1.7) repeated below:

$$\dot{x}(t) = g(x, t). \tag{1.8}$$

Let x^* an equilibrium point (thus $g(x^*) = 0$), the linear system

$$\dot{x} = Ax \text{ with } A = Jg(x^*) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \frac{\partial g_n}{\partial x_2} & \cdots & \frac{\partial g_n}{\partial x_n} \end{pmatrix}_{x=x^*}, \quad (1.9)$$

is called the linearization of the system at point x^* and $Jg(x^*)$ Jacobian matrix.

By a change of coordinates, the fixed point of (1.9) is reduced to the origin ($g(0) = 0$) and the Taylor series expansion of g around $x = 0$ gives:

$$g(x) = Jg(0)x + \frac{1}{2!}Jg(0)(x, x) + \frac{1}{3!}Jg(0)(x, x, x) + \dots \quad (1.10)$$

The indirect Lyapunov method, to study the stability around an equilibrium point x^* , consists of studying the linear system $\dot{x} = Ax$.

In the case where the matrix $A = Jg(0)$ has n distinct eigenvalues $\lambda_i; i = 1, \dots, n$, the solution to (1.9) is:

$$x(t) = \sum_{i=1}^n k_i v^{(i)} e^{\lambda_i t},$$

where $v^{(i)}$ is the eigenvector corresponding to the eigenvalue λ_i and the $k_i; i = 1, \dots, n$ are constants (determined by the initial conditions).

Theorem 22. (i) The equilibrium point is asymptotically stable if every eigenvalue of matrix A has a negative real component.

(ii) The equilibrium point is considered unstable if there is an eigenvalue of matrix A with a positive real component.

(iii) The equilibrium point will be stable, although not asymptotically, if there is at least one eigenvalue that is entirely imaginary or zero and at least one other eigenvalue that has a negative real portion.

Definition 23. (Classification of the equilibrium points of linear systems)[57]

1. If the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$ are reals and of the same sign, the equilibrium point $x = 0$ is called node.
2. If the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$ are real, non-zero and of a different sign, the equilibrium point $x = 0$ is called a saddle.

3. If the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$ are complex with $Re(\lambda_k) \neq 0, k = 1, \dots, m$ the equilibrium point $x = 0$ is called focus.
4. If the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$ are complex with $Re(\lambda_k) = 0, k = 1, \dots, m$ the equilibrium point $x = 0$ is called center.

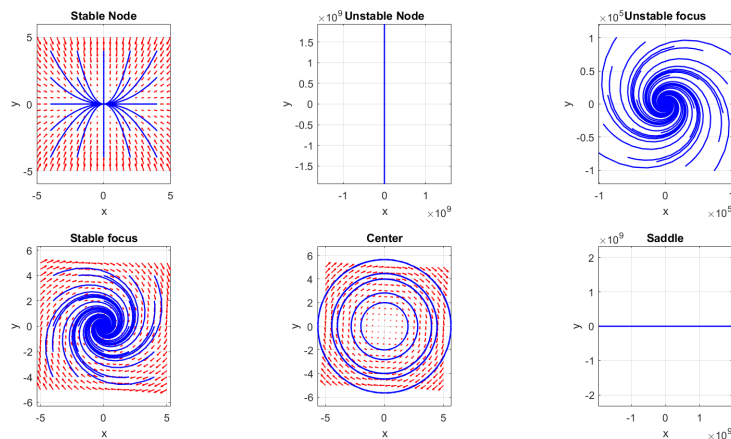


Figure 1.6: Equilibrium points classification.

Topological equivalence of systems

We will define in this section a notion of equivalence between two systems, very important for the study of dynamic systems, especially complex behaviors.

Two flows φ_t and ψ_t are considered to have the same topology in the neighborhoods of equilibrium points if there exists a homeomorphism (a continuous and bijective function) h that couples the two flows by sending their respective equilibrium points to each other, that is, $h \circ \varphi_t = \psi_t \circ h$.

Theorem of Hartman–Grobman

We propose to compare, in the neighborhood of an equilibrium point x^* , the solutions of the system $\dot{x} = g(x)$, and those of its linearization (1.9).

Definition 24. The equilibrium point x^* of the system $\dot{x} = g(x)$ is considered to be hyperbolic when every eigenvalue of the matrix $\frac{\partial g}{\partial x}(x^*)$ has a non-zero real part.

Theorem 25. Suppose a system $\dot{x} = g(x)$, with flow χ_t . If x^* is a hyperbolic equilibrium point, hence, it's present a neighborhood V of x^* on which the flow χ_t is topologically equivalent to the flow of the linearized system at x^* .

Consequently, we have the following theorem:

Theorem 26. Let x^* an equilibrium point of system (1.7).

- (i) The equilibrium point is asymptotically stable if every eigenvalue of the matrix $Jg(x^*)$ has a negative real component.
- (ii) The equilibrium point is considered unstable if, for every $Jg(x^*)$ matrix, there is at least one eigenvalue with a positive real component.

1.11.3 Stable manifold theorem

Invariant manifolds

The eigenvalues λ_i of the matrix A , which are often complex and not distinct, and the corresponding characteristic vector subspaces E_λ are taken into account when a linear system $\dot{x} = Ax$ in $\mathbb{R}^n$ is considered. Here, we clarify

- The stable subspace $E^s = \oplus_{\text{Re}(\lambda) < 0} E_\lambda$.
- The unstable subspace $E^u = \oplus_{\text{Re}(\lambda) > 0} E_\lambda$.
- The central subspace $E^c = \oplus_{\text{Re}(\lambda) = 0} E_\lambda$.

We have

$$E^s \oplus E^u \oplus E^c = \mathbb{R}^n.$$

and $w \in E^s \Rightarrow \lim_{t \rightarrow +\infty} e^{tA}w = 0$, $w \in E^u \Rightarrow \lim_{t \rightarrow -\infty} e^{tA}w = 0$; Consequently, all solutions originating from initial conditions in the stable subspace are attracted towards the origin, while those originating from initial conditions in the unstable subspace are repelled by the origin.

- At the equilibrium point x^* of the system (1.7), there exists a differentiable manifold W^s that is tangent to the stable subspace E^s of the linearized system (1.9) at x^* . for all solutions starting from W^s to approach x^* as t approaches $+\infty$.
- All solutions emanating from the unstable manifold W^u gravitate towards x^* as t approaches $-\infty$. This manifold is differentiable and tangent to the unstable subspace E^u at x^* .

- Along the center subspace E^c lies the central manifold W^c . The linearization of the system in x^* does not dictate the asymptotic behavior of the orbits included in the central manifold.

Theorem 27. Let's consider a system with 0 as an equilibrium point. Let E^s , E^u , and E^c be the subspaces: stable, unstable, and center, respectively, of the linearized system at 0. Then the system admits invariant manifolds W^s , W^u , and W^c passing through the point 0 and tangent respectively to the subspaces E^s , E^u , and E^c .

The solutions originating from W^s (resp W^u) tend exponentially towards 0 as t approaches $+\infty$ (resp t approaches $-\infty$). The nonlinear terms dictate how the solutions behave in the manifold W^c .

1.12 Bifurcation theory

We consider a continuous system

$$\dot{x} = g(x, \nu)$$

where $\nu \in \mathbb{R}$.

In bifurcation theory, the topological characteristics of a flow, such as the number of stationary points and periodic orbits, are discussed in relation to how these characteristics change when one or more parameters are altered.. Differential equations can be understood through various approaches, including analytic descriptions and topological classifications of possible behaviors under small perturbations.

Saddle-Node Bifurcation

A saddle-node bifurcation is when there are two equilibrium points (stable and unstable) before the bifurcation. After the bifurcation, there are no equilibrium points. Consider the following planar system:

$$\begin{cases} \dot{x} = \nu - x^2, \\ \dot{y} = -y, \end{cases}$$

the system has two equilibrium points of $(\sqrt{\nu}, 0)$ and $(-\sqrt{\nu}, 0)$. The linearisation around $(\sqrt{\nu}, 0)$ gives the following Jacobian matrix

$$\begin{pmatrix} -2\sqrt{\nu} & 0 \\ 0 & -1 \end{pmatrix}.$$

So the point $(\sqrt{\nu}, 0)$ is a stable node. The linearisation around $(-\sqrt{\nu}, 0)$ gives the following Jacobian matrix

$$\begin{pmatrix} 2\sqrt{\nu} & 0 \\ 0 & 1 \end{pmatrix}.$$

So the point $(-\sqrt{\nu}, 0)$ is a col (instable), qualitative change.

Transcritical Bifurcation

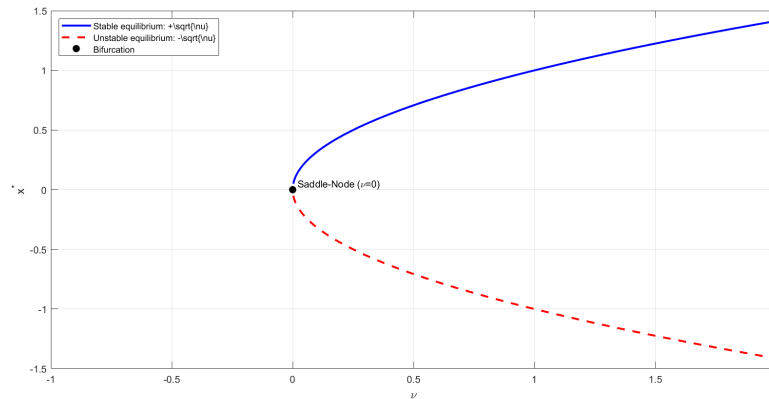


Figure 1.7: Saddle-Node bifurcation.

There are many different types of bifurcation, and one of them is called the transcritical bifurcation. In this sort of bifurcation, the stability characteristics of the equilibrium points are altered for different values of the parameters. We have the following planar system :

$$\begin{cases} \dot{x} = \nu x - x^2, \\ \dot{y} = -y, \end{cases}$$

it has two equilibrium points: $(0; 0)$ and $(\nu; 0)$. The Jacobian matrix $(0; 0)$ is equal to:

$$\begin{pmatrix} -\nu & 0 \\ 0 & -1 \end{pmatrix}.$$

Which shows that $(0; 0)$ is a stable node for $\nu < 0$ and a col for $\nu > 0$. On the other hand, the Jacobian matrix in $(\nu; 0)$ is:

$$\begin{pmatrix} -\nu & 0 \\ 0 & -1 \end{pmatrix}.$$

Which shows that $(\nu; 0)$ is a col for $\nu < 0$ and a stable node for $\nu > 0$.

Pitchfork Bifurcation

In pitchfork bifurcations, a stable equilibrium point transitions to an unstable state, resulting

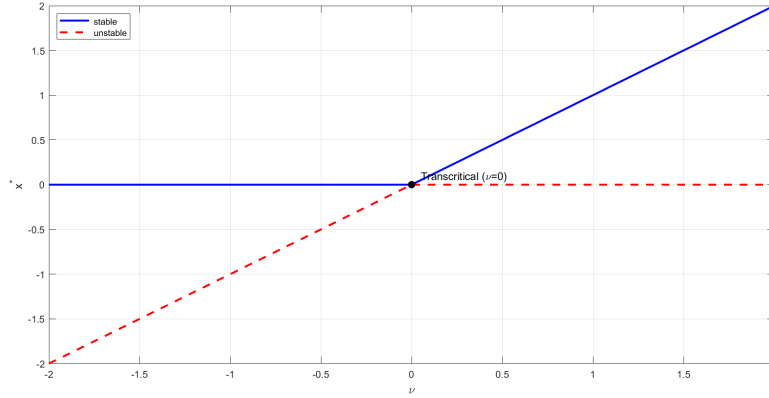


Figure 1.8: Transcritical bifurcation.

in the emergence of two stable equilibrium points. This transition may occur in either a supercritical or subcritical state.

1. Supercritical Pitchfork

This bifurcation indicates a modification in the quantity of equilibrium points at the specified bifurcation value. It can then have the shape of a fork. Consider a two-dimensional system represented by:

$$\begin{cases} \dot{x} = \nu x - x^3, \\ \dot{y} = -y. \end{cases} \quad (1.11)$$

For $\nu < 0$, The system possesses a singular equilibrium point at the origin $(0,0)$. The Jacobian matrix at this equilibrium point is provided by:

$$J_{(0,0)} = \begin{pmatrix} \nu & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalues $J_{(0,0)}$ are $\nu, -1$, indicating that the fixed point origin is a stable node. For $\nu > 0$, there are three equilibrium points $(0;0)$, $(\sqrt{\nu};0)$ and $(-\sqrt{\nu};0)$. The Jacobian matrix of (1.10) calculated at these fixed points are given by:

$$J_{(0,0)} = \begin{pmatrix} \nu & 0 \\ 0 & -1 \end{pmatrix}, J_{(\sqrt{\nu},0)} = \begin{pmatrix} -2\nu & 0 \\ 0 & -1 \end{pmatrix}, J_{(-\sqrt{\nu},0)} = \begin{pmatrix} -2\nu & 0 \\ 0 & -1 \end{pmatrix}.$$

So, the equilibrium point $(0;0)$ is a saddle, $(\sqrt{\nu};0)$ and $(-\sqrt{\nu};0)$ are stable nodes. At the point of bifurcation $\nu = 0$, the stable node bifurces into three points of equilibrium: one saddle and two stable nodes. As the parameter increases, the amplitudes of the newly formed stable nodes also increase.

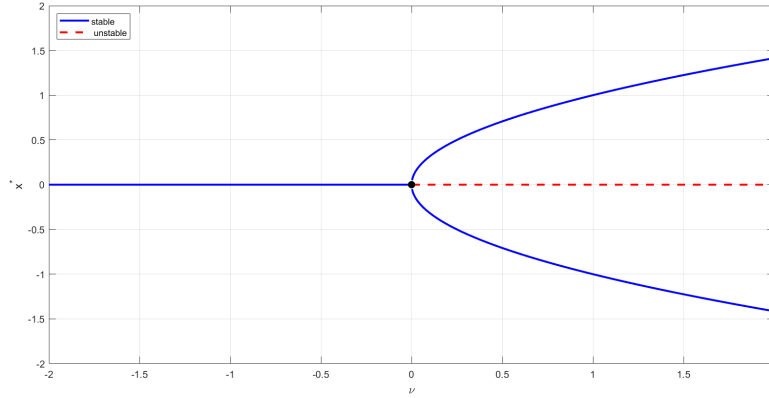


Figure 1.9: Supercritical pitchfork.

2. Subcritical Pitchfork

Examine a parameter-dependent two-dimensional system characterized by:

$$\begin{cases} \dot{x} = \nu x + x^3, \\ \dot{y} = -y, \end{cases} \quad (1.12)$$

with the parameter $\nu \in \mathbb{R}$. When $\nu < 0$, the system (1.12) possesses three distinct fixed points, namely $(0; 0)$, $(\sqrt{\nu}; 0)$ and $(-\sqrt{\nu}; 0)$. The Jacobians of the system assessed at these fixed points are provided by:

$$J_{(0,0)} = \begin{pmatrix} \nu & 0 \\ 0 & -1 \end{pmatrix}, J_{(\sqrt{\nu},0)} = \begin{pmatrix} 4\nu & 0 \\ 0 & -1 \end{pmatrix}, J_{(-\sqrt{\nu},0)} = \begin{pmatrix} 4\nu & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalues $J_{(0,0)}$ are ν and -1 , both of the same sign. Consequently, the fixed point origin is a stable node for $\nu < 0$. In a similar manner, by calculating the eigenvalues of the remaining two Jacobian matrices of the system, it can be observed that the fixed points $(\pm\sqrt{\nu}, 0)$ are saddle points. For $\nu > 0$, the system possesses a solitary fixed point at the origin, which is a saddle point. Upon constructing the phase portrait of the system, it becomes evident that when the parameter surpasses the bifurcation point $\nu = 0$, the stable node at the origin aligns with the saddles and then bifurcates into a saddle.

Hopf Bifurcation

A Hopf bifurcation occurs when a stable equilibrium point transitions to instability, resulting in a limit cycle, or when the reverse happens. Supercritical and subcritical Hopf bifurcations are the two different kinds of these bifurcations. A supercritical Hopf bifurcation

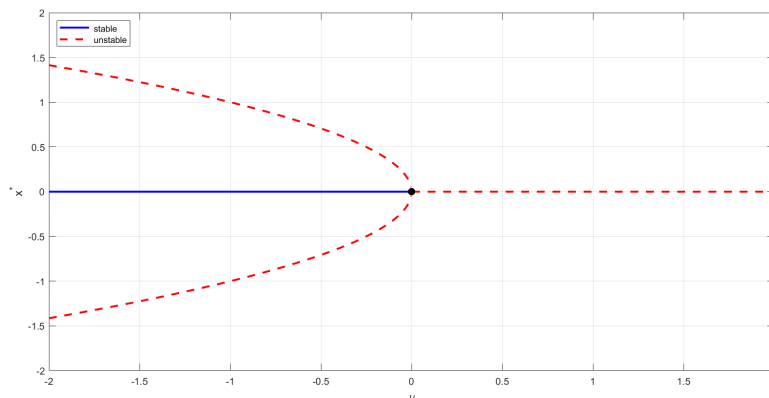


Figure 1.10: Subcritical pitchfork.

transpires when stable limit cycles emerge from an unstable equilibrium point. Subcritical Hopf bifurcations arise when an unstable limit cycle is produced from a stable equilibrium point. Another name for it is a hard bifurcation, when there is subcritical Hopf bifurcation, nonzero solutions may trend to infinity and a steady-state solution may become unstable as the parameter changes.

1. Supercritical Hopf Bifurcation

Consider a two-dimensional system with parameter $\nu \in \mathbb{R}$,

$$\begin{cases} \dot{x} = \nu x - y - x(x^2 + y^2), \\ \dot{y} = x + \nu y - y(x^2 + y^2). \end{cases}$$

The system has a unique fixed point at the origin. In polar coordinates, the system can be written as:

$$\begin{cases} \dot{r} = \nu r - r^3, \\ \dot{\theta} = 1. \end{cases}$$

When $\nu < 0$, the fixed point at the origin $r = 0$ is a stable spiral, with all trajectories converging towards it in a counterclockwise orientation. For $\nu = 0$, the origin remains a stable spiral, albeit very weakly. For $\nu > 0$, the origin constitutes an unstable spiral, and in this scenario, a stable limit cycle exists at $r = \sqrt{\nu}$. We are now keen to see the behavior of the eigenvalues as the parameter varies. The Jacobian matrix at the fixed point, the origin, is computed as

$$J_{(0,0)} = \begin{pmatrix} \nu & -1 \\ 1 & \nu \end{pmatrix}.$$

That possesses the eigenvalues $\nu \pm i$. The origin is a stable spiral when $\nu < 0$ and an unstable spiral when $\nu > 0$. Consequently, as anticipated, the eigenvalues traverse the imaginary axis from left to right when the parameter transitions from negative to positive values. A supercritical Hopf bifurcation transpires when a stable spiral transitions into an unstable spiral encircled by a limit cycle.

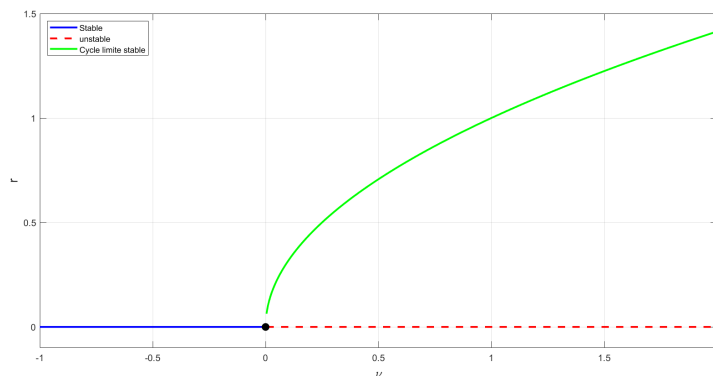


Figure 1.11: Supercritical Hopf bifurcation.

2. Subcritical Hopf Bifurcation

Consider a two-dimensional system represented by:

$$\begin{cases} \dot{x} = \nu x - y + x(x^2 + y^2) - x(x^2 + y^2)^2, \\ \dot{y} = x + \nu y + y(x^2 + y^2) - y(x^2 + y^2)^2. \end{cases} \quad (1.13)$$

Where $\nu \in \mathbb{R}$ is the parameter. In polar coordinates, the system can be transformed as:

$$\begin{cases} \dot{r} = \nu r + r^3 - r^5, \\ \dot{\theta} = 1. \end{cases}$$

This system possesses a distinct fixed point at the origin. When $\nu > 0$, the fixed point at the origin $r = 0$ is a stable spiral, with all trajectories converging towards it in an anti-clockwise orientation, and the system possesses only a stable limit cycle. For $\nu < 0$, the fixed point at the origin is an unstable spiral, and the system exhibits two limit cycles, one of which is stable while the other is unstable.

1.13 Conclusion

This chapter addresses certain aspects of dynamical systems theory: the concepts of stability and analysis techniques (the Lyapunov function and linearization for fixed points, as well as

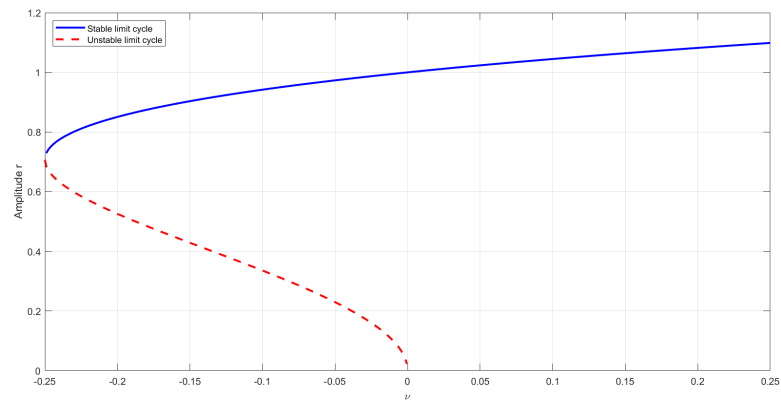


Figure 1.12: Subcritical Hopf bifurcation.

the Poincaré section for limit cycles). It also covers the four categories of local bifurcations of codimension one (node-saddle, transcritical, pitchfork, and Hopf).

Chapter 2

Chaos theory

The concept of chaos was first formalized in discrete dynamical systems by Li and Yorke [42] in 1975, revealing that deterministic systems can exhibit highly irregular and unpredictable behavior. This discovery stimulated extensive research into nonlinear dynamics, leading to multiple definitions and frameworks for chaos. Initially confined to specialized research literature, the field gained broader recognition with Devaney's influential monograph [16], which provided a rigorous and accessible formulation of chaos and helped integrate it into mainstream mathematical discourse. By the early 1990s, chaos theory had become a standard topic in both theoretical and applied sciences.

2.1 Approaches described chaos

Theorem 28. Li-Yorke theorem:[42]

Let $J \subset \mathbb{R}$ and $g : J \rightarrow J$ be a continuous map. Assume there exists a point $b \in J$ such that $c_1 = g(b), c_2 = g^2(b), c_3 = g^3(b)$, and the following condition holds:

$$c_3 \leq b < c_1 < c_2 \quad (\text{or } c_3 \geq b > c_1 > c_2).$$

Then:

1. For every integer $i = 1, 2, \dots$, there exists a i -periodic point x^* such that $g^i(x^*) = x^*$.
2. There exists an uncountable set $S \subset J$, containing no periodic points, such that:
 - (i) For every $x, y \in S$ with $x \neq y$:

$$\limsup_{n \rightarrow \infty} |g^n(x) - g^n(y)| > 0, \quad \liminf_{n \rightarrow \infty} |g^n(x) - g^n(y)| = 0.$$

(ii) For each $x \in S$ and each periodic point $x_{\text{per}} \in J$ with $x \neq x_{\text{per}}$:

$$\limsup_{n \rightarrow \infty} |g^n(x) - g^n(x_{\text{per}})| = 0.$$

Definition 29. (Dense set)

A subset A of E is said to be a dense set if $\bar{A} = E$, where (E, ρ) is a topological space.

Definition 30. (Topologically transitive)

Let $g : J \rightarrow J$ be a continuous map of the interval J and let Λ be an invariant set. If for all open sets U_1 and U_2 such that $U_1 \cap \Lambda \neq \emptyset$ and $U_2 \cap \Lambda \neq \emptyset$ there exists $k > 0$ such that $g^k(U_1) \cap U_2 \neq \emptyset$, then g is topologically transitive on Λ .

Definition 31. (Sensitive dependence on initial conditions)

A map $g : J \rightarrow J$ is characterized by sensitive dependence on initial conditions if there exists $\eta > 0$ such that for all $x \in J$ and any neighborhood $N_\epsilon(x) = (x - \epsilon, x + \epsilon)$ there exists $y \in N_\epsilon(x)$ and an integer $k > 0$ such that the property $|g^k(x) - g^k(y)| > \eta$ is satisfied.

Definition 32. Devany's definition:[\[16\]](#)

A map $g : E \rightarrow E$ (where E is either a topological space or a metric space) is classified as a chaotic map on an invariant subset $A \subset E$ if it fulfills the following criteria.

- (i) g has sensitive dependence on initial conditions on A ;
- (ii) g is topologically transitive on A ;
- (iii) The subset of g - periodic points is dense in A .

2.2 Chaos characterization

2.2.1 Non-linearity

A chaotic system is a nonlinear dynamical system. A linear system is incapable of exhibiting chaos. The erratic behavior of a nonlinear dynamic system arises from its non-linearities. To forecast phenomena produced by dynamic systems, one often constructs a mathematical model that delineates the relationship between a collection of causes and various outcomes. If this relationship constitutes an operation of proportionality, the phenomena is linear. In a non-linear phenomenon, the outcome is not commensurate with the cause.

2.2.2 Determinism

Determinism refers to the capacity to forecast the future of a phenomenon based on a previous or present event. The erratic progression of a chaotic system's behavior is attributable to nonlinearity. In stochastic events, predicting the track of any particle is unfeasible. Conversely, a chaotic system governed by deterministic and non-probabilistic underlying principles.

2.2.3 Sensitivity to initial conditions

Most chaotic systems exhibit sensitivity to initial conditions; for two arbitrary initial conditions very close initially; the two trajectories corresponding to these initial data differ exponentially, therefore both trajectories are incomparable.

The two very similar initial conditions are attributed to the chaotic Lorenz system presented below. Initially, the two systems develop in the same way; however, their behavior diverges rapidly. This is represented in the illustration (2.1).

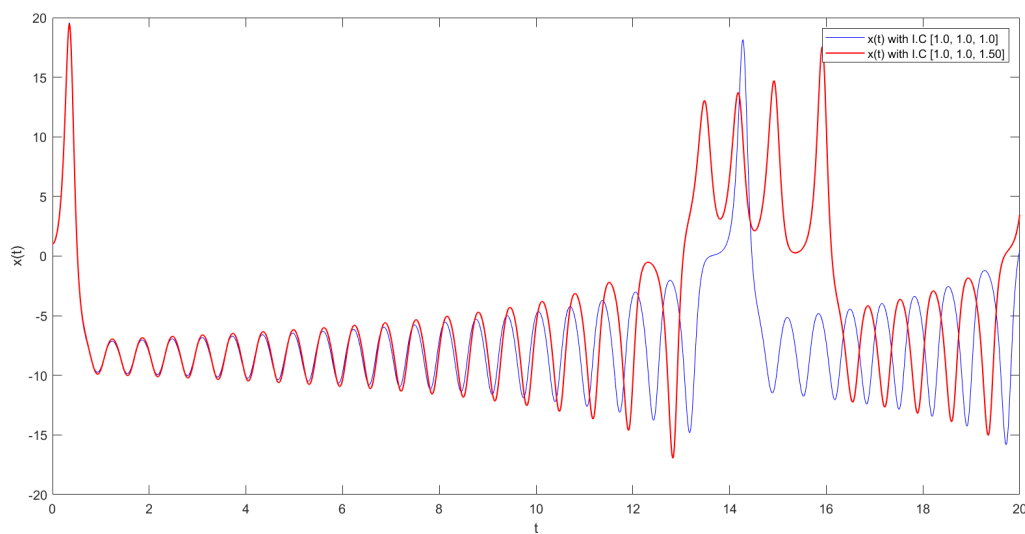


Figure 2.1: Evolution over time for two initial conditions.

2.2.4 Strange attractors

A chaotic system exhibits a distinctive pattern in phase space known as a strange attractor, which differs greatly from those in simpler systems. This attractor combines two opposing behaviors: the convergence of trajectories due to dissipative forces, and their divergence due

to extreme sensitivity to initial conditions. Although nearby trajectories initially separate exponentially, the limited dimension of the attractor prevents infinite divergence. Instead, the system undergoes a repeated process of contraction, stretching, and folding, forming a horseshoe-like structure. This dynamic results in a fractal geometry—self-similar at every scale and characterized by a non-integer dimension. Because of this fractal structure, the attractor is termed “strange,” and it serves as a hallmark of chaotic behavior. Strange attractors provide a way to recognize and mathematically describe chaos, whether in discrete-time or continuous-time systems.

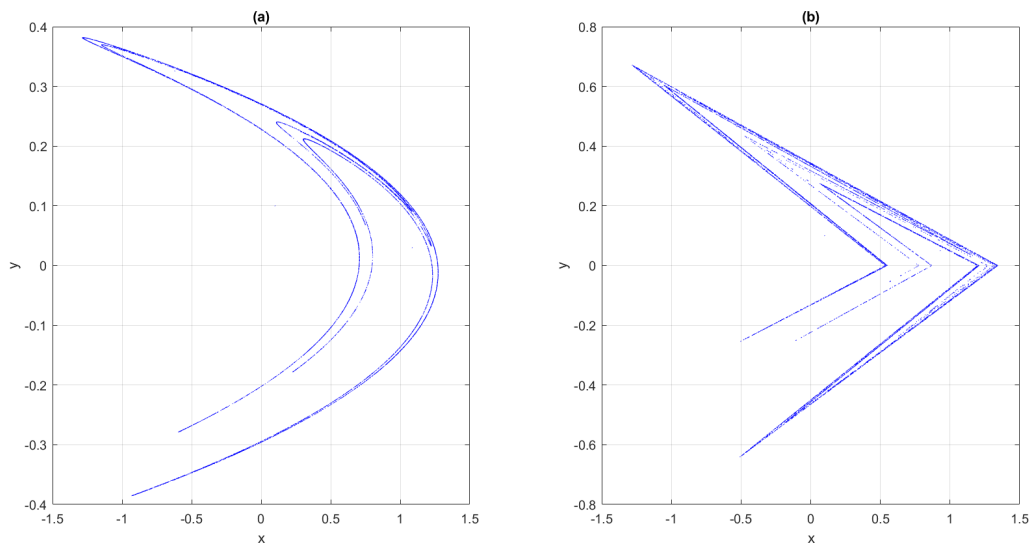


Figure 2.2: Strange attractors for some discrete-time chaotic systems

(a): Hénon attractor;

(b): Lozi attractor.

2.2.5 Dissipativity

In most cases, trajectories in chaotic systems are eventually limited to a subset of zero measure called strange attractor since these systems are dissipative. The system’s long-term behavior is controlled by this attractor, and its geometry often displays fractal patterns. Although the paths initially diverge, dissipativity guarantees that the system’s global dynamics are limited and drawn to a lower-dimensional subset of phase space.

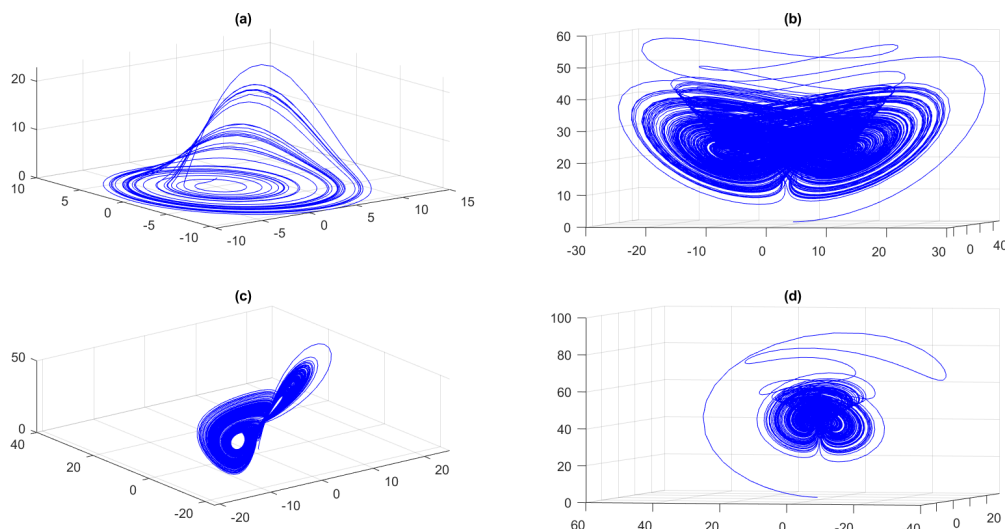


Figure 2.3: Strange attractors for some continuous-time chaotic systems

(a): Rössler attractor. (b): Chen attractor. (c): Lü attractor. (d): Liu attractor.

2.3 Routes of chaos

There are several paths that a system can take before degenerating into a chaotic regime. Chaos is a multiscale, multiplicative phenomena. A chaotic system possessing a topological mixing characteristic is very susceptible to even the slightest alteration in its initial conditions. It has an unlimited number of periodic cycles and has achieved an infinite dense orbit at infinity. Some of the so-called routes of chaos qualitative order or regularities to disorder have been discussed. There are countless routes. Among the common routes are periodic windows, intermittency, crisis, period-doubling cascades, etc. Numerous systems, including as electric circuits, lasers, fluid turbulence convection, population dynamics, and chemical and biological processes, exhibit these routes. We'll quickly review a few routes.

2.3.1 Period-doubling route

This is the best recognized pathway of chaos and manifests in several one-dimensional systems. Numerous natural systems demonstrate this pathway. The systems exhibit dissipative characteristics. The route starts with a limit cycle. A limit cycle may arise from a bifurcation involving a node or other types of fixed points inside the system. As the system parameter fluctuates, the limit cycle becomes unstable, resulting in the emergence of a period-2 limit cycle. Upon further parameter modifications, the period-2 limit cycle becomes unstable,

resulting in the emergence of period-4 cycles. The period-doubling cascades may continue until the period reaches infinity. This indicates that the trajectory is non-repetitive. The orbit is thereafter chaotic.

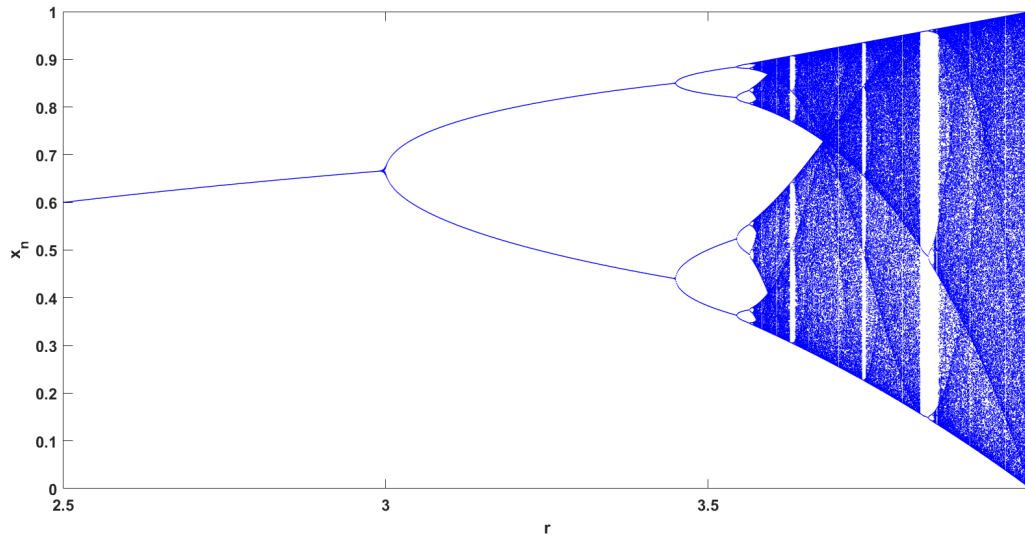


Figure 2.4: Period-doubling route shown in bifurcation diagram of logistic map.

2.3.2 Quasi-periodic

In the quasi-periodic framework, the system resumes with a limit cycle trajectory. Upon alteration of a control parameter, a secondary periodicity emerges in the system's behavior. This bifurcation event generalizes the Hopf bifurcation. If the ratio of the period of the second kind of motion to that of the first is irrational, we designate the motion as quasi-periodic, as previously explained. Under some conditions, if the control parameter is altered further, the motion becomes chaotic. This pathway is sometimes referred to as the Ruelle-Takens scenario.

2.3.3 Intermittency

The intermittency path to chaos is defined by dynamics with sporadic bursts of chaotic action intermingled with phases of seemingly periodic behavior. As a control parameter of the system is altered, the chaotic bursts lengthen and increase in frequency until the whole time series becomes chaotic. In figure (2.6) the trajectory exhibits periodic behavior after a brief initial transient phase for $\rho = 165$, $\beta = \frac{8}{3}$, and $\sigma = 10$. In the lower half, $\rho = 167$,

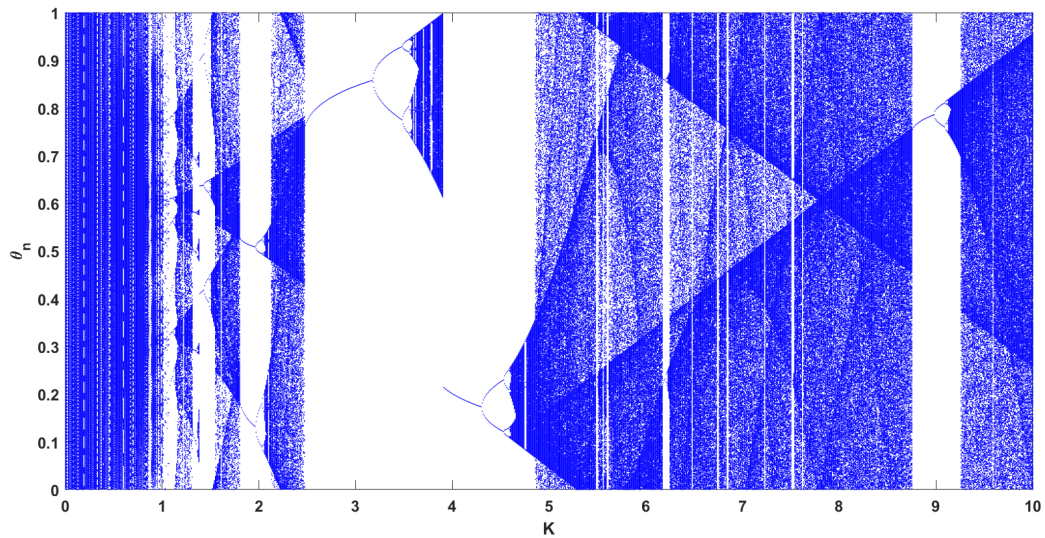


Figure 2.5: Quasi-periodic route shown in bifurcation diagram of Sine-circle map.

and the trajectory exhibits intermittent behavior, alternating between periodicity and brief episodes of chaotic activity (around $t = 7.5$ and $t = 11$). The intermittency would be more apparent over an extended time frame.

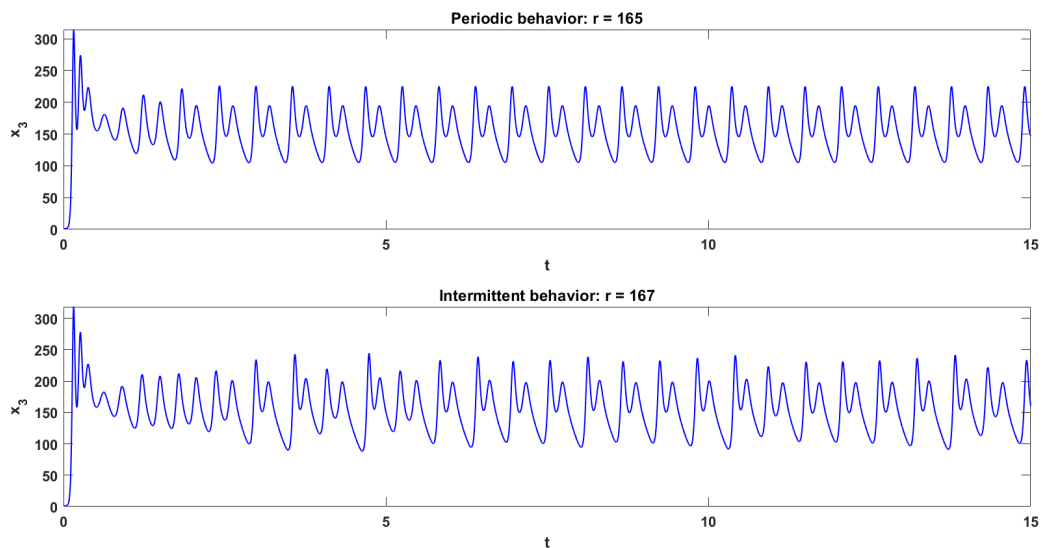


Figure 2.6: The temporal dependency of the x_3 variable in the Lorenz model.

2.3.4 Crisis

A crisis is a bifurcation event when a chaotic attractor and its basin of attraction either vanish or abruptly increase in size. The previous kind of crisis is often referred to as a

border crisis for reasons that will soon be evident. The abrupt growth or contraction of a chaotic attractor is referred to as an internal crisis. In both instances, the crisis transpires when an unstable fixed point or an unstable limit cycle intersects with the chaotic attractor due to a variation in a control parameter of the system. A metamorphosis, another kind of crisis event, is characterized by the emergence or abrupt expansion of fractal structures inside a basin border.

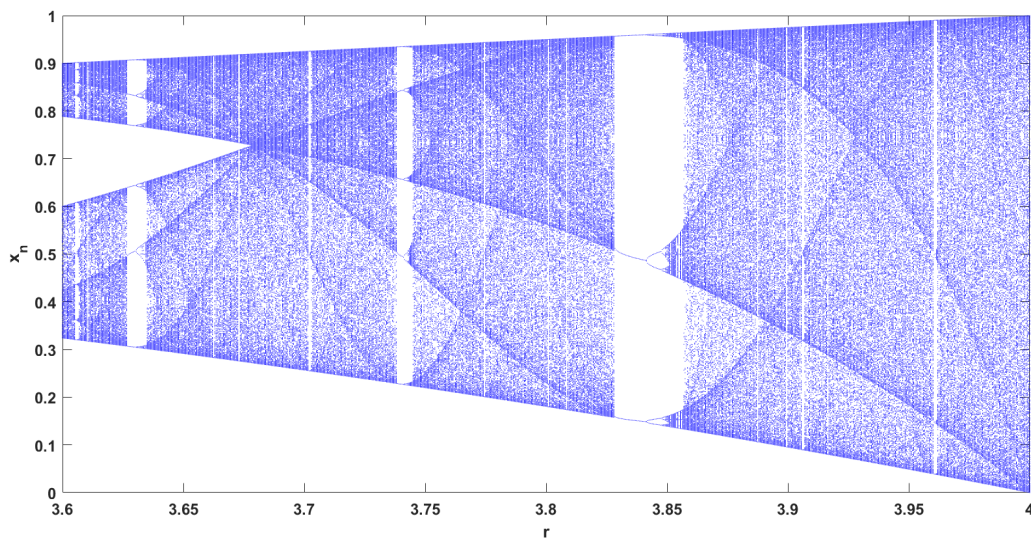


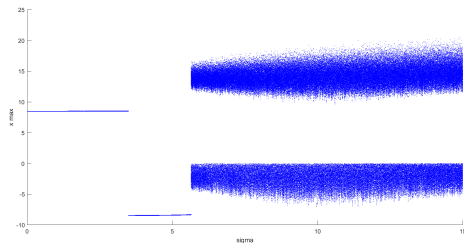
Figure 2.7: Crisis bifurcation in the logistic map illustrated by a sudden change in the size of the chaotic attractor.

2.4 Quantifying chaos

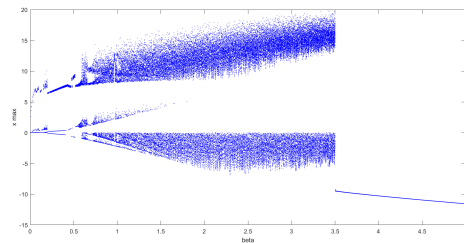
2.4.1 Bifurcation diagram

The progression from a fixed point to a period-2 limit cycle, followed by a period-4 limit cycle, is a significant occurrence in the dynamics of a system. A bifurcation is stated to occur when a qualitative shift in the solutions arises from the modification of a parameter. The graphs depicting these bifurcations are aptly termed bifurcation diagrams. This concept is fundamental to the examination of chaos. When analyzing such graphs, one must focus on the axes. One axis represents the parameter, while the other denotes the state variable, so constituting the parametric space. A bifurcation diagram illustrates regions of the parametric space where the system's qualitative behavior is analogous. A rapid succession of period-

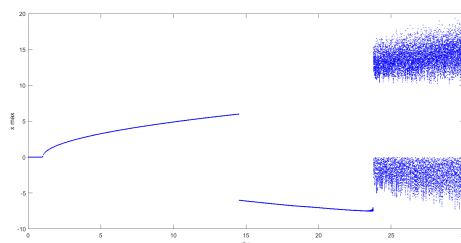
doubling is also observed, resulting in a specific condition. The phenomenon of period doubling is a pathway to chaos. Figure (2.8) illustrates the bifurcation diagrams of the Lorenz system.



(a) Variation depends on σ .



(b) Variation depends on β .



(c) Variation depends on ρ .

Figure 2.8: Bifurcation diagrams of the Lorenz system.

Chaotic activity makes a kind of unpredictability and diminishes knowledge on initial conditions, which may elucidate complicated behavior (or at least aspects of it) in actual systems. We want a precise, quantifiable method for identifying chaos and distinguishing "true" chaos from just noisy or unpredictable behavior resulting from complexity.

2.4.2 Lyapunov exponents

Lyapunov exponents are coefficients that measure sensitivity to the initial conditions of a time series. By definition, a Lyapunov exponent is the average exponential rate of divergence of adjacent phase space trajectories. It measures the local rate of expansion of the space in which the expansion is maximum, i.e., generally towards the attraction. The number of Lyapunov exponents is equal to the dimension of the phase space.

Computation of Lyapunov exponents

$$|g^k(x_0 + \epsilon) - g^k(x_0)| \cong \epsilon \exp(\lambda k).$$

The distance between two originally near states after t steps is represented on the left, while the assumption that the distance increases exponentially with time is represented on the right. The Lyapunov exponent is the exponent λ that is measured over an extended period of time (preferably $t \rightarrow \infty$). Small distances extend infinitely over time if $\lambda > 0$, indicating the stretching mechanism's operation. Alternatively, if $\lambda < 0$, then short distances don't increase infinitely; instead, the system gradually settles into a periodic trajectory. Keep in mind that while stretching is the sole mechanism of chaos that the Lyapunov exponent characterizes, this is not the only mechanism. Remember that this Lyapunov exponent does not capture the folding mechanism. So

$$k\lambda \cong \log \frac{|g^k(x_0 + \epsilon) - g^k(x_0)|}{\epsilon},$$

for $\epsilon \rightarrow 0$ we have

$$\begin{aligned} k\lambda &\cong \log \left| \frac{dg^k(x_0)}{dx_0} \right|, \\ \lambda &\cong \frac{1}{k} \log \left| \frac{dg^k(x_0)}{dx_0} \right|, \\ &\cong \frac{1}{k} \log \left| \frac{dg^k(x_0)}{dg^{k-1}(x_0)} \times \frac{dg^{k-1}(x_0)}{dg^{k-2}(x_0)} \times \dots \times \frac{dg^1(x_0)}{dx_0} \right|, \\ &\cong \frac{1}{k} \log \left| \frac{dg(x_{k-1})}{dg(x_{k-2})} \times \frac{dg(x_{k-2})}{df(x_{k-3})} \times \dots \times \frac{dg(x_0)}{dx_0} \right|, \\ &\cong \frac{1}{k} \sum_{i=0}^{k-1} \log \left| \frac{g'(x_i)}{x_i} \right|. \end{aligned}$$

Finally for $k \rightarrow +\infty$

$$\lambda = \lim_{k \rightarrow +\infty} \frac{1}{k} \sum_{i=0}^{k-1} \log |g'(x_i)|.$$

<i>Attractors types</i>	<i>Lyapunov exponents</i>
Fixed point	$0 > \lambda_1 > \lambda_2 > \dots > \lambda_k$
Limit cycle	$\lambda_1 = 0, 0 > \lambda_2 > \lambda_3 > \dots > \lambda_k$
Torus	$\lambda_1 = \lambda_2 = 0, 0 > \lambda_3 > \lambda_4 > \dots > \lambda_k$
j-torus	$\lambda_1 = \lambda_2 = \dots = \lambda_j = 0, 0 > \lambda_{j+1} > \lambda_{j+2} > \dots > \lambda_k$
Chaotic attractor	$\lambda_1 \geq 0, \sum_{i=1}^k \lambda_i < 0$

Table 2.1: Classification of permanent schemes according to Lyapunov exponents

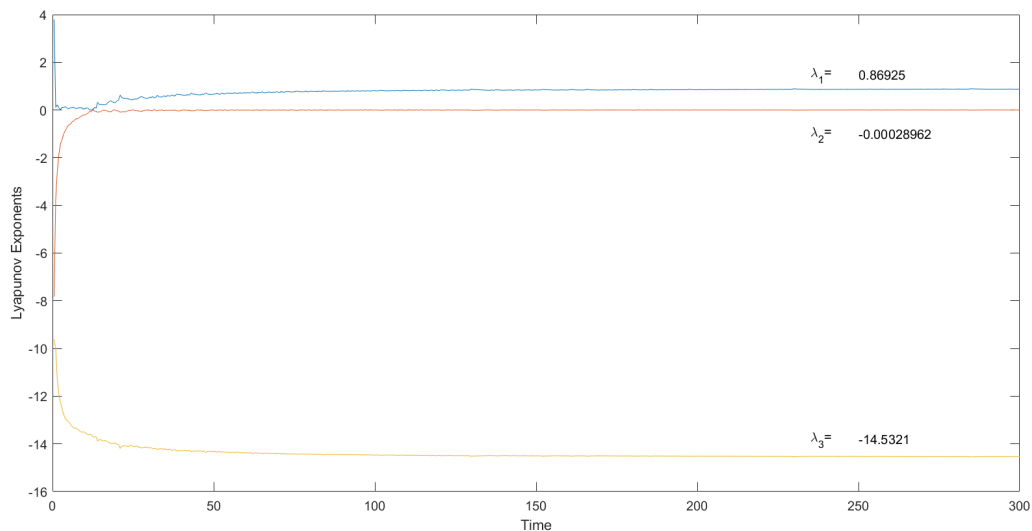


Figure 2.9: Lyapunov exponents of Lorenz system.

2.4.3 Fractal dimensions

Chaotic attractors display highly irregular shapes and patterns that repeat at multiple scales. This intricate, scale-dependent structure is described using fractal dimensions, numerical indicators that characterize how an attractor fills space as one zooms in. Unlike ordinary geometric objects, whose dimensions are whole numbers, a fractal's dimension is usually fractional, capturing its intermediate level of complexity. Over time, several distinct definitions of fractal dimension have been proposed to analyze different aspects of this complexity. The main categories are:

Kolmogorov dimension

The Kolmogorov dimension known also by box-counting dimension (or capacity dimension) is a simple measure of the geometrical complexity of an object, first applied to dynamics by Kolmogorov. It is defined by covering the object with boxes (or cells) of side length S . In 1D, the boxes are line segments; in 2D, squares; in 3D, cubes; and so on. Let $N(S)$ denote the minimum number of boxes required to cover all points of the object. As S decreases, $N(S)$ increases. The box-counting dimension D_b satisfies:

$$N(S) = lS^{-D_b},$$

where l is a proportionality constant. Taking the logarithm, we obtain:

$$D_b = \lim_{S \rightarrow 0} \left(-\frac{\log N(S)}{\log S} + \frac{\log l}{\log S} \right).$$

For sufficiently small S , the second term vanishes, giving:

$$D_b = -\lim_{S \rightarrow 0} \frac{\log N(S)}{\log S}.$$

Hausdorff dimension

The Hausdorff dimension is a rigorous measure of the complexity of a set, generalizing the notion of dimension to non-integer values. For a set X , the d -dimensional Hausdorff measure is defined as

$$\mathcal{H}^d(X) = \lim_{\delta \rightarrow 0} \inf \left\{ \sum_i (\text{diam}(U_i))^d : X \subseteq \bigcup_i U_i, \text{diam}(U_i) < \delta \right\},$$

where $\{U_i\}$ is a countable covering of X by sets of diameter less than δ .

The Hausdorff dimension D_H is the critical value of d at which the measure transitions from infinity to zero:

$$D_H = \inf\{d \geq 0 : \mathcal{H}^d(X) = 0\} = \sup\{d \geq 0 : \mathcal{H}^d(X) = \infty\}.$$

This dimension captures the fine structure and self-similarity of fractals and chaotic attractors more precisely than simpler measures like the box-counting dimension.

Correlation dimension

It quantifies the geometrical complexity of an attractor. Given N points from a trajectory on the attractor, for each point i we count the number of points within a distance S , excluding the point itself. Denoting this relative count by $p_i(S)$, the correlation sum is defined as:

$$C(S) = \frac{1}{N} \sum_{i=1}^N p_i(S) = \frac{1}{N(N-1)} \sum_{i=1}^N \sum_{j \neq i} \Theta(S - |x_i - x_j|),$$

where Θ is the Heaviside step function

$$\Theta(z) = \begin{cases} 1, & z \geq 0, \\ 0, & z < 0. \end{cases}$$

$C(S)$ represents the probability that two points are separated by less than S . The correlation dimension D_c is then given by

$$D_c = \lim_{S \rightarrow 0} \frac{\log C(S)}{\log S},$$

which measures how the attractor scales at small distances.

Kaplan-Yorke dimension (D_{KY})

Introduced in 1979, this dimension provides a way to condense the information carried by

the Lyapunov exponents into a single indicator of the attractor's complexity. Consider the ordered Lyapunov exponents $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. Identify the integer k such that

$$\sum_{i=1}^k \lambda_i \geq 0 \quad \text{but} \quad \sum_{i=1}^{k+1} \lambda_i < 0.$$

The Kaplan-Yorke dimension is then calculated as:

$$D_{KY} = k + \frac{\sum_{i=1}^k \lambda_i}{|\lambda_{k+1}|}.$$

2.5 Conclusion

This chapter provides both qualitative and quantitative techniques for recognizing and comprehending chaotic behavior. We have established the groundwork for comprehending the several situations that, when influenced by one or more system control factors, might cause a nonlinear dynamic system that does not exhibit chaotic behavior to degenerate into chaotic dynamics. Among other fractal dimensions, we have explained several of their categories, including the Hausdorff dimension and the correlation dimension. Finally, we give a description of the Lyapunov exponents, which are a crucial quantitative description of chaos. They do, in fact, concentrate on determining sensitivity to initial conditions.

Chapter 3

Control of chaotic systems

Chaos control is a field at the intersection of dynamical systems and control theory, concerned with the stabilization of systems that exhibit chaotic behavior. It primarily involves transforming chaotic trajectories into periodic ones by stabilizing unstable periodic orbits or equilibrium points. Rooted in nonlinear control theory, chaos control gained momentum after the introduction of the OGY method by Ott, Grebogi, and Yorke in 1990, which marked a turning point in the field.

Chaos control techniques are generally divided into closed-loop (feedback) and open-loop (non-feedback) methods. Feedback approaches, such as the OGY method, time-delayed feedback control (TDFC), and occasional proportional feedback (OPF), aim to gently stabilize the system using internal state information. Open-loop strategies, on the other hand, rely on external perturbations but often lack precision. This chapter focuses on key methods including OGY, TDFC, adaptive control, and other control techniques.

3.1 Controlled systems

In the context of chaos control, dynamical systems are typically modeled by ordinary differential equations of the form:

$$\dot{x} = g(x, \mu),$$

where $x \in \mathbb{R}^n$ is the state vector and $\mu \in \mathbb{R}^m$ is the control input. The existence and uniqueness of solutions are guaranteed by the assumption that the function g is continuously differentiable. In certain instances, the output is represented as:

$$y(t) = f(x(t)),$$

but when not explicitly given, it is generally presumed that the full state vector $x(t)$ is observable. For systems subject to time-varying conditions or disturbances, the model becomes non-stationary:

$$\dot{x} = g(x, \mu, t),$$

which indicates the system dynamics may change with respect to time or input.

Control strategies aim to guide the system away from chaotic behavior by applying small, well-designed perturbations to stabilize it and induce periodic motion. This can be modeled using affine control structures:

$$\dot{x} = g(x) + h(x)\mu,$$

or in discrete-time form as:

$$x_{i+1} = G(x_i, \mu_i),$$

where x_i , μ_i denote the state and control input at the i -th step. Whether in continuous or discrete form, chaos control involves adjusting accessible parameters to suppress or regulate chaotic behavior. This chapter presents a selection of the most established and reliable methods used to achieve this goal.

3.2 Controlling chaos

3.2.1 Methods of chaos suppression

This category of chaos control methods involves converting chaotic behaviour into a desired periodic behavior. In such algorithms, the objective is to ensure that the flow $x(t)$ follows a predefined dynamic $f(t)$. This requires adding a control term $\mu(t)$ to the equation of motion:

$$\dot{x} = g(x),$$

such that:

$$|x(t) - f(t)| \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty,$$

the control input is chosen as:

$$u(t) = \frac{df}{dt} - g(f(t)).$$

Several methods are used to suppress chaos, including periodic or stochastic perturbations that force the system toward unstable periodic orbits. Although not ideal, these perturbations alter the system's dynamics and can result in consistent behaviour. Rajasekar and Lakshmanan demonstrated that Gaussian noise, periodic forcing, and adaptive algorithms can induce order. Other approaches include coupling with a periodic system, generalized predictive control (GPC), the Melnikov method, ARMA models, and Dubois' anticipatory control, which relies on predicting the system's future state [18]. The expression for the controlled chaotic system is:

$$x(i+1) = g(x(i), \mu(i)),$$

where the control input is given by:

$$\mu(i) = h(x(i), x(i+1)).$$

This means the chaotic system includes a predictive model of itself. To illustrate, consider the logistic equation:

$$x(i+1) = qx(i)(1 - x(i)),$$

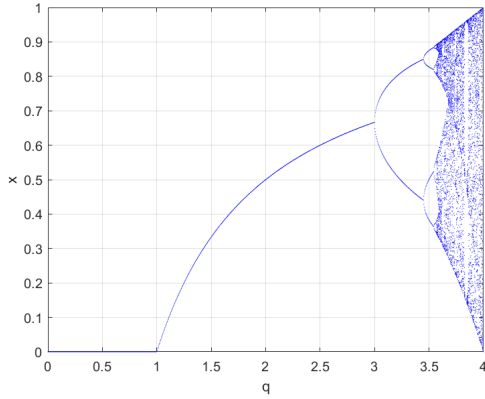
where $q = 4$. Under anticipatory control, a state variable is computed using a predicted value as follows:

$$x(i+1) = qx(i)(1 - x(i+1)).$$

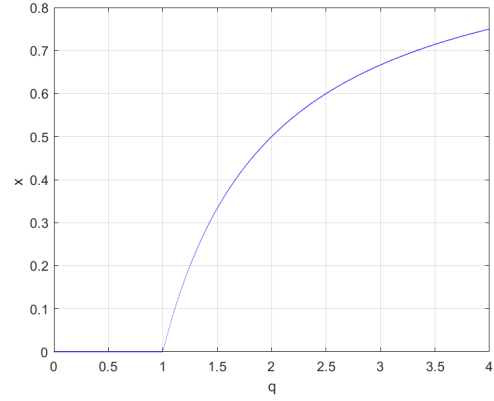
This control effectively suppresses chaos and stabilizes the system at a fixed value. However, the stabilized value is neither an unstable periodic orbit nor an unstable fixed point of the original system. The results of this control are shown in figure (3.1).

3.2.2 Chaotification methods

Anticontrol or the chaotification of dynamic systems is a control method that allows a non-chaotic system to exhibit chaotic behavior. Since the emergence of this new concept, researchers have been increasingly interested in this aspect. As with the control of nonlinear dynamic systems with chaotic behaviour, anti-control is a difficult task. This section introduces a control strategy that involves introducing little adjustments to the system's most sensitive parameter. The applied disturbance must be bounded and very small compared to the perturbed parameter.



(a) Bifurcation diagram without control.



(b) Bifurcation diagram with incursive anticipatory control.

Figure 3.1: Suppression of chaos through anticipatory control.

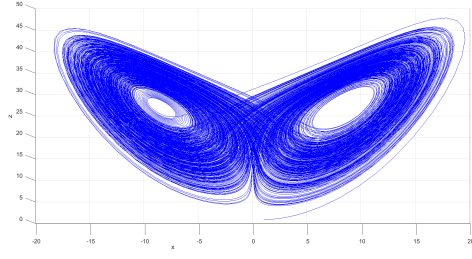
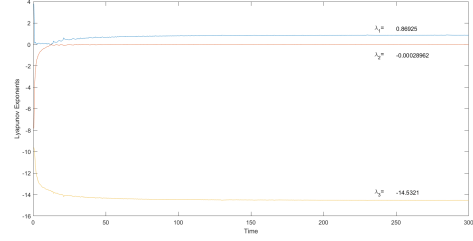
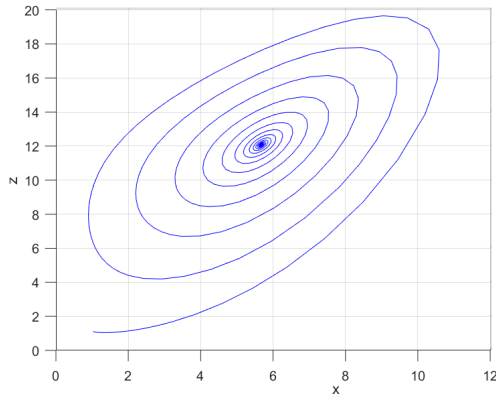
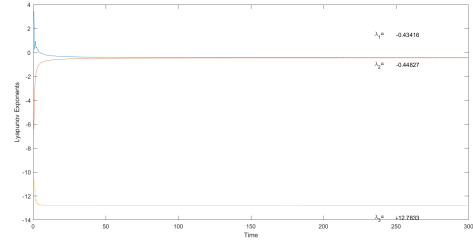
Example 33.

$$\begin{cases} \dot{x}_1 = \sigma(x_2 - x_1), \\ \dot{x}_2 = \rho x_1 - x_2 - x_1 x_3, \\ \dot{x}_3 = x_1 x_2 - \beta x_3. \end{cases} \quad (3.1)$$

The parameter ρ exhibits the highest sensitivity within the Lorenz system; hence, the analysis will focus on its fluctuation. We use the phase diagram, and Lyapunov exponent to ascertain the many stability zones and the dynamic development of the system inside each zone. The attractor and the Lyapunov exponents are illustrated in figure (3.2) for $\rho = 28$, and figure (3.3) for $\rho = 13$. For $\rho = 28$, the Lorenz system exhibits the well-known chaotic behavior, characterized by a strange attractor with a complex butterfly-shaped structure and the presence of a positive Lyapunov exponent, indicating strong sensitivity to initial conditions. In contrast, for $\rho = 13$, the system displays a non-chaotic regime, where the trajectories converge toward a regular spiral structure around a stable equilibrium point, which is confirmed by non-positive Lyapunov exponents.

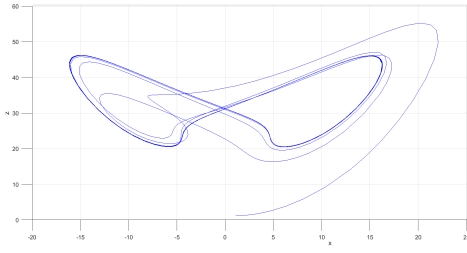
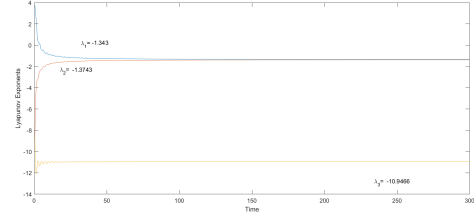
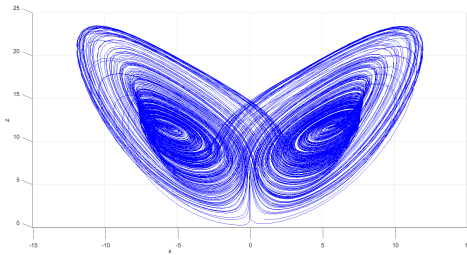
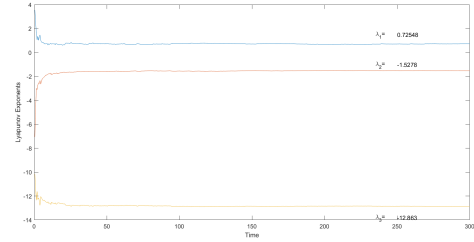
The goal of the control method is to eliminate chaos in chaotic zones and generate chaos in stable parts, for example, making the system chaotic for $\rho = 13$ and stabilizing it for $\rho = 28$. We apply a sinusoidal perturbation to the parameter ρ as follows:

$$\begin{cases} \dot{x}_1 = \sigma(x_2 - x_1), \\ \dot{x}_2 = \rho(1 + \varepsilon \sin(\omega t))x_1 - x_2 - x_1 x_3, \\ \dot{x}_3 = x_1 x_2 - \beta x_3. \end{cases} \quad (3.2)$$

(a) Attractor of system (3.1) for $\rho = 28$ (b) Lyapunov exponents of system (3.1) for $\rho = 28$ Figure 3.2: System (3.1) for $\rho = 28$ before chaotification.(a) Attractor of system (3.1) for $\rho = 13$ (b) Lyapunov exponents of system (3.1) for $\rho = 13$ Figure 3.3: System (3.1) for $\rho = 13$ before chaotification.

ε, ω are perturbation parameters.

The introduction of the sinusoidal perturbation $\varepsilon \sin(\omega t)$ transforms the autonomous Lorenz system into a non-autonomous one. The parameters are chosen as $\varepsilon = 0.3$, $\omega = 10$, and $\rho \in [2, 30]$. The corresponding phase portraits and Lyapunov exponents are shown in figures (3.4) and (3.5). After introducing the sinusoidal perturbation, a significant modification of the system dynamics is observed. In particular, the perturbation induces a chaotic behavior for $\rho = 13$, whereas for $\rho = 28$, the system transitions from its classical chaotic regime to a non-chaotic one. This inversion of dynamical behavior highlights the strong influence of the sinusoidal perturbation on the stability properties of the Lorenz system.

(a) Attractor of system (3.2) for $\rho = 28$.(b) Lyapunov exponents of system (3.2) for $\rho = 28$.Figure 3.4: System (3.2) for $\rho = 28$ after chaotification.(a) Attractor of system (3.2) for $\rho = 13$.(b) Lyapunov exponents of system(3.2) for $\rho = 13$.Figure 3.5: System (3.2) for $\rho = 13$ after chaotification.

3.2.3 Methods of chaos stabilization

The second group, the most intriguing, encompasses approaches that attain chaos control by stabilising the unstable periodic orbits of a chaotic system.

The OGY method

The Ott, Grebogi and Yorke (OGY) method known as OGY is a state-back control method that uses chaos in a dynamic system to stabilize an unstable periodic orbit. The method is based on the following idea:

1. Determine some unstable periodic orbits, examine them, and then choose one that represents the performance of the system.
2. Modify the perturbation settings promptly to stabilise the unstable periodic orbit.

Consider the following system:

$$\dot{x} = g(x, r) \quad \forall x \in \mathbb{R}^n, \quad (3.3)$$

where r is a control parameter. The Poincaré section of the system is represented by:

$$x_{n+1} = G(x_n, r) \quad \forall x_n \in \mathbb{R}^{n-1}. \quad (3.4)$$

In accordance with the concept of using a little disturbance to regulate the system, the permissible fluctuations of a parameter must be minimal.

$$|r - r^*| < \eta, \quad (3.5)$$

where r^* is the nominal value of the parameter, and η represents the admissible scope of parameter variation. The parameter r is modified to stabilise a chaotic trajectory upon its entry into the vicinity of the target periodic orbit. Let $x_f(r^*)$ denote a fixed point of the map (3.4) at the nominal value r^* that we want to stabilise. The position of the fixed point in the phase space is contingent upon the control parameter r . Upon the introduction of a little perturbation δr , we obtain $r = r^* + \delta r$. Given that δr is minimal, we anticipate that $x_f(r)$ will approximate $x_f(r^*)$ closely. We compose:

$$x_f(r) = x_f(r^*) + h\delta r, \quad (3.6)$$

with h a vector defined by:

$$h \equiv \left(\frac{\partial x_f}{\partial r} \right) \approx \frac{x_f(r) - x_f(r^*)}{\delta r}. \quad (3.7)$$

In a small neighborhood of the fixed point $x_f(r)$, one can use the linear approximation for the map:

$$[x_{n+1} - x_f(r)] = A(x_f(r)) \cdot [x_n - x_f(r)], \quad (3.8)$$

where $A(x_f(r))$ is the Jacobian matrix of the map $G(x_n, r)$ evaluated at the fixed point $x_f(r)$, which is defined as follows:

$$A(x_f(r)) = \left(\frac{\partial G}{\partial r} \right) |_{x_f(r)}. \quad (3.9)$$

By replacing equations (3.6) and (3.9) with equation (3.8), we obtain:

$$[x_{n+1} - x_f(r^*)] \approx h\delta r + A(x_f(r^*)) \cdot [x_n - x_f(r^*) - h\delta r]. \quad (3.10)$$

In equation (3.10), matrix A has two own directions one is unstable eu (associated eigenvalue strictly greater than 1 in absolute value), and the other is stable es (associated eigenvalue strictly less than 1 in absolute value). So the corrections are applied to the unstable direction. If the following ratings are used

- λ_u : eigenvalue $|\lambda_u| > 1$.
- λ_s : eigenvalue $|\lambda_s| < 1$.
- e_u : eigenvector correspond to the eigenvalue λ_u .
- e_s : eigenvector correspond to the eigenvalue λ_s .

So the matrix A can be written in the form

$$A = \lambda_s e_s v_s^T + \lambda_u e_u v_u^T,$$

where v_s and v_u represent covariance vectors, with

$$e_s^T v_s = e_u^T v_u = 1 \text{ and } e_s^T v_u = e_u^T v_s = 0$$

A little perturbation of parameter r is performed at time n to gently move the fixed point when the point x_n reaches the small neighbourhood of the intended fixed point $x_f(r)$. This way, during the following iteration $(n+1)$, x_{n+1} falls on the stable direction of $x_f(r)$. To do this, we must have

$$v_u [x_{n+1} - x_f(r^*)] = 0. \quad (3.11)$$

We can replace equation (3.10) into equation (3.11) to get

$$\delta r_n = \frac{\lambda_u v_u [x_n - x_f(r^*)]}{(\lambda_u - 1) v_u^T h} \equiv k [x_n - x_f(r^*)], \quad (3.12)$$

with

$$k = \frac{\lambda_u v_u}{(\lambda_u - 1) v_u^T h}.$$

The goal of the control is to verify the condition (3.5) η being a default parameter that determines the neighborhood of $x_f(r^*)$. So, the increase in control is given by

$$\delta r_n = \begin{cases} k [x_n - x_f(r^*)] & \text{if } |k [x_n - x_f(r^*)]| < \eta, \\ 0 & \text{else.} \end{cases}$$

Application of OGY method on Lorenz system

Let consider the Lorenz system (3.1), figure 3.6 shows the Lorenz time series. The Poincaré section considered here is the set of points of the attractor such that $z = z_{max}$ as shown in figure (3.7). We are focused in the first return map with respect to the third coordinate, figure (3.8). We observe that this curve is particularly simple, which is why the simplified

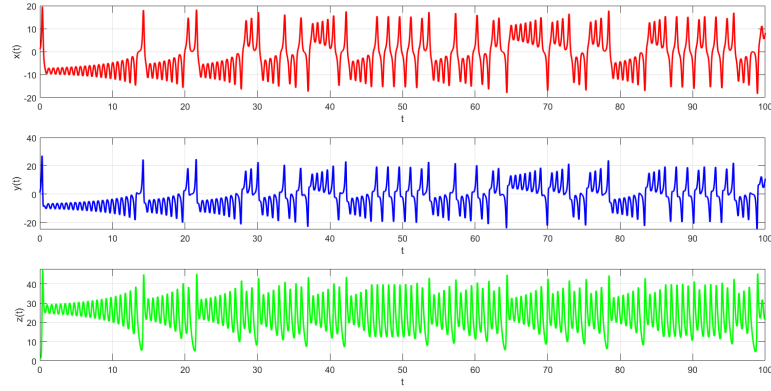


Figure 3.6: Time series for Lorenz system.

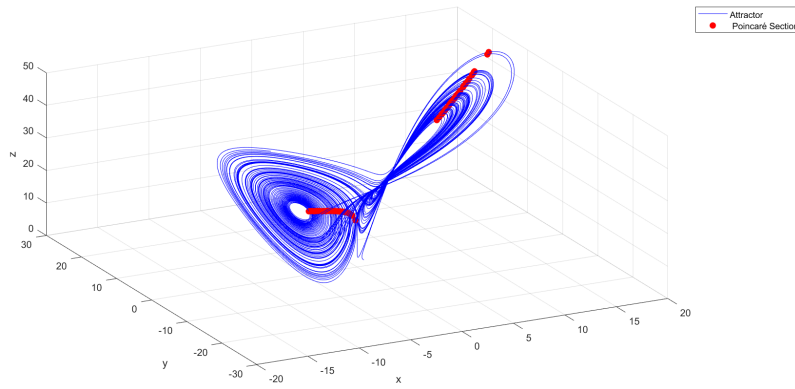
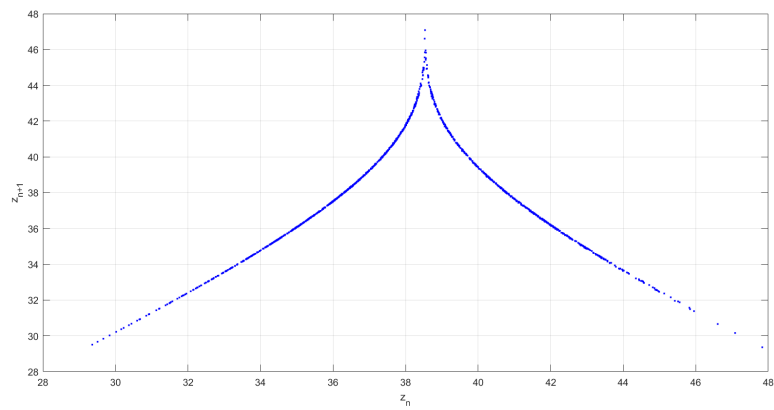
Figure 3.7: Lorenz attractor with the Poincaré section corresponding to $z = z_{max}$ highlighted in red.

Figure 3.8: First return map relative to the third coordinate for the Lorenz system.

OGY method is used. Control is executed in the vicinity of the fixed point, therefore, it is essential to ascertain it. Given the structure of the attractor, finding the value of the third

coordinate of the fixed point allows us to determine the fixed point in the Poincaré section. Therefore, the first return map with respect to the third coordinate is used, figure (3.9).

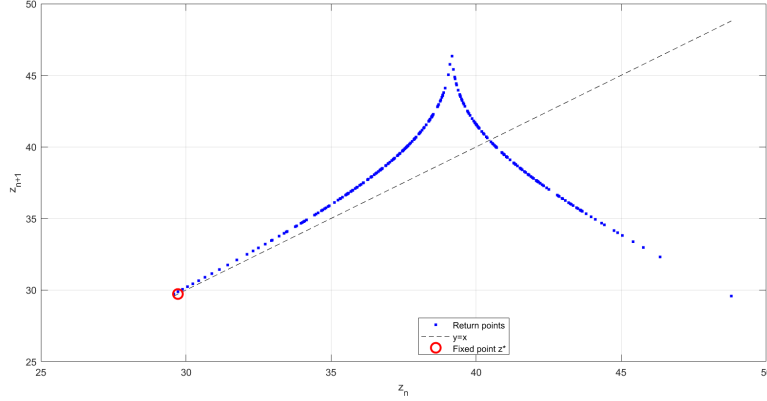


Figure 3.9: Finding the third coordinate of the fixed point.

$$x_{3F} = 29.50674$$

So

$$x_{1F} = -9.17111, \quad x_{2F} = -8.24388.$$

The OGY method requires determining the directions and eigenvalues of the Jacobian matrix A . Here, the Poincaré sections can be treated as curves: we only focus on a single eigenvalue and directly obtain the correction. To do this, we determine the influence of the control parameter ρ , for the Lorenz system, on the fixed point. For several values of the parameter, we search for the fixed point. By simulation, we then generate several first-return applications for different values of ρ , close to the one for which we want to control the system. Figure (3.10) shows the first return applications for $\rho = 28$ (in blue), and $\rho = 28.5$ (in green). The fixed point corresponding to is

$$x_{3F} = 29.72325, x_{1F} = -9.13389, x_{2F} = -8.31668,$$

Understanding how the parameter affects the fixed point, as well as the nearby points, allows us to determine the necessary parameter adjustment to guide the current point back to the fixed point in the Poincaré section on its next crossing. This forms the basis for deriving the control law.

$$\begin{aligned} \delta r_n &= \frac{\partial r}{\partial x_{3F}} = \frac{28.5 - 28}{29.72325 - 29.50674} (x_{3n} - x_{3F}) \\ &= 2.30(x_{3n} - x_{3F}), \end{aligned} \tag{3.13}$$

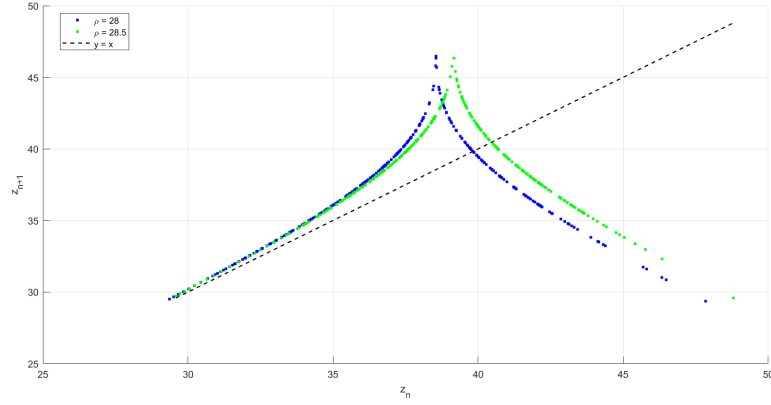


Figure 3.10: Superposition of two first return maps of the Lorenz system ($\rho = 28$ in blue and $\rho = 28.5$ in green).

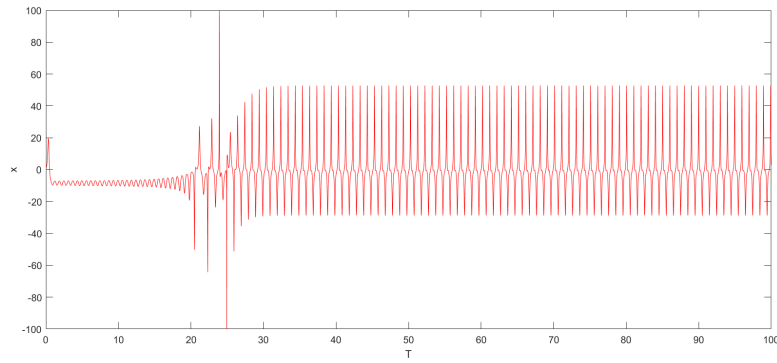


Figure 3.11: Time history of the periodic evolution of state variable x .

Pyragas method

Pyragas [60] established a method to stabilise unstable periodic orbits by applying small continuous control to a system's parameter during its continuous evolution, rather than using discrete control at the intersection of a surface. This is referred to as delayed feedback control. Examine the subsequent autonomous nonlinear system:

$$\dot{x} = g(x(t)),$$

with $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ a continuous function. The main objective of this method is to apply an external control defined by

$$\begin{cases} \dot{x} = g(x(t)) + \mu(t), \\ \mu(t) = q(x(t) - x(t - \tau)), \end{cases}$$

where q represents the gain, and τ is the time delay.

So the controllable system is given as follows:

$$\dot{x} = g(x(t)) + q(x(t) - x(t - \tau)). \quad (3.14)$$

From equation (3.14), we conclude that the systems control depends on a good selection of q , and τ . To illustrate the principle of this method for the control of continuous systems, we will apply it to the Lorenz system.

Example 34. The controllable Lorenz system is defined by:

$$\begin{cases} \dot{x}_1 = \sigma(x_2 - x_1) + q_1(x_1(t) - x_1(t - \tau_1)), \\ \dot{x}_2 = \rho x_1 - x_2 - x_1 x_3 + q_2(x_2(t) - x_2(t - \tau_2)), \\ \dot{x}_3 = x_1 x_2 - \beta x_3 + q_3(x_3(t) - x_3(t - \tau_3)). \end{cases} \quad (3.15)$$

With q_i are the feedback gains, and τ_i are the time delays.

Before applying the (DCF) method to the Lorenz system, we set $\sigma = 10$, $\beta = 8/3$, and $\rho = 28$, meaning that the application of the control method on the Lorenz system will be done in the zone where the system is completely chaotic. In this zone, we will study the influence of the feedback gains q_i and the time delays τ_i on the control procedure.

We set $q_1 = 2.5$, $q_2 = q_3 = 0$, $\tau_2 = \tau_3 = 0$, and vary τ_1 in the interval $[0.5, 0.65]$ with a step of 0.001. Figure (3.12) represents the bifurcation diagram of $x(t)$ on function of τ_1 ;

The bifurcation diagram highlights the system's transitions between stability and chaos as the parameter τ_1 varies. In this figure, we observe initially for small τ_1 , the system is stable around a fixed point. As τ_1 increases, periodic oscillations emerge, indicating a bifurcation. Further increases lead to chaotic behavior, with a dense structure suggesting period-doubling cascades. Around $\tau_1 \approx 0.63$, a sudden jump occurs, possibly indicating a crisis or attractor switching. Beyond this point, the system exhibits strong chaotic behaviour. Figure (3.13), shows the phase diagram of the system for $\tau_1 = 0.58$.

Other control methods

1 Adaptive control

Adaptive control is a recognised and effective way for managing systems with unknown constant or gradually varying parameters. Adaptive control fundamentally involves utilising a parameter adaptation strategy to estimate unknown parameters and substituting these estimates into the feedback controller. Adaptive control is fundamentally

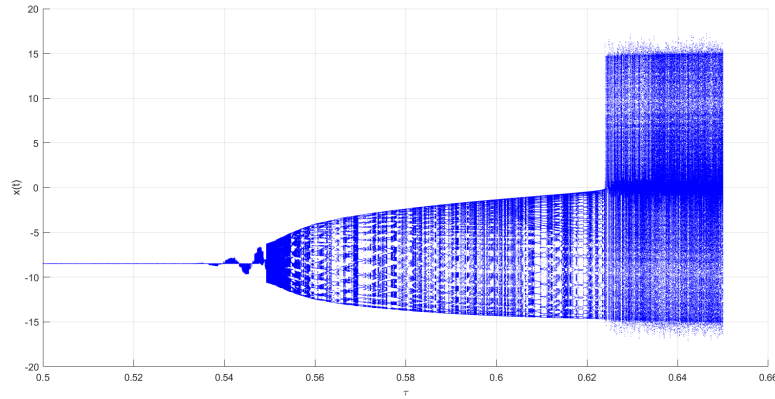


Figure 3.12: Bifurcation diagram of state variable $x(t)$ as function of τ_1 .

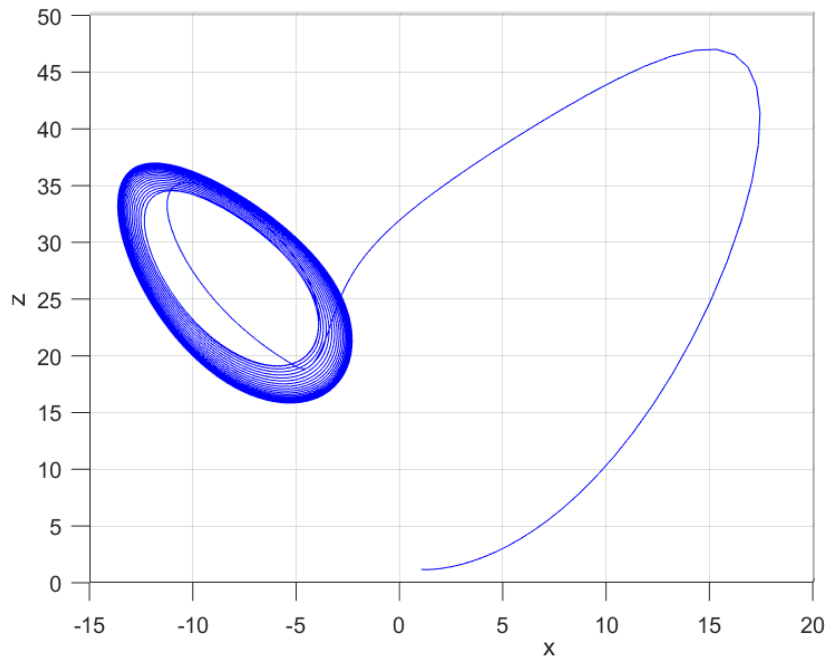


Figure 3.13: Lorenz attractor with $\tau_1 = 0.58$.

nonlinear control since it incorporates a dynamic parameter adaptation law, regardless of its application to linear or nonlinear systems. Regarding nonlinear systems.

The model of the system is represented in a parametric form:

$$\begin{cases} \dot{x} = G(x, \theta, \mu), \\ y = h(x). \end{cases} \quad (3.16)$$

Where $\theta \in \mathbb{R}^p$ is the vector of unknown parameters. As indicated in (3.16) the law of control is also expressed in a parametric form:

$$\mu = U(x, \vartheta),$$

where ϑ depends on θ i.e. $\vartheta = \psi(\theta)$. The process is acquired by system states $x(t)$ measurement, or the outputs $y = h(x(t))$ are used, to assess adaptation parameters that are estimates, either $\theta(t)$ the estimated values of unknown parameters $\theta(t)$ or controller parameters $\vartheta(t)$. A wide variety of adaptation methods exists such as the gradient and inclined graded methods, the smallest squares, maximum probability, and so on it can be employed to formulate adaptive control algorithms and conduct parametric identification. Their validation often relies on the Lyapunov function..

Example 35. An adaptive controller will be constructed in order to stabilize the Lorenz system, whose parameters are unknown. The controlled Lorenz system is

$$\begin{cases} \dot{x}_1 = \sigma(x_2 - x_1) + \mu_1, \\ \dot{x}_2 = \rho x_1 - x_2 - x_1 x_3 + \mu_2, \\ \dot{x}_3 = x_1 x_2 - \beta x_3 + \mu_3. \end{cases} \quad (3.17)$$

Such that x_1, x_2, x_3 are the state variables, σ, ρ, β are constants and unknown parameters, and μ_1, μ_2, μ_3 are adaptive controls.

Let's consider the adaptive controller defined as follows:

$$\begin{cases} \mu_1 = -\hat{\sigma}(x_2 - x_1) - q_1 x_1, \\ \mu_2 = -\hat{\rho} x_1 + x_2 + x_1 x_3 - q_2 x_2, \\ \mu_3 = -x_1 x_2 + \hat{\beta} x_3 - q_3 x_3, \end{cases} \quad (3.18)$$

where q_1, q_2, q_3 are positive constants and $\hat{\sigma}, \hat{\rho}, \hat{\beta}$ for the unknown parameters σ, ρ, β respectively. Introducing (3.18) into (3.17), we obtain:

$$\begin{cases} \dot{x}_1 = (\sigma - \hat{\sigma})(x_2 - x_1) - q_1 x_1, \\ \dot{x}_2 = (\rho - \hat{\rho})x_1 - q_2 x_2, \\ \dot{x}_3 = -(\beta - \hat{\beta})x_3 - q_3 x_3. \end{cases} \quad (3.19)$$

Now, let's define the error estimation parameters as:

$$\begin{cases} e_\sigma = \sigma - \hat{\sigma}, \\ e_\rho = \rho - \hat{\rho}, \\ e_\beta = \beta - \hat{\beta}. \end{cases} \quad (3.20)$$

Using (3.20); then introducing into (3.19); and according to the simplification, we obtain the following equations:

$$\begin{cases} \dot{x}_1 = e_\sigma(x_2 - x_1) - q_1x_1, \\ \dot{x}_2 = e_\rho x_1 - q_2x_2, \\ \dot{x}_3 = -e_\beta x_3 - q_3x_3. \end{cases} \quad (3.21)$$

Differentiating (3.20) with respect to t , we obtain:

$$\begin{cases} \dot{e}_\sigma = -\dot{\hat{\sigma}}, \\ \dot{e}_\rho = -\dot{\hat{\rho}}, \\ \dot{e}_\beta = -\dot{\hat{\beta}}. \end{cases} \quad (3.22)$$

We use Lyapunov's stability theory to find a suitable update for the parameter estimates. Let's consider a Lyapunov quadratic function given by:

$$P = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + e_\sigma^2 + e_\rho^2 + e_\beta^2),$$

which implies

$$\begin{aligned} \dot{P} &= \dot{x}_1x_1 + \dot{x}_2x_2 + \dot{x}_3x_3 + \dot{e}_\sigma e_\sigma + \dot{e}_\rho e_\rho + \dot{e}_\beta e_\beta \\ &= e_\sigma(x_2 - x_1)x_1 - q_1x_1^2 + e_\rho x_1x_2 - q_2x_2^2 - e_\beta x_3^2 - q_3x_3^2 - \dot{\hat{\sigma}}e_\sigma - \dot{\hat{\rho}}e_\rho - \dot{\hat{\beta}}e_\beta \\ &= -q_1x_1^2 - q_2x_2^2 - q_3x_3^2 + e_\sigma((x_2 - x_1)x_1 - \dot{\hat{\sigma}}) + e_\rho(x_1x_2 - \dot{\hat{\rho}}) + e_\beta(-x_3^2 - \dot{\hat{\beta}}). \end{aligned}$$

Then, we choose

$$\begin{cases} \dot{\hat{\sigma}} = (x_2 - x_1)x_1, \\ \dot{\hat{\rho}} = x_1x_2, \\ \dot{\hat{\beta}} = -x_3^2. \end{cases} \quad (3.23)$$

So

$$\dot{P} = -q_1x_1^2 - q_2x_2^2 - q_3x_3^2.$$

Theorem 36. The Lorenz system is globally and exponentially stabilized by the adaptive control law (3.18) and the update law (3.23), such that q_1, q_2, q_3 are positive constants.

2 Active control

Active control is a suitable and efficient method of controlling systems of known parameters. Let's take the following system:

$$\dot{x} = Ax + g(x) + \mu, \quad (3.24)$$

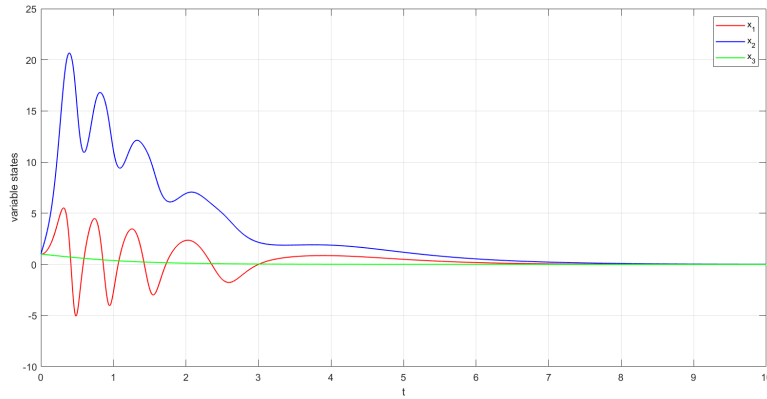


Figure 3.14: Time response of the controlled system states x_1 , x_2 , and x_3 using adaptive control.

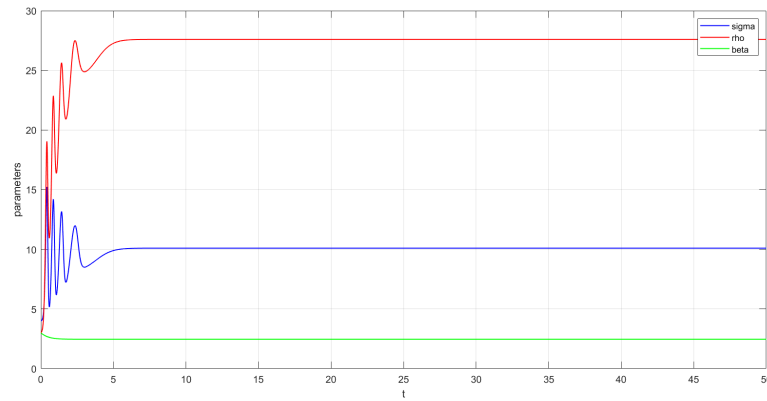


Figure 3.15: Parameters estimation of system (3.17).

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, A is a constant system matrix and $g(x)$ is a sequence function that is nonlinear, and μ is the controller vector.

Controller $\mu(t)$ is used to delete non-linear section. That is

$$\mu(t) = v(t) - g(x), \quad (3.25)$$

where $v(t) = Kx$, K is a constant matrix. From equation (3.25) and equation (3.24), we get

$$\dot{x} = (A + K)x. \quad (3.26)$$

Proposition 37. To satisfy the conditions for the diagonal matrix $(A + K)$, it is required that $\text{Re}(\lambda_i) < 0$ if λ_i are complex, and $\lambda_i < 0$ if λ_i are real, where λ_i represents the eigenvalue of the matrix $(A + K)$. Consequently, the state vectors of system (3.26) will asymptotically converge to 0.

Example 38. We implement active control to globally stabilise the trajectories of the Lorenz system.

The controlled Lorenz system is described by the 3D dynamics:

$$\begin{cases} \dot{x}_1 = 10(x_2 - x_1) + \mu_1, \\ \dot{x}_2 = 28x_1 - x_2 - x_1x_3 + \mu_2, \\ \dot{x}_3 = x_1x_2 - \frac{8}{3}x_3 + \mu_3. \end{cases} \quad (3.27)$$

Re-defining the control functions as follows:

$$\begin{cases} \mu_1 = v_1, \\ \mu_2 = x_1x_3 + v_2, \\ \mu_3 = -x_1x_2 + v_3. \end{cases} \quad (3.28)$$

The system (3.27) becomes:

$$\begin{cases} \dot{x}_1 = 10(x_2 - x_1) + v_1, \\ \dot{x}_2 = 28x_1 - x_2 + v_2, \\ \dot{x}_3 = -\frac{8}{3}x_3 + v_3. \end{cases} \quad (3.29)$$

Active control technique requires selecting a constant matrix A to regulate the system (3.29) such that:

$$[v_1, v_2, v_3]^T = A[x_1, x_2, x_3]^T. \quad (3.30)$$

So (3.29) becomes:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} a_{11} - 10 & 10 + b_{11} & c_{11} \\ a_{21} + 28 & -1 + b_{21} & c_{21} \\ a_{31} & b_{31} & -\frac{8}{3} + c_{31} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \quad (3.31)$$

Multiple selections of A can fulfil system (3.30). We select the subsequent matrix that fulfils the Routh-Hurwitz criteria for the stability of the controlled system:

$$A = \begin{pmatrix} 0 & -15 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.32)$$

Then, the active control of the Lorenz system is:

$$\begin{cases} \mu_1 = -15x_2, \\ \mu_2 = x_1x_3, \\ \mu_3 = -x_1x_2. \end{cases}$$

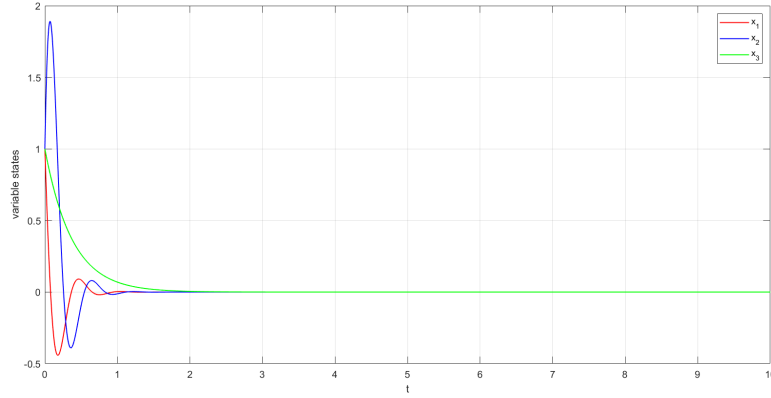


Figure 3.16: Time response of the controlled system states x_1 , x_2 , and x_3 using active control.

3 Backstepping control

Backstepping is a methodical approach for designing nonlinear control applicable to a broad category of systems. The name denotes the recursive characteristics of the procedure's design. A tiny subsystem is initially analysed to develop a virtual command rule, which is subsequently expanded through multiple stages until the comprehensive command law for the entire system is formulated. A Lyapunov function for the controlled system is systematically developed during the design process. This technique uses successive causal relationships to construct iteratively and systematically a command law and a stabilizing Lyapunov function. In addition, and unlike linear locking, backstepping is the ability to retain in the locking the stabilizing nonlinearities. In order for it to apply, the nonlinear system must be in the parametric form of a strict return loop. Examine the system presented in the parametric format of the specified strict return loop type:

$$\begin{cases} \dot{x}_i = g_i(x_i, t)x_{i+1} + f_i(x_i, t), & i = 1, \dots, n-1, \\ \dot{x}_n = g_n(x_n, t)\mu + f_n(x_n, t), \\ y = x_1, \\ x(t_0) = x_0. \end{cases} \quad (3.33)$$

Where $\bar{x}_i = [x_1; x_2; \dots; x_i] \in \mathbb{R}^i (i = 1; \dots; n-1); \mu \in \mathbb{R}; y \in \mathbb{R}$ are the vectors of the input and output statuses, respectively.. $g_i(\cdot) \neq 0, f_i(\cdot), i = 1; \dots; n-1$ are contained and derivable nonlinear functions. $g_n(\cdot) \neq 0, f_n(\cdot)$ are contained nonlinear functions. The aim is to develop a controller in the counter-reaction for the system (3.33) ensuring overall stability and forcing the system to the point of balance i.e. $x(t) \rightarrow 0$ when

$t \rightarrow \infty$. The design of the backstepping procedure contains n steps, during the i^{th} step an intermediate command law also called stabilizing function or virtual control law α_i must be determined using an appropriate Lyapunov function P_i .

When $i = 1$, in the first step, the first sub-system with a passive output z_1 and an x_2 input, is passive by a virtual control law $\alpha_1(x_1)$, the second sub-system with a passive outcome $z_2 = x_2 - \alpha_1$ and an input x_3 , is passively by a virtualized command law $\alpha_2(x_1; x_2)$, this procedure is applicable up to the n^{th} under system with an output $z_n = x_n - \alpha_{n-1}$ and an entry u . The z_n output is made passive and globally asymptotically stable by a final command law u . At each stage a Lyapunov function is constructed which is also considered a square type storage function and can be determined by both Krasovskii methods or the variable gradient.

First step

Let us define the first variable of the procedure as z_1 , which is the virtual output of the first under the system $z_1 = x_1$. The time-ratio derivative is:

$$\dot{z}_1 = \dot{x}_1 = g_1 z_2 + g_1 \alpha_1 + f_1,$$

and a second variable of Backstepping $z_2 = x_2 - \alpha_1$, with α_1 a virtual command law that will be determined later. To find this command law we construct a partial Lyapunov function of the quadratic type:

$$P_1(z_1) = \frac{1}{2} z_1^2,$$

its derivation with respect to time is

$$\dot{P}_1 = z_1 \dot{z}_1 = g_1 z_1 z_2 + z_1 (g_1 \alpha_1 + f_1),$$

with $-c_1 z_1 = g_1 \alpha_1 + f_1$. Nonlinearity is replaced by a linear control and α_1 is chosen so that P_1 is negative defined

$$\alpha_1 = \frac{1}{g_1} (-c_1 z_1 - f_1),$$

where $c_1 > 0$ is a positive design constant. Noting that α_1 was chosen to eliminate nonlinearity f_1 and to have $\dot{P}_1 < 0$. We replace α_1 in $\dot{P}_1$ and in $\dot{z}_1$ we find:

$$\dot{P}_1 = -c_1 z_1^2 + g_1 z_1 z_2,$$

$$\dot{z}_1 = -c_1 z_1 + g_1 z_2.$$

Second step

In this step we use a new variable $z_3 = x_3 - \alpha_2$ as an output for the second sub system with x_3 as an input and rewriting the second system (3.33) equation:

$$\dot{z}_2 = \dot{x}_2 - \dot{\alpha}_1 = g_2 z_3 + g_2 \alpha_2 + f_2 - \dot{\alpha}_1,$$

since $\alpha_1(x_1)$ is the stabilizing function of the subsystem then

$$\dot{\alpha}_1 = \frac{\partial \alpha_1}{\partial t} = \frac{\partial \alpha_1}{\partial x_1} \dot{x}_1 = \frac{\partial \alpha_1}{\partial x_1} (g_1 x_2 + f_1),$$

replacing $\dot{\alpha}_1$ in $\dot{z}_2$

$$\dot{z}_2 = g_2 z_3 + g_2 \alpha_2 + f_2 - \frac{\partial \alpha_1}{\partial x_1} (g_1 x_2 + f_1),$$

now we need to find a Lyapunov P_2 function for the second sub system, Because z_2 is a passive output, a possible choice of the storage function:

$$P_2(z_1, z_2) = P_1(z_1) + \frac{1}{2} z_2^2,$$

its derivation with respect to time is:

$$\dot{P}_2 = \dot{P}_1 + z_2 \dot{z}_2 = -c_1 z_1^2 + g_2 z_2 z_3 + z_2 (g_1 z_1 + g_2 \alpha_2 + f_2 - \frac{\partial \alpha_1}{\partial x_1} (g_1 x_2 + f_1)),$$

with $-c_2 z_2 = g_1 z_1 + g_2 \alpha_2 + f_2 - \frac{\partial \alpha_1}{\partial x_1} (g_1 x_2 + f_1)$. The key property of this expression is that all potentially undefined terms appeared to be multiplied by z_2 . Hence, our virtual control $x_3 = \alpha_2(x_1; x_2)$ can be chosen to do $\dot{P}_2$ negative defined a possible design is:

$$\alpha_2 = \frac{1}{g_2} (-c_2 z_2 - g_1 z_1 - f_2 + \frac{\partial \alpha_1}{\partial x_1} (g_1 x_2 + f_1)),$$

where c_2 is a positive constant. We replace α_2 in $\dot{P}_2$ and in $\dot{z}_2$ find

$$\dot{P}_2 = -c_1 z_1^2 - c_2 z_2^2 + g_2 z_2 z_3,$$

$$\dot{z}_2 = -c_2 z_2 + g_2 z_3 - g_1 z_1.$$

Similarly, for overall stability, the last term $g_2 z_2 z_3$ in $\dot{P}_2$ will be removed in the next step.

The third step

Let's start with the third variable of the process either $z_3 = x_3 - \alpha_2$ as an output for the third subsystem with x_4 as an input, unleashing a new variable $z_4 = x_4 - \alpha_3$, α_3 is

the stabilizing function of the third subsystem and rewriting the third equation of the system (3.33). The time ratio derivative is:

$$\dot{z}_3 = \dot{x}_3 - \dot{\alpha}_2 = g_3 z_4 + g_3 \alpha_3 + f_3 - \frac{\partial \alpha_2}{x_2}(g_2 x_3 + f_2) - \frac{\partial \alpha_1}{x_1}(g_1 x_2 + f_1),$$

because z_3 is a passive output, a possible choice of the storage function (Lyapunov):

$$P_3 = P_1 + P_2 + \frac{1}{2} z_3^2,$$

its derivation with relating to time is:

$$\begin{aligned} \dot{P}_3 &= \dot{P}_1 + \dot{P}_2 + z_3 \dot{z}_3, \\ &= -c_1 z_1^2 - c_2 z_2^2 + g_3 z_3 z_4 + z_3 (g_3 \alpha_3 + g_2 z_2 + f_3 - \frac{\partial \alpha_2}{x_2}(g_2 x_3 + f_2) - \frac{\partial \alpha_1}{x_1}(g_1 x_2 + f_1)), \end{aligned}$$

with

$$-c_3 z_3 = g_3 \alpha_3 + g_2 z_2 + f_3 - \frac{\partial \alpha_2}{x_2}(g_2 x_3 + f_2) - \frac{\partial \alpha_1}{x_1}(g_1 x_2 + f_1),$$

the virtual control $x_4 = \alpha_3(x_1, x_2, x_3)$ can be chosen by making negative P_3 defined a possible design is:

$$\alpha_3 = \frac{1}{g_3} (-c_3 z_3 - g_2 z_2 - f_3 + \frac{\partial \alpha_2}{x_2}(g_2 x_3 + f_2) + \frac{\partial \alpha_1}{x_1}(g_1 x_2 + f_1)),$$

α_3 is chosen so that P_3 is negative, c_3 is a positive constant. We replace α_3 in $\dot{P}_3$ and in $\dot{z}_3$ we find:

$$\begin{aligned} \dot{P}_3 &= -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 + g_3 z_3 z_4, \\ \dot{z}_3 &= -c_3 z_3 + g_3 z_4 - g_2 z_2. \end{aligned}$$

For overall stability, the last term $g_3 z_3 z_4$ in $\dot{P}_3$ will be excluded in the next step.

The n^{th} step

This is the closing design step, as long as the final μ control presents itself in the derivative of z_n

$$\dot{z}_n = \dot{x}_n - \dot{\alpha}_{n-1} = g_n \mu - f_{nth},$$

with

$$f_{nth} = f_n - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{x_{n-1}} (f_k + g_k x_{k+1}),$$

the μ control law is constructed to make negative determined the derivative of the global Lyapunov function:

$$P_n = P_{n-1} + \frac{1}{2} z_n^2,$$

its derivative with relating to time is

$$\dot{P}_n = \dot{P}_{n-1} + z_n \dot{z}_n,$$

what leads to

$$\dot{P}_n = - \sum_{k=1}^{n-1} c_k z_k^2 + z_n (g_{n-1} z_{n-1} + f_{nth} + g_n \mu),$$

with

$$-c_n z_n = g_{n-1} z_{n-1} + f_{nth} + g_n \mu,$$

The selection of the current control is

$$\mu = \frac{1}{g_n} (-c_n z_n - g_{n-1} z_{n-1} - f_{nth}), \quad (3.34)$$

we have now designed a command law in the counter reaction for system stabilization (3.33) of order n . u exists and is a regular stabilization function in the return loop, according to the Artstein theorem the dynamic system (3.33) has a law of control based on the Lyapunov function. We replace u in $\dot{z}_n$ and $\dot{P}_n$ we find:

$$\begin{aligned} \dot{P}_n &= - \sum_{k=1}^n c_k z_k^2, \\ \dot{z}_n &= -c_n z_n - g_{n-1} z_{n-1}. \end{aligned}$$

$\dot{P}_n < 0 \ \forall z \neq 0$, then the equilibrium point $x = x^*$ of (3.33) is asymptotically stable because the function $V(z)$ is continuously differentiable having the properties quoted in the LaSalle theorem, hence the global system in the coordinates $(z_1; \dots; z_n)$.

$$\begin{cases} \dot{z}_1 = -c_1 z_1 - g_1 z_2, \\ \dot{z}_i = -c_i z_i - g_{i-1} z_{i-1} + g_i z_{i+1} \quad i = 2, \dots, n-1, \\ \dot{z}_n = -c_n z_n - g_{n-1} z_{n-1}. \end{cases}$$

Example 39. Here, we acquire backstepping control for globally stabilizing the trajectories of the Lorenz system. To obtain a system of the strict feedback loop type, we perform a translation of the system's origin to this equilibria point $(\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho-1)$ as follows:

$$\begin{cases} x_1 = x_1 + \sqrt{\beta(\rho-1)}, \\ x_2 = x_2 + \sqrt{\beta(\rho-1)}, \\ x_3 = x_3 + \rho - 1. \end{cases}$$

With $\sigma = 10$, $\beta = 3$, and $\rho = 28$. So, the Lorenz system illustrated by

$$\begin{cases} \dot{x}_1 = 10(x_2 - x_1), \\ \dot{x}_2 = 28x_1 - x_2 - x_1x_3, \\ \dot{x}_3 = x_1x_2 - 3x_3. \end{cases} \quad (3.35)$$

The system (3.35) transforms into the type system strict feedback loop:

$$\begin{cases} \dot{x}_1 = 10x_2 - 10x_1, \\ \dot{x}_2 = x_1 - x_2 - x_3(x_1 + 9), \\ \dot{x}_3 = 9(x_1 + x_2) - 3x_3 + x_1x_2 + \mu. \end{cases} \quad (3.36)$$

Where μ is a backstepping control to be identified.

We initiate with the first subsystem $\dot{x}_1 = 10x_2 - 10x_1$, we set

$$\begin{cases} x_1 = y_1, \\ x_2 = y_2 + \alpha_1. \end{cases}$$

We set the Lyapunov function as expressed below:

$$P_1 = \frac{1}{2}y_1^2,$$

we get

$$\dot{P}_1 = \dot{y}_1 y_1 = (10y_2 + 10\alpha_1 - 10y_1)y_1 = -10y_1^2 + 10y_2y_1,$$

a stabilizing function $\alpha_1 = 0$, and $\dot{P}_1$ is negative definite, the last term will be suppressed in the next step. So the equation in coordinates (y_1, y_2) and the value of the Lyapunov function of the first subsystem are:

$$\begin{cases} \dot{y}_1 = 10y_2 - 10y_1, \\ \dot{P}_1 = -10y_1^2 + 10y_2y_1. \end{cases}$$

For the second subsystem $\dot{x}_2 = x_1 - x_2 - x_3(x_1 + 9)$, we define:

$$\begin{cases} x_1 = y_1, \\ x_2 = y_2, \\ x_3 = y_3 + \alpha_2. \end{cases}$$

And the Lyapunov function

$$P_2 = P_1 + \frac{1}{2}y_2^2,$$

we get

$$\dot{P}_2 = -10y_1^2 + 10y_2y_1 + \dot{y}_2y_2,$$

$$\dot{P}_2 = -10y_1^2 + 10y_2y_1 + y_2(y_1 - y_2 - (9 + y_1)y_3 - \alpha_2(9 + y_1)),$$

$$\dot{P}_2 = -10y_1^2 - y_2^2 + y_2(11y_1 - (9 + y_1)y_3 - \alpha_2(9 + y_1)).$$

To make $\dot{P}_2$ negative definite, α_2 is chosen as

$$\alpha_2 = \frac{11y_1}{(9 + y_1)},$$

so

$$\begin{cases} \dot{y}_2 = -y_3(9 + y_1) + (9y_1 + y_2), \\ \dot{P}_2 = -10y_1^2 - y_2^2 - (9 + y_1)y_3y_2. \end{cases}$$

In the following stage, the third term will be removed for overall stability.

Being aware that $x_3 = y_3 + \alpha_2$ and that the third equation rewrite contains u

$$\begin{cases} \dot{y}_3 = \dot{x}_3 - \dot{\alpha}_2, \\ \dot{y}_3 = 9(y_1 + y_2) - 3y_3 - 3\alpha_2 + y_1y_2 + u - \frac{99(10y_2 - 10y_1)}{(9 + y_1)^2}. \end{cases}$$

The last subsystem is $\dot{x}_3 = 9(x_1 + x_2) - 3x_3 + x_1x_2 + \mu$, the appropriate Lyapunov function of the subsystem is

$$P_3 = P_2 + \frac{1}{2}y_3^2$$

with a time derivative:

$$\begin{aligned} \dot{P}_3 &= -10y_1^2 - y_2^2 - (9 + y_1)y_3y_2 + y_3\dot{y}_3, \\ &= -10y_1^2 - y_2^2 - (9 + y_1)y_3y_2 + y_3(9(y_1 + y_2) - 3y_3 - 3\alpha_2 + y_1y_2 + \mu \\ &\quad - \frac{99(10y_2 - 10y_1)}{(9 + y_1)^2}), \\ &= -10y_1^2 - y_2^2 - 3y_3^2 + y_3(-(9 + y_1)y_2 + 9(y_1 + y_2) - 3\alpha_2 + y_1y_2 + \mu \\ &\quad - \frac{99(10y_2 - 10y_1)}{(9 + y_1)^2}). \end{aligned}$$

the control law is chosen as:

$$\mu = 9y_1 - \frac{33y_1}{(9 + y_1)} + \frac{99(10y_2 - 10y_1)}{(9 + y_1)^2},$$

So

$$\dot{P}_3 = -10y_1^2 - y_2^2 - 3y_3^2 \leq 0,$$

it indicates that the system (3.36) is stable in the origin.

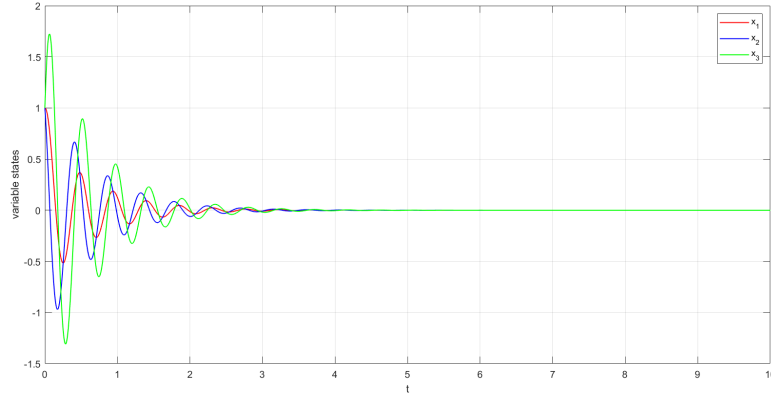


Figure 3.17: Time response of the controlled system states x_1 , x_2 , and x_3 using backstepping control.

4 Slide mode integral control

Sliding mode controllers are designed with stability concerns in mind and employ a three-step process that consistently produces good performance.

The selection of the sliding surface pertains not only to the requisite quantity of these surfaces but also to their configuration. Based on the application and the desired outcome. Generally, for a system characterised by the subsequent state equation:

$$\dot{x} = Ax + g(x) + \mu,$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $A \in \mathbb{R}^n \times \mathbb{R}^n$, A is a constant system matrix and $g(x)$ is a sequence function that is nonlinear, and μ is the controller vecteur.

Based on the equation of Mathews and DeCarlo, which proposes a general formula to determine the sliding surface that guarantees the convergence of a variable towards its intended value. We define the integral sliding surface as

$$s_i(x) = x_i + k_i \int_0^t x_i(\tau) d\tau, \text{ for } i = 1, \dots, n,$$

k_i are the weight for the integral term.

Next, the convergence criteria enable the system dynamics in the phase plane to approach the sliding surface. We retain two conditions from the literature: It is a matter of formulating a positive scalar function $P(x) > 0$ for the system state variables and choosing a control law that will make this function decrease $\dot{P}(x) < 0$. The Lyapunov function must be as follows:

$$P(x) = \frac{1}{2} s^2(x),$$

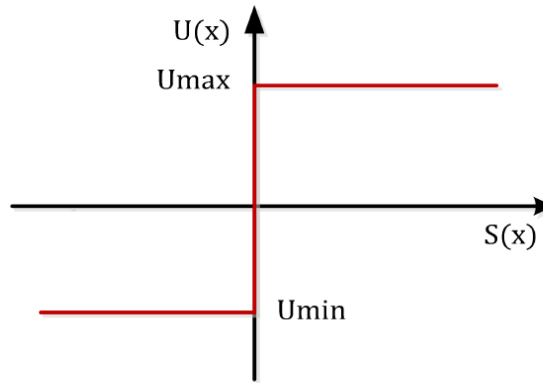


Figure 3.18: Command applied to the system.

for a good system dynamics and suitable convergence, it is sufficient to ensure that:

$$\dot{P}(x) = \dot{s}(x)s(x) < 0. \quad (3.37)$$

Ultimately, after selecting the sliding surface and convergence speed, it is essential to ascertain the requisite control to guide the variable towards this surface and subsequently to its equilibrium point (the origin of the phase plane), while preserving the conditions for the existence of the sliding mode. A fundamental hypothesis in the design of variable structure systems for sliding mode control is that the control must transition instantly between μ_{max} and μ_{min} (infinite frequency) based on the sign of the sliding surface, figure (3.18). The control law is given by:

$$\mu_i = -Ax_i - f(x) - k_i x_i - q_i \text{sign}(s_i), i = 1, \bar{n},$$

it is assumed that $k_i > 0$, $q_i > 0$, and $p_i > 0$ for $i = 1, \bar{n}$

Example 40. The controlled Lorenz system is described as follow:

$$\begin{cases} \dot{x}_1 = 10(x_2 - x_1) + \mu, \\ \dot{x}_2 = 28x_1 - x_2 - x_1x_3, \\ \dot{x}_3 = x_1x_2 - \frac{8}{3}x_3. \end{cases} \quad (3.38)$$

Next, we define:

$$s_1(x) = x_1 + k_1 \int_0^t x_1(\tau) d\tau. \quad (3.39)$$

The integral sliding surface s_1 in equation (3.39) is linked to the variable x_1 . A simple calculation results

$$\dot{s}_1 = \dot{x}_1 + k_1 x_1, \quad (3.40)$$

next, we define the integral sliding mode controls by

$$\mu = -10(x_2 - x_1) - k_1 x_1 - q_1 \text{sign}(s_1), \quad (3.41)$$

it is assumed that $k_1 > 0$, and $q_1 > 0$. Substituting the ISMC defined by (3.41) into the system dynamics (3.38), we obtain

$$\begin{cases} \dot{x}_1 = -k_1 x_1 - q_1 \text{sign}(s_1), \\ \dot{x}_2 = 28x_1 - x_2 - x_1 x_3, \\ \dot{x}_3 = x_1 x_2 - \frac{8}{3} x_3. \end{cases}$$

We regard the Lyapunov function described by the following equation:

$$P = \frac{1}{2} s_1^2,$$

by dividing P with reference to time, we get

$$\dot{P} = s_1(-q_1 \text{sign}(s_1)) = -q_1 |s_1| < 0,$$

since P is a positive definite function on $\mathbb{R}^3$ and $\dot{P}$ is a negative definite function on $\mathbb{R}^3$, we conclude from Lyapunov stability theory that $s_1 \rightarrow 0$, for as $t \rightarrow \infty$. This shows that $x_i \rightarrow 0$, for $i = 1, 2, 3$ as $t \rightarrow \infty$, this completes the proof.

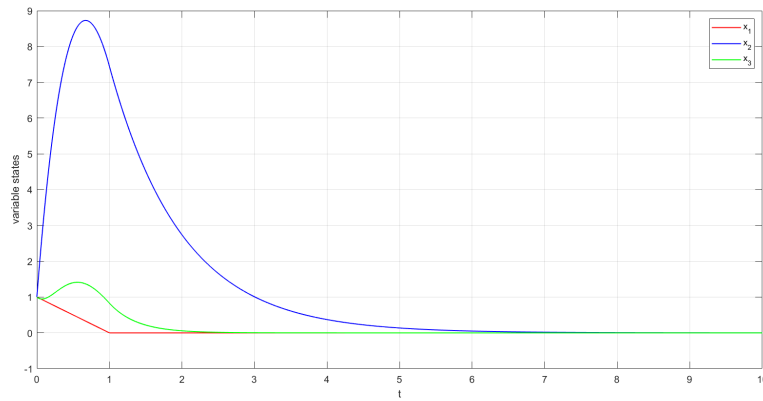


Figure 3.19: Time response of the controlled system states x_1 , x_2 , and x_3 using slide mode control.

3.3 Conclusion

In this chapter, we presented a comprehensive review of the main methods for chaos control as proposed in the literature, highlighting their significance in managing nonlinear dynamical

systems. We discussed concepts such as clarification, stabilization, and suppression of chaos, and explored various control strategies, including the OGY and Pyragas methods, adaptive control, active control, as well as modern techniques like backstepping and sliding mode control.

Through this study, we demonstrated how these methods can serve as effective tools for guiding the dynamics of nonlinear systems toward stable or desired behavior while preserving their internal complexity.

Chapter 4

Chaos synchronization

When two or more chaotic systems finally proceed in all future stages in the same way for various initial circumstances, this is known as synchronization of chaotic systems. In 1990, Pecora and Carroll presented a technique for synchronising two identical chaotic systems with varied beginning conditions [25].

Since then, one of the main processes in the control of complicated systems has been the synchronization between two chaotic systems. This process has a wide range of possible applications in several domains, including social science, biological research, chemical reactions, and secure communication, among many others.

A notable outcome is the identification of several synchronisation occurrences, including complete synchronisation. (CS), phase synchronization (PS), lag synchronization (LS) or anticipated synchronization (AS), and generalized synchronization (GS).

4.1 General definition

One of the first things we learned about human connections is their synchronization; a newborn who smiles at his mother just copies the expressions on her face.

Definition 41. (Larousse) The Greek term "synchronization" is split into two parts: "Syn," which means "together," and "Chrono," which means "time." Depending on the time, it is the process of phasing in to provide simultaneity between many processes.

Definition 42. (General) Synchronization is one method of preserving a periodic movement (or chaotic). When two dynamic systems are synchronized, one system changes by imitating the other's actions.

4.2 Mathematical definition

A comprehensive concept that takes into consideration all varieties of synchronization between finite dimensional systems is put by Brown and Kocarev [12]. Their findings may be adopted to N continuous-time or N discrete-time systems, even though they specifically address synchronization between two continuous-time dynamical systems. Assume that a sizable stable deterministic dynamical system is split up into distinct two sub-systems,

$$\begin{cases} \dot{x} = f(x, t), \\ \dot{y} = g(y, t). \end{cases} \quad (4.1)$$

Here, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ are vectors that may have different dimensions.

First, the large-system dynamics is decomposed into the sub-system dynamics. Let $\phi(z_0)$ represent the trajectory of the massive dynamical system, provided by (4.1), where $z_0 = [x_0^T, y_0^T] \in \mathbb{R}^{n+m}$ is the starting condition.

Projecting the x and y components away from this trajectory yields the curves $\phi_x(z_0)$ and $\phi_y(z_0)$, respectively. We denote the trajectories of the first and second sub-systems of (4.1) as $\phi_x(z_0)$ and $\phi_y(z_0)$. Instead of creating $\phi(z_0)$ from $\phi_x(z_0)$ and $\phi_y(z_0)$, we have detached the trajectories $\phi_x(z_0)$ and $\phi_y(z_0)$ from $\phi(z_0)$ in this context. The first subsystem's trajectories are denoted by ζ , whereas the second subsystem's trajectories are represented by χ .

We consider the mapping

$$g_x : \zeta \times \mathbb{R} \rightarrow \mathbb{R}^k$$

and

$$g_y : \chi \times \mathbb{R} \rightarrow \mathbb{R}^k$$

In order for g_x and g_y to explicitly refer to time, the first $\mathbb{R}$ is added, which symbolises time. The mapping, g_x and g_y , are a property of the first and second sub-systems. The image of $[\phi_x(z_0), t] \in \zeta \times \mathbb{R}$ under the mapping g_x is the outcome of measuring the attribute of the primary subsystem.

It will be indicated as $g(x) \in \mathbb{R}^k$, and $[\phi_x(z_0), t] \in \chi \times \mathbb{R}$ under the mapping g_y is the result of measuring the property of the second sub-system. It will be denoted by $g(y) \in \mathbb{R}^k$.

Finally, Brown and Kocarev define the function $h : \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ that compares the measured properties of the two sub-systems, and there is a time agreement between the two measurements if and only if $h[g(x), g(y)] = 0$.

Definition 43. The sub-systems, denoted by (4.1), are synchronised on the trajectory $\phi(z_0)$, with regard to the attributes g_x and g_y , provided that there exists a function h that is independent of time. $\mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ such that $\|h[g(x), g(y)]\| = 0$, where $\|\cdot\|$ is some norm.

With the selection of g_x , g_y , and h , we can ascertain the type of synchronization. This method raises the possibility that multiple synchronization forms might be included inside the same formalism.

4.3 Types of synchronization

In the sense of an extensive bibliographic study, we have gathered different types and patterns of synchronization.

We consider a pair of master-slave systems represented by:

$$\begin{aligned}\dot{x} &= f(x(t)), \\ \dot{y} &= g(y(t)) + \mu,\end{aligned}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $g : \mathbb{R}^m \rightarrow \mathbb{R}^m$, and $\mu = (\mu_i)_{i=1}^m \in \mathbb{R}^m$ is the control vector.

Complete synchronization

The error of complete synchronization noted e_{cs} is defined by:

$$e_{cs}(t) = y(t) - x(t) \iff n = m, \text{ and } \lim_{t \rightarrow +\infty} \|e_{cs}(t)\| = 0 \quad \forall (x(0), y(0)).$$

- If $f = g$, it is a complete identical synchronization.
- If $f \neq g$, it is a complete non-identical synchronization.

Anti synchronization

Definition 44. [12]

When the absolute values of the state vectors in the master and slave systems are the same but the signs are opposite, and when the total of these vectors approaches 0 as time approaches infinity, we say that the two systems are in anti-synchronization. Consequently, the anti-synchronization error noted by e_{as} manifests as:

$$e_{as}(t) = y(t) + x(t) \text{ and } \lim_{t \rightarrow +\infty} \|e_{as}(t)\| = 0 \quad \forall (x(0), y(0)).$$

Delayed synchronization

Definition 45. It is said that two non-identical chaotic dynamical systems exhibit delayed synchronization if and only if the time is delayed.

Definition 46. [69] If the state variables $y(t)$ of the slave chaotic system converge towards the state variables $x(t)$ of the master system with a time delay, it is said that there is delayed synchronization. So, the relation of delayed synchronization error noted by e_{ds} is given by:

$$e_{ds} = y(t) - x(t - \tau) \text{ and } \lim_{t \rightarrow +\infty} \|e_{ds}(t)\| = 0 \quad \forall (x(0), y(0)).$$

Projective synchronization

Definition 47. [12]

It is said that we have a projective synchronization if the state variables $y_i(t)$ of the slave chaotic system $y(t) = (y_i(t))_{1 \leq i \leq n}$ synchronizes with a constant multiple of the state $x_i(t)$ of the master chaotic system $x(t) = (x_i(t))_{1 \leq i \leq n}$. The projective synchronization error noted by e_{ps} is given by :

$$\exists \alpha_i \neq 0, \quad e_{ps} = y(t) - \alpha_i x(t) \text{ and } \lim_{t \rightarrow +\infty} \|e_{ps}\| = 0, \quad \forall (x(0), y(0)).$$

- A complete synchronisation situation is when all of the α_i are exactly equal to 1.
- The anti-synchronization situation is represented by all α_i being equal to -1 .

Generalized synchronization

Generalized synchronization is considered a generalization of complete synchronization, anti-synchronization, and projective synchronization in the case of chaotic systems of different dimensions and models. It is demonstrated by the functional connection between the two chaotic systems. In comparison with identical synchronization, generalized synchronization, denoted by GS, can provide a richer dynamics, as it can also consider certain desynchronized cases due to disparities in parameters, distortions in transmission channels, and others. As a result, the possibilities of applying GS may be broader than identical synchronization.

Definition 48. [69] If there exists a function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$, such that all the trajectories of the master system and the slave system satisfy:

$$\lim_{t \rightarrow +\infty} \|y(t) - \varphi(x(t))\| = 0 \quad \forall (x(0), y(0)).$$

So, the master-slave systems (4.51) synchronize in a generalized sense with respect to the function φ and $e_{gs} = y(t) - \varphi(x(t))$ represents the generalized synchronization error.

$R - S$ synchronization

$R - S$ synchronisation is thought to be a generalisation of all earlier synchronisation kinds. We will say that a master system, n -dimensional, $x(t)$, and a slave system, m -dimensional, $y(t)$, are in $R - S$ synchronization in dimension d , if there exists a controller $\mu = (\mu_i)_{1 \leq i \leq m}$ and two functions $R : \mathbb{R}^n \rightarrow \mathbb{R}^d$, $S : \mathbb{R}^m \rightarrow \mathbb{R}^d$ such that the synchronization error noted by e_{RS} :

$$\lim_{t \rightarrow +\infty} \|e_{RS}(t)\| = 0; \quad e_{RS}(t) = R(x(t)) - S(y(t)), \quad \forall (x(0), y(0)).$$

Phase synchronization

This type of synchronization comes from the classical notion of synchronization for two periodic systems whose phases are θ_1 and θ_2 , which is expressed by the following relation:

$$|n\theta_1 - m\theta_2| < c,$$

where m and n are natural integers and c is a positive constant. This notion of synchronization has been extended to chaotic systems, and the analytical approach is one of the solutions that allows for defining the phase of a chaotic system. To express the phase of a system by representing its analytical signal $\Psi(t)$ in the form of a complex function defined by:

$$\Psi(t) = r(t) + j\tilde{r}(t) = A(t)e^{i\theta(t)},$$

such that $\tilde{r}(t)$ is the Hilbert transform of the time series $r(t)$, $A(t)$ is the amplitude of the signal $\Psi(t)$ and $\theta(t)$ is its phase. The phase synchronization between two coupled chaotic systems occurs if:

$$|n\theta_1(t) - m\theta_2(t)| < c.$$

It is worth noting that in this case, the amplitudes remain uncorrelated.

4.4 Application of control and synchronization methods on the SIR model

The SIR model, first introduced by Kermack and McKendrick in 1927 [32], is a fundamental mathematical framework for understanding the spread of infectious diseases within a population. The model divides the population into three compartments: susceptible (S), infectious (I), and recovered (R).

The susceptible compartment denotes individuals at risk of infection, the infectious compartment includes those currently infected and able to transmit the disease, and the recovered compartment comprises individuals who have recuperated and acquired immunity or are currently not infectious.

To account for seasonal or periodic changes in the transmission dynamics, the SIR model can be adapted to a forced version, where the transmission rate α becomes time-dependent. A common formulation is given by

$$\alpha(t) = \alpha_0 \left(1 + \delta \sin \left(\frac{2\pi t}{T} \right) \right)$$

where α_0 is the mean transmission rate, δ represents the degree of seasonal variation, and T is the period of these variations, typically one year. This modification allows the model to capture seasonal effects such as changes in weather conditions or human behavior, making it a powerful tool for analyzing the periodic dynamics of infectious diseases and designing efficient intervention techniques.

4.4.1 Description of SIR model

The following system of differential equations gives the forced SIR model [7, 17] :

$$\begin{cases} \dot{x} &= \gamma - \alpha(t)xy - \sigma x, \\ \dot{y} &= \alpha(t)xy - (\sigma + \beta)y, \\ \dot{z} &= \beta y - \sigma z. \end{cases} \quad (4.2)$$

Such that γ is the rate of birth, σ is the rate of death and β is the rate of recovery and, the rate of transmission is defined by $\alpha(t) = \alpha_0(1 + \delta \sin(\frac{2\pi}{T}t))$. Also x represents susceptible, y represents infected people and, z represents recovered people.

The system (4.2) is chaotic when the parameter systems take the following values .

$$\gamma = 0.01, \quad \sigma = 0.01, \quad \beta = 50, \quad \alpha_0 = 1510, \quad T = 1, \quad \delta \in [0.0960, 0.1073]. \quad (4.3)$$

The system possesses two equilibrium points when we take $\alpha(t) = \alpha_0$:

1. the disease- free equilibrium point:

$$(x^0, y^0, z^0) = \left(\frac{\gamma}{\sigma}, 0, 0\right),$$

pertaining to the population devoid of diseased persons;

2. the endemic equilibrium point:

$$(x^1, y^1, z^1) = \left(\frac{\beta + \sigma}{\alpha_0}, \frac{\sigma}{\alpha_0}(R_{ep} - 1), \frac{\beta}{\alpha_0}(R_{ep} - 1)\right),$$

reflecting the existence of a significant cohort of infected persons within the community.

The dynamics of the illness are contingent upon a parameter.

$$R_{ep} = \frac{\alpha_0 \gamma}{\sigma(\beta + \sigma)},$$

referred to as the fundamental reproduction number, it may be understood as the "disease-spreading impact" of a single sick person within a community comprised only of susceptible people.

For the numerical simulations, we take:

$$(x_0, y_0, z_0) = (0.0331, 0.0001, 0.9668)$$

The chaotic attractor of the system (4.2) is represented in figure (4.1). The Lyapunov exponents of system (??) are depicted in Figure 4.2. Bifurcation diagrams for the variables x and y are given in figures (4.3) and (4.4)

4.4.2 Anti-synchronization of SIR model using multiple types of control

In this section, we investigate the anti-synchronization behavior between two identical chaotic SIR models under different control strategies. Anti-synchronization occurs when the corresponding state variables of the drive and response systems are equal in magnitude but opposite in sign, leading to a complete phase opposition. To achieve this objective, several control techniques are applied and compared, namely active control, sliding mode control, backstepping control, and adaptive control. Each method is designed to ensure that the

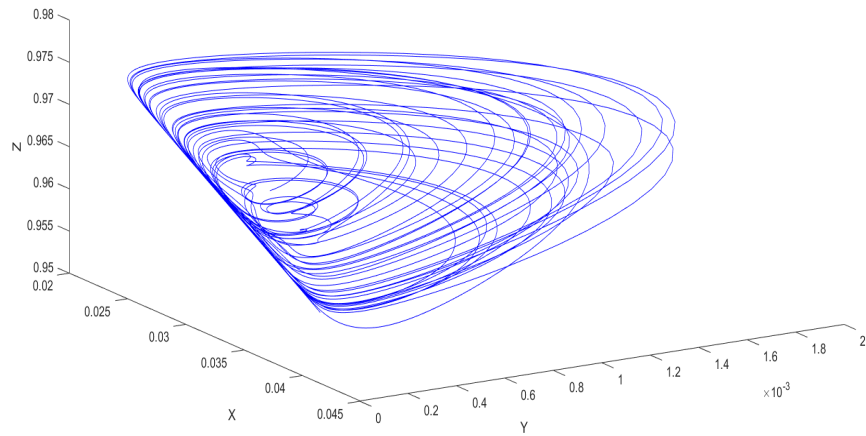


Figure 4.1: SIR attractor of system (4.2).

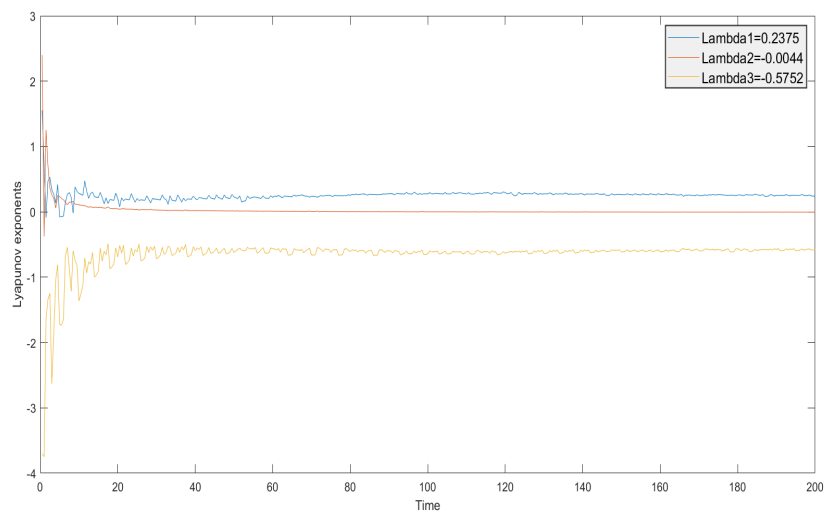


Figure 4.2: The Lyapunov coefficients of the SIR system (4.2).

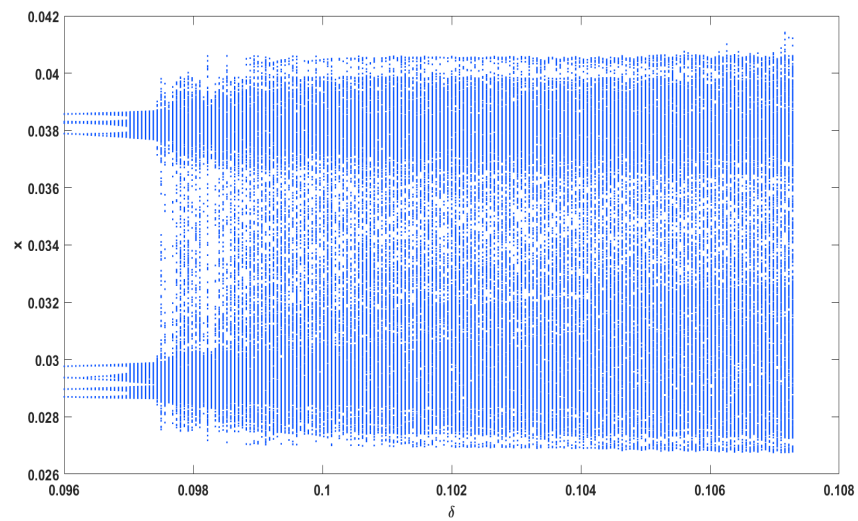


Figure 4.3: Bifurcation diagram for the variable x .

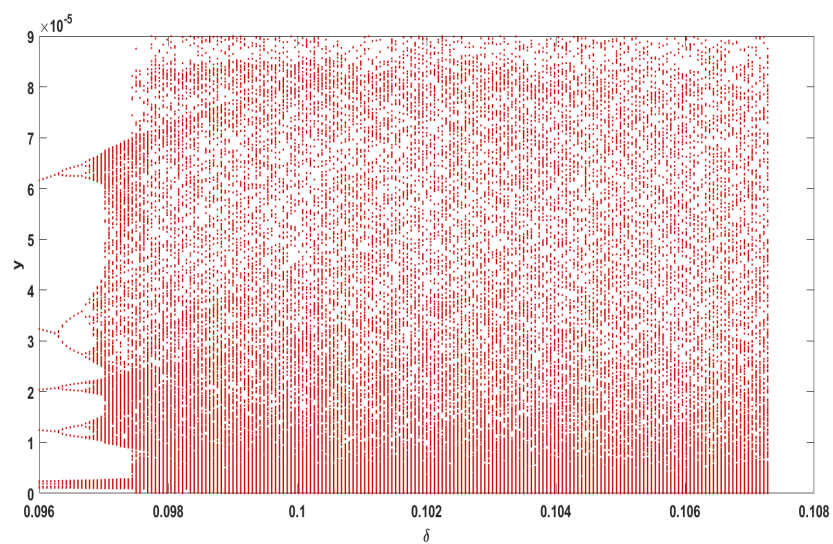


Figure 4.4: Bifurcation diagram for the variable y .

synchronization errors converge asymptotically to zero. We reformed system (4.2) to be autonomous. By substituting $w = \frac{2\pi}{T}t$, it follows that $\dot{w} = \frac{2\pi}{T}$. Therefore, the new system is:

$$\begin{cases} \dot{x} &= \gamma - \alpha_0(1 + \delta \sin(w))xy - \sigma x, \\ \dot{y} &= \alpha_0(1 + \delta \sin(w))xy - (\sigma + \beta)y, \\ \dot{z} &= \beta y - \sigma z, \\ \dot{w} &= \frac{2\pi}{T}. \end{cases} \quad (4.4)$$

Active control

The master-slave systems are represented as follows:

$$\begin{cases} \dot{x}_1 &= \gamma - \alpha_0(1 + \delta \sin(w_1))x_1y_1 - \sigma x_1, \\ \dot{y}_1 &= \alpha_0(1 + \delta \sin(w_1))x_1y_1 - (\sigma + \beta)y_1, \\ \dot{z}_1 &= \beta y_1 - \sigma z_1, \\ \dot{w}_1 &= \frac{2\pi}{T}. \end{cases} \quad (4.5)$$

$$\begin{cases} \dot{x}_2 &= \gamma - \alpha_0(1 + \delta \sin(w_2))x_2y_2 - \sigma x_2 + \mu_x, \\ \dot{y}_2 &= \alpha_0(1 + \delta \sin(w_2))x_2y_2 - (\sigma + \beta)y_2 + \mu_y, \\ \dot{z}_2 &= \beta y_2 - \sigma z_2 + \mu_z, \\ \dot{w}_2 &= \frac{2\pi}{T} + \mu_w. \end{cases} \quad (4.6)$$

Where μ_x, μ_y, μ_z , and μ_w indicates the active control functions that will be present. Equation (4.4.2) gives the definition of anti-synchronization error between systems (4.5) and (4.6):

$$\begin{cases} e_x &= x_1 + x_2, \\ e_y &= y_1 + y_2, \\ e_z &= z_1 + z_2, \\ e_w &= w_1 + w_2. \end{cases}$$

A basic calculation provides the error dynamics formula:

$$\begin{cases} \dot{e}_x &= 2\gamma - \sigma e_x - \alpha_0((1 + \delta \sin(w_2))x_2y_2 + (1 + \delta \sin(w_1))x_1y_1) + \mu_x, \\ \dot{e}_y &= -(\sigma + \beta)e_y + \alpha_0((1 + \delta \sin(w_2))x_2y_2 + (1 + \delta \sin(w_1))x_1y_1) + \mu_y, \\ \dot{e}_z &= \beta e_y - \sigma e_z + \mu_z, \\ \dot{e}_w &= \frac{4\pi}{T} + \mu_w. \end{cases} \quad (4.7)$$

The active control laws for each μ_x, μ_y, μ_z , and μ_w are developed in accordance with the scheme outlined in [67], to ensure that the error system (4.7) are global asymptotic stable.

Control functions outlined below:

$$\begin{cases} \mu_x &= -2\gamma + \alpha_0((1 + \delta \sin(w_2))x_2y_2 + (1 + \delta \sin(w_1))x_1y_1) + v_x, \\ \mu_y &= -\alpha_0((1 + \delta \sin(w_2))x_2y_2 + (1 + \delta \sin(w_1))x_1y_1) + v_y, \\ \mu_z &= -\beta e_y + v_z, \\ \mu_w &= -\frac{4\pi}{T} - e_w + v_w. \end{cases} \quad (4.8)$$

System(4.7) turns into:

$$\begin{cases} \dot{e}_x &= -\sigma e_x + v_x, \\ \dot{e}_y &= -(\sigma + \beta)e_y + v_y, \\ \dot{e}_z &= -\sigma e_z + v_z, \\ \dot{e}_w &= -e_w + v_w. \end{cases} \quad (4.9)$$

The matrix form (4.9) is described below:

$$[\dot{e}_x \ \dot{e}_y \ \dot{e}_z \ \dot{e}_w]^T = (D + F)[e_x \ e_y \ e_z \ e_w]^T,$$

where

$$[v_x \ v_y \ v_z \ v_w]^T = F[e_x \ e_y \ e_z \ e_w]^T,$$

in the active control strategy, F is selected to manage the dynamics of the error, whereas D is a constant matrix.

$$F = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}.$$

So

$$\begin{pmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_z \\ \dot{e}_w \end{pmatrix} = \begin{pmatrix} -\sigma + a_1 & b_1 & c_1 & d_1 \\ a_2 & -(\beta + \sigma) + b_2 & c_2 & d_2 \\ a_3 & b_3 & -\sigma + c_3 & d_3 \\ a_4 & b_4 & c_4 & -1 + d_4 \end{pmatrix} \begin{pmatrix} e_x \\ e_y \\ e_z \\ e_w \end{pmatrix}.$$

The error dynamics can be satisfied by a variety of matrix F choices. Nevertheless, it is possible to introduce a matrix F that satisfies the Routh-Hurwitz criterion to guarantee the

stability of the error system in the vicinity of zero. Consequently,

$$F = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -50 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, it is immediate that the anti-synchronization error $e(t) \rightarrow 0$ is asymptotically stable.

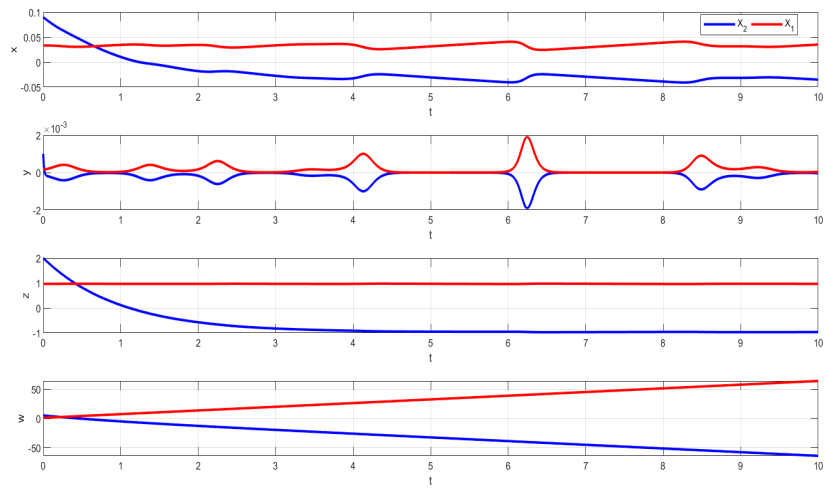


Figure 4.5: Anti-synchronization of the states of systems (4.5) and (4.6) using active control

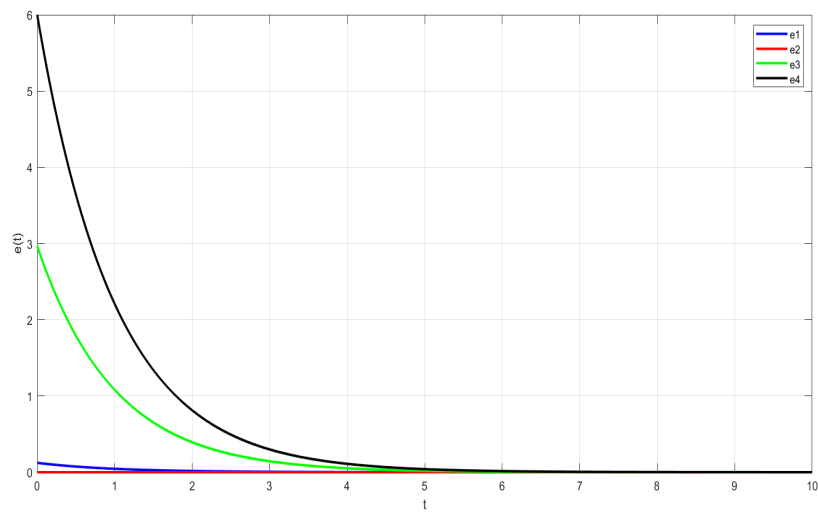


Figure 4.6: Time-history of the anti-synchronization errors using active control

Sliding mode control

We will provide an integral sliding mode control solution to achieve anti-synchronization of systems (4.5) and (4.6). The anti-synchronization error of systems (4.5) and (4.6) is given by (4.4.2). The ISMC approach requires to select a suitable sliding surfaces, that confirm the stability of the error dynamics and facilitate the convergence of system (4.4.2) to zero. So, we select the sliding surfaces in the following manner:

$$\begin{cases} s_x &= e_x + h_1 \int_0^t e_x d\tau, \\ s_y &= e_y + h_2 \int_0^t e_y d\tau, \\ s_z &= e_z + h_3 \int_0^t e_z d\tau, \\ s_w &= e_w + h_4 \int_0^t e_w d\tau. \end{cases} \quad (4.10)$$

Where the variables $h_i, i = 1, 2, 3, 4$ are all positive. So, we get

$$\begin{cases} \dot{s}_x &= \dot{e}_x + h_1 e_x, \\ \dot{s}_y &= \dot{e}_y + h_2 e_y, \\ \dot{s}_z &= \dot{e}_z + h_3 e_z, \\ \dot{s}_w &= \dot{e}_w + h_4 e_w. \end{cases} \quad (4.11)$$

Next, we define the integral sliding mode controls by:

$$\begin{cases} \mu_x &= -2\gamma - \alpha_0((1 + \delta \sin(w_2))x_2 y_2 + (1 + \delta \sin(w_1))x_1 y_1) + (\sigma - h_1)e_x, \\ &-p \cdot \text{sign}(s_x) \\ \mu_y &= -\alpha_0((1 + \delta \sin(w_2))x_2 y_2 + (1 + \delta \sin(w_1))x_1 y_1) + (\sigma + \beta - h_2)e_y, \\ &-p \cdot \text{sign}(s_y) \\ \mu_z &= -\beta e_y + (\sigma - h_3)e_z - p \cdot \text{sign}(s_z), \\ \mu_w &= -\frac{4\pi}{T} - h_4 e_w - p \cdot \text{sign}(s_w). \end{cases} \quad (4.12)$$

We replace the control laws defined in (4.12) in (4.7), we obtain

$$\begin{cases} \dot{e}_x &= -h_1 e_x - p \cdot \text{sign}(s_x), \\ \dot{e}_y &= -h_2 e_y - p \cdot \text{sign}(s_y), \\ \dot{e}_z &= -h_3 e_z - p \cdot \text{sign}(s_z), \\ \dot{e}_w &= -h_4 e_w - p \cdot \text{sign}(s_w). \end{cases} \quad (4.13)$$

Let the following Lyapunov function be:

$$P(s) = \frac{1}{2}(s_x^2 + s_y^2 + e_z^2 + e_w^2),$$

which implies that:

$$\dot{P} = s_x \dot{s}_x + s_y \dot{s}_y + s_z \dot{s}_z + s_w \dot{s}_w,$$

substituting (4.11) into (4.4.2), we find:

$$\dot{P} = s_x(\dot{e}_x + h_1 e_x) + s_y(\dot{e}_y + h_2 e_y) + s_z(\dot{e}_z + h_3 e_z) + s_w(\dot{e}_w + h_4 e_w),$$

which implies that:

$$\dot{P} = s_x(-p \cdot \text{sign}(s_x)) + s_y(-p \cdot \text{sign}(s_y)) + s_z(-p \cdot \text{sign}(s_z)) + s_w(-p \cdot \text{sign}(s_w)),$$

$$\dot{P} \leq -p|s_x| - p|s_y| - p|s_z| - p|s_w|.$$

By selecting $p > 0$, $\dot{P}$ is ensured to be negative definite. Depending on Lyapunov stability theorem, the sliding surface is guaranteed to converge to zero. As a result, the error systems are asymptotically stable and will eventually converge to zero.

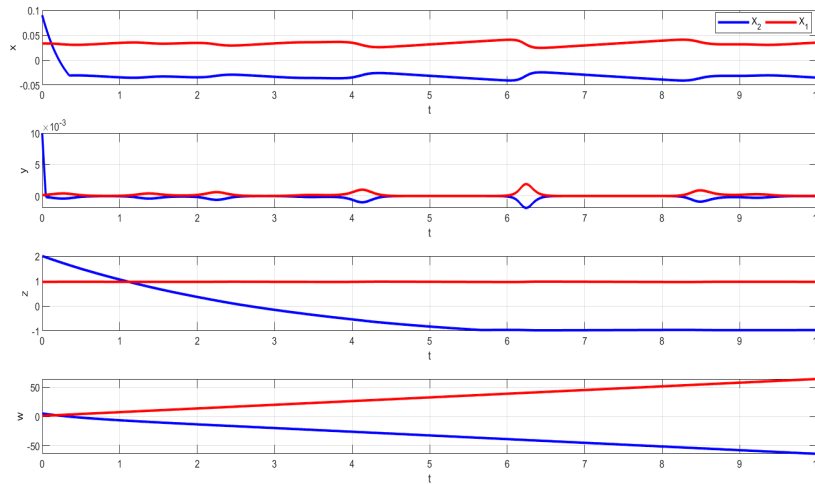


Figure 4.7: Anti-synchronization of system (4.5) and (4.6) using sliding mode control.

Active backstepping control

We replace $\gamma, \sigma, \beta, \alpha, T = 1$ by their values presented in (4.3), and we apply a rearrangement of variables:

$$\begin{cases} x = w, \\ y = z, \\ z = y, \\ w = x. \end{cases} \Rightarrow \begin{cases} \dot{x} = \dot{w}, \\ \dot{y} = \dot{z}, \\ \dot{z} = \dot{y}, \\ \dot{w} = \dot{x}. \end{cases}$$

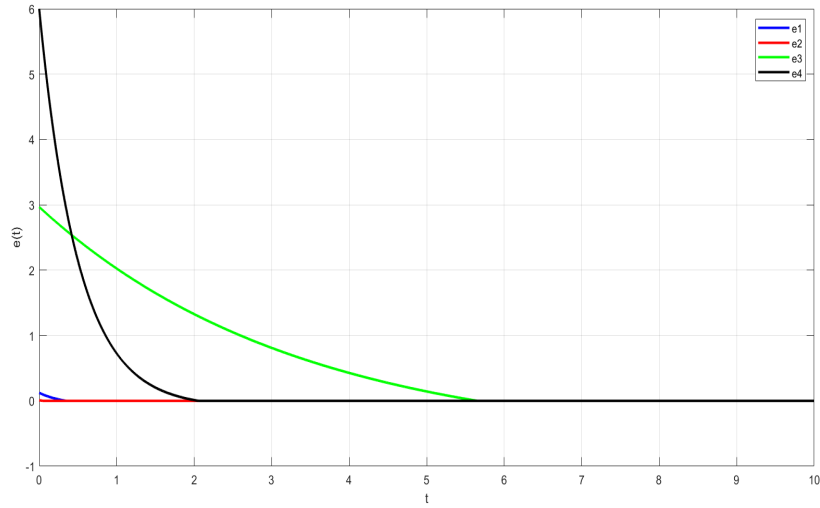


Figure 4.8: Time-history of the anti-synchronization errors using sliding mode control

Thus, system (4.5) can be defined as:

$$\begin{cases} \dot{x}_1 = 2\pi, \\ \dot{y}_1 = -0.01y_1 + 50z_1, \\ \dot{z}_1 = -50.01z_1 + 1510(1 + \delta \sin(x_1))z_1w_1, \\ \dot{w}_1 = 0.01 - 0.01w_1 - 1510(1 + \delta \sin(x_1))z_1w_1. \end{cases} \quad (4.14)$$

So system (4.6) becomes:

$$\begin{cases} \dot{x}_2 = 2\pi + \mu_x, \\ \dot{y}_2 = -0.01y_2 + 50z_2 + \mu_y, \\ \dot{z}_2 = -50.01z_2 + 1510(1 + \delta \sin(x_2))z_2w_2 + \mu_z, \\ \dot{w}_2 = 0.01 - 0.01w_2 - 1510(1 + \delta \sin(x_2))z_2w_2 + \mu_w. \end{cases} \quad (4.15)$$

In this case, the feedback control laws u_x, u_y, u_z, u_w need to be found by backstepping. The error system is defined by (4.4.2) which implies that:

$$\begin{cases} \dot{e}_x = 4\pi + \mu_x, \\ \dot{e}_y = -0.01e_y + 50e_z + \mu_y, \\ \dot{e}_z = -50.01e_z + 1510((1 + \delta \sin(x_2))z_2w_2 + (1 + \delta \sin(x_1))z_1w_1) + \mu_z, \\ \dot{e}_w = 0.02 - 0.01e_w - 1510((1 + \delta \sin(x_2))z_2w_2 + (1 + \delta \sin(x_1))z_1w_1) + \mu_w. \end{cases} \quad (4.16)$$

First, feedback control restructures system (4.16) to facilitate the utilization of backstepping control. The process starts with an examination of the feedback control law:

$$\begin{cases} \mu_x &= -e_x - e_y - 4\pi, \\ \mu_y &= -0.99e_y, \\ \mu_z &= 0.01e_z - 1510((1 + \delta \sin(x_2))z_2w_2 + (1 + \delta \sin(x_1))z_1w_1) + e_w, \\ \mu_w &= -0.99e_w + 1510((1 + \delta \sin(x_2))z_2w_2 + (1 + \delta \sin(x_1))z_1w_1) - 0.02 + \mu. \end{cases} \quad (4.17)$$

Here, μ denotes the control command to be determined for feedback purposes. Replacing (4.17) in (4.16) leads to the updated system in the following form:

$$\begin{cases} \dot{e}_x &= -e_x - e_y, \\ \dot{e}_y &= -e_y + 50e_z, \\ \dot{e}_z &= -50e_z + e_w, \\ \dot{e}_w &= -e_w + \mu. \end{cases} \quad (4.18)$$

The first Lyapunov function is:

$$P_1(v_x) = \frac{1}{2}v_x^2,$$

where $v_x = e_x$, we get:

$$\dot{P}_1(v_x) = v_x \dot{v}_x = v_x(-e_x - e_y) = -v_x^2 - v_x e_y, \quad (4.19)$$

next, we define:

$$v_y = -e_y, \quad (4.20)$$

using (4.20), we can express (4.19) as:

$$\dot{P}_1(v_x) = -v_x^2 + v_x v_y,$$

the second Lyapunov function is:

$$P_2(v_x, v_y) = P_1(v_x) + \frac{1}{2}v_y^2,$$

we acquire:

$$\dot{P}_2(v_x, v_y) = -v_x^2 + v_x v_y + v_y \dot{v}_y = -v_x^2 - v_y^2 + v_y(v_x + v_y + \dot{v}_y). \quad (4.21)$$

We establish:

$$v_z = v_x + v_y + \dot{v}_y = e_x - 50e_z,$$

thus, (4.21) becomes:

$$\dot{P}_2(v_x, v_y) = -v_x^2 - v_y^2 + v_y v_z,$$

the third Lyapunov function is:

$$P_3(v_x, v_y, v_z) = P_2(v_x, v_y) + \frac{1}{2}v_z^2,$$

we get

$$\dot{P}_3(v_x, v_y, v_z) = -v_x^2 - v_y^2 + v_y v_z + v_z \dot{v}_z = -v_x^2 - v_y^2 - v_z^2 + v_z(v_y + v_z + \dot{v}_z). \quad (4.22)$$

We define:

$$\begin{aligned} v_w &= v_y + v_z + \dot{v}_z = -e_y + e_x - 50e_z + \dot{e}_x - 50\dot{e}_z = -e_y + e_x - 50e_z - e_x - e_y, \\ &+ 2500e_z - 50e_w = -2e_y + 2450e_z - 50e_w. \end{aligned}$$

Hence, (4.22) becomes:

$$\dot{P}_3(v_x, v_y, v_z) = -v_x^2 - v_y^2 - v_z^2 + v_z v_w,$$

the fourth Lyapunov function is:

$$P_4(v_x, v_y, v_z, v_w) = P_3(v_x, v_y, v_z) + \frac{1}{2}v_w^2,$$

we obtain:

$$\dot{P}_4(v_x, v_y, v_z, v_w) = -v_x^2 - v_y^2 - v_z^2 - v_w^2 + v_w(v_z + v_w + \dot{v}_w), \quad (4.23)$$

we set:

$$\begin{aligned} -mv_w &= v_z + v_w + \dot{v}_w = e_x - 50e_z - 2e_y + 2450e_z - 50e_w - 2\dot{e}_y + 2450\dot{e}_z - 50\dot{e}_w, \\ &= e_x - 50e_z - 2e_y + 2450e_z - 50e_w - 2(-e_y + 50e_z) + 2450(-50e_z + e_w), \\ &-50(-e_w + \mu) = e_x - 120200e_z + 2450e_w - 50\mu. \end{aligned} \quad (4.24)$$

We define μ as:

$$\mu = 0.02e_x - 2404e_z + 49e_w + \frac{m}{50}v_w, \quad (4.25)$$

where $m > 0$ by substituting (4.24) into (4.23), we obtain:

$$\dot{P}_4(v_x, v_y, v_z, v_w) = -v_x^2 - v_y^2 - v_z^2 - (1 + m)v_w^2.$$

The derivative $\dot{P}_4$ is negatively definite. Consequently, the stability principle of Lyapunov theory guarantees the asymptotic stability of system (4.18) and the anti-synchronization of

systems (4.5) and (4.6). The control laws are then defined as follows:

$$\begin{cases} \mu_x &= -4\pi - e_x - e_y, \\ \mu_y &= -0.99e_y, \\ \mu_z &= 0.01e_z - 1510((1 + \delta \sin(x_2))z_2w_2 + (1 + \delta \sin(x_1))z_1w_1) + e_w, \\ \mu_w &= -0.99e_w + 1510((1 + \delta \sin(x_2))z_2w_2 + (1 + \delta \sin(x_1))z_1w_1) + 0.02e_x, \\ &\quad -2404e_z + 49e_w - \frac{m}{50}v_w - 0.02. \end{cases}$$

With $v_w = -2e_y + 2450e_z - 50e_w$,

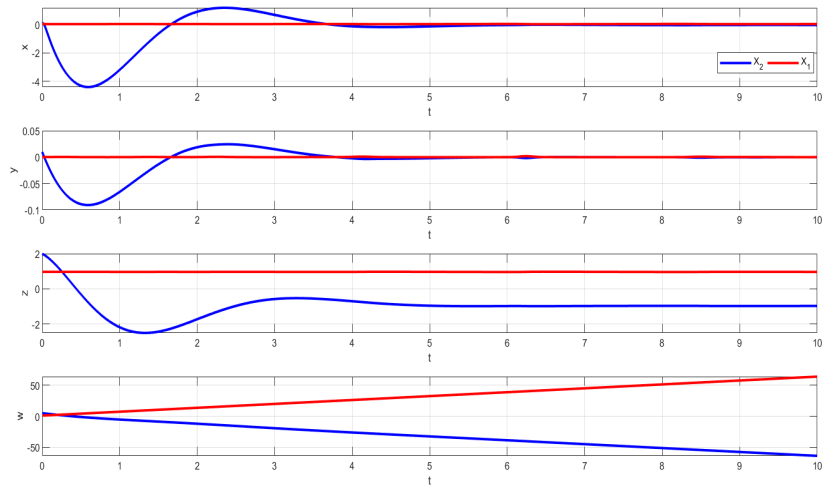


Figure 4.9: Anti-synchronization of system (4.5) and (4.6) using active backstepping control.

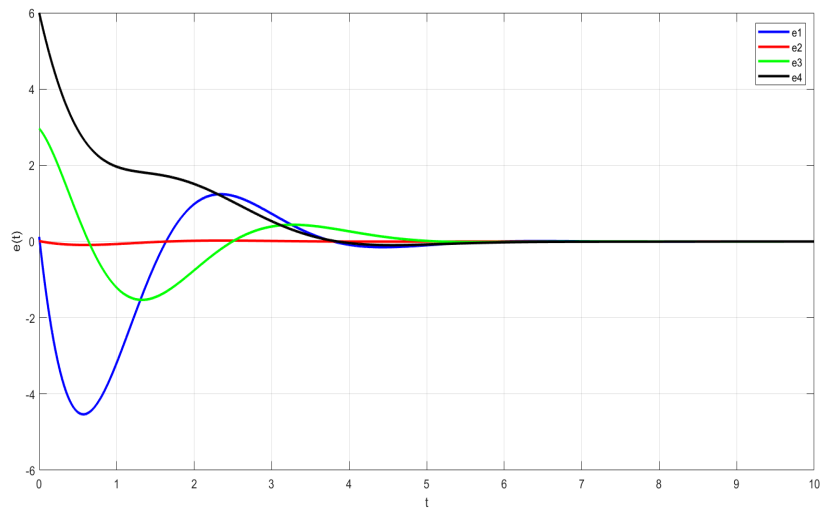


Figure 4.10: Time-history of anti-synchronization errors using active backstepping control

Adaptive control

Now our parameters are unknown, we rewrite systems (4.5) and (4.6) in the manner that ensues:

$$\begin{cases} \dot{x} = j(x) + J(x)\Lambda \\ \dot{y} = j(y) + J(y)\Lambda + \mu. \end{cases}$$

Where

$$\dot{x} = \begin{pmatrix} \dot{x}_1 & \dot{y}_1 & \dot{z}_1 & \dot{w}_1 \end{pmatrix}^T, \quad \dot{y} = \begin{pmatrix} \dot{x}_2 & \dot{y}_2 & \dot{z}_2 & \dot{w}_2 \end{pmatrix}^T,$$

and

$$j(x) = j(y) = \begin{pmatrix} 0 & 0 & 0 & 2\pi \end{pmatrix}^T,$$

$$J(x) = \begin{pmatrix} 1 & -x_1 & -(1 + \delta \sin(w_1))x_1y_1 & -\alpha \sin(w_1)x_1y_1 & 0 \\ 0 & -y_1 & (1 + \delta \sin(w_1))x_1y_1 & \alpha \sin(w_1)x_1y_1 & -y_1 \\ 0 & -z_1 & 0 & 0 & y_1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$J(y) = \begin{pmatrix} 1 & -x_2 & -(1 + \delta \sin(w_2))x_2y_2 & -\alpha \sin(w_2)x_2y_2 & 0 \\ 0 & -y_2 & (1 + \delta \sin(w_2))x_2y_2 & \alpha \sin(w_2)x_2y_2 & -y_2 \\ 0 & -z_2 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\Lambda = \begin{pmatrix} \gamma & \sigma & \alpha & \delta & \beta \end{pmatrix}^T; \mu = \begin{pmatrix} \mu_x & \mu_y & \mu_z & \mu_w \end{pmatrix}^T.$$

The anti-synchronization error between (4.29) and (4.30) is provided by:

$$\dot{e}_i = 2j(x) + (J(y) + J(x))\Lambda + \mu, \quad (4.26)$$

we define adaptative control as:

$$\mu = -2j(x) - (J(x) + J(y))\tilde{\Lambda} - pe, \quad (4.27)$$

where

$$\tilde{\Lambda} = \begin{pmatrix} \tilde{\gamma} & \tilde{\sigma} & \tilde{\alpha} & \tilde{\delta} & \tilde{\beta} \end{pmatrix}^T, e = \begin{pmatrix} e_x & e_y & e_z & e_w \end{pmatrix}^T,$$

and $p = \text{diag}(p_i) \in \mathbb{R}^{4 \times 4}, p_i > 0$. So

$$\begin{cases} \mu_x &= -2\tilde{\gamma}e_x + \tilde{\sigma}e_x + \tilde{\alpha}((1 + \delta \sin(w_2))x_2y_2 + (1 + \delta \sin(w_1))x_1y_1) + \alpha\tilde{\delta}(\sin(w_2)x_2y_2 \\ &\quad + \sin(w_1)x_1y_1) - p_1e_x, \\ \mu_y &= \tilde{\sigma}e_y - \tilde{\alpha}((1 + \delta \sin(w_2))x_2y_2 + (1 + \delta \sin(w_1))x_1y_1) - \alpha\tilde{\delta}(\sin(w_2)x_2y_2 \\ &\quad + \sin(w_1)x_1y_1) + \tilde{\beta}e_y - p_2e_y, \\ \mu_z &= \tilde{\sigma}e_z - \tilde{\beta}e_y - p_3e_z, \\ \mu_w &= -4\pi - p_4e_w. \end{cases}$$

In accordance with the parameter estimation adjustment laws:

$$\dot{\tilde{\Lambda}} = [J(y) + J(x)]^T e + \epsilon(\Lambda - \tilde{\Lambda}),$$

hence

$$\begin{cases} \dot{\tilde{\gamma}} &= 2e_x + \epsilon(\gamma - \tilde{\gamma}), \\ \dot{\tilde{\sigma}} &= -(e_x^2 + e_y^2 + e_z^2) + \epsilon(\sigma - \tilde{\sigma}), \\ \dot{\tilde{\alpha}} &= (e_y - e_x)((1 + \delta \sin(w_2))x_2y_2 + (1 + \delta \sin(w_1))x_1y_1) + \epsilon(\alpha - \tilde{\alpha}), \\ \dot{\tilde{\delta}} &= \delta(e_y - e_x)(\sin(w_2)x_2y_2 + \sin(w_1)x_1y_1) + \epsilon(\delta - \tilde{\delta}), \\ \dot{\tilde{\beta}} &= -e_y^2 + e_ye_z + \epsilon(\beta - \tilde{\beta}). \end{cases} \quad (4.28)$$

Where

$$\epsilon = \text{diag}(\eta_i), \eta_i > 0, i = 1 \dots 5,$$

The Lyapunov's function is taken in the following manner:

$$P(e) = e_x^2 + e_y^2 + e_z^2 + e_w^2 + e_\gamma^2 + e_\sigma^2 + e_\alpha^2 + e_\delta^2 + e_\beta^2,$$

given the parameters chosen (4.28), the function $\dot{P}$ has a negative definite. Thus, in accordance with Lyapunov stability theory, the error dynamics (4.26) exhibit global asymptotic stability.

4.4.3 Synchronization of SIR model using multiple types of control

In this section, we study the synchronization behavior between two identical chaotic SIR models under various control strategies. Synchronization refers to the process by which the corresponding state variables of the drive and response systems evolve to become identical over time. To achieve this goal, several control techniques are implemented and compared,

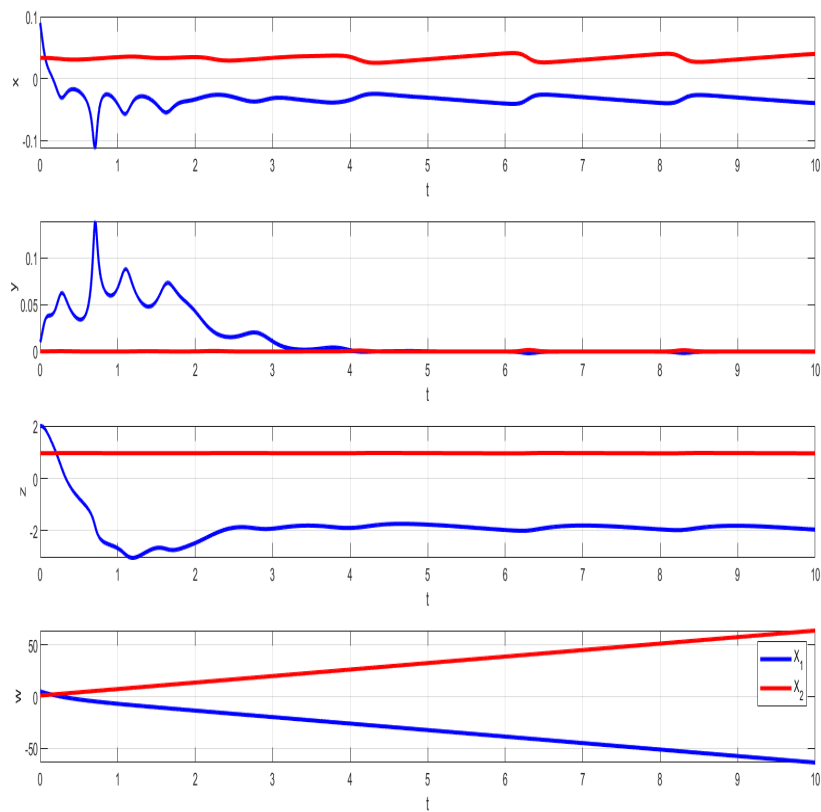


Figure 4.11: Anti-synchronization of system (4.5) and (4.6) using adaptive control.

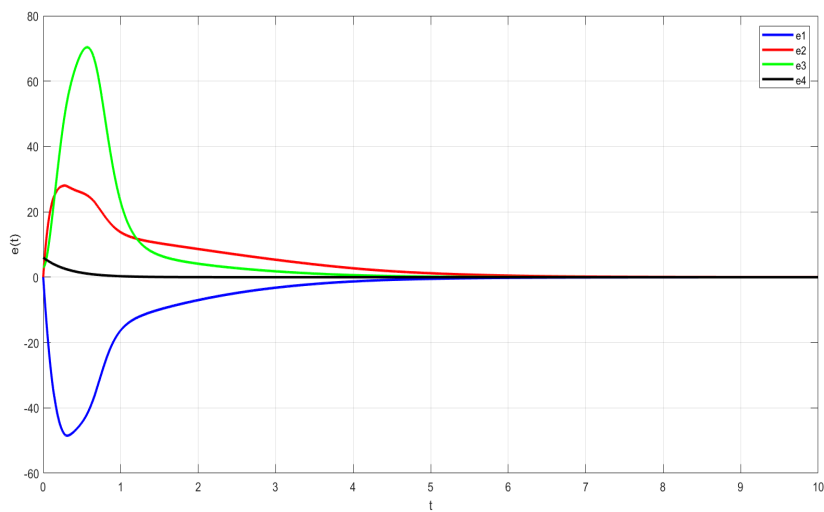


Figure 4.12: Time-history of anti-synchronization errors using adaptive control

namely active control, sliding mode control, backstepping control, and adaptive control. Each method is designed to ensure that the synchronization errors asymptotically converge

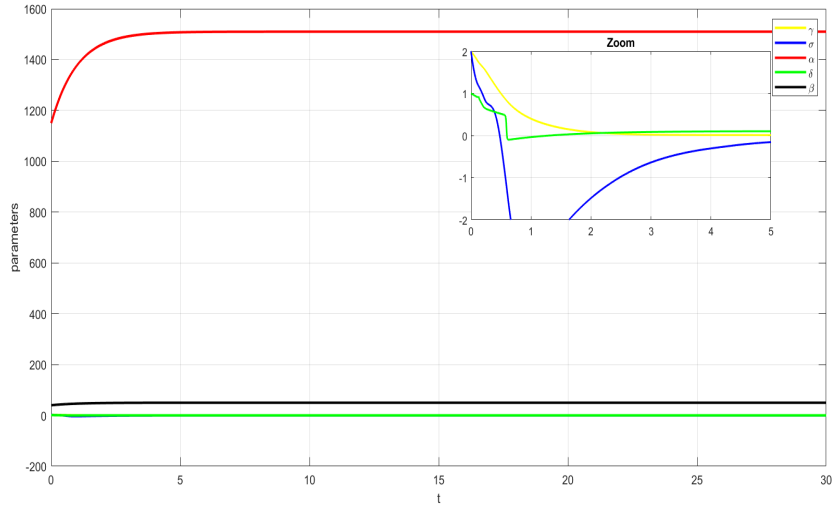


Figure 4.13: Parameters estimation in anti-synchronization

to zero, thereby demonstrating the effectiveness and robustness of the proposed approaches in managing the nonlinear and chaotic dynamics of the epidemic model.

Active control

The master and slave systems involved in the active control scheme are described as follows:

$$\begin{cases} \dot{x}_1 = \gamma - \alpha_0(1 + \delta \sin(w_1)x_1y_1 - \sigma x_1, \\ \dot{y}_1 = \alpha_0(1 + \delta \sin(w_1))x_1y_1 - (\sigma + \beta)y_1, \\ \dot{z}_1 = \beta y_1 - \sigma z_1, \\ \dot{w}_1 = \frac{2\pi}{T}. \end{cases} \quad (4.29)$$

$$\begin{cases} \dot{x}_2 = \gamma - \alpha_0(1 + \delta \sin(w_2)x_2y_2 - \sigma x_2 + \mu_x, \\ \dot{y}_2 = \alpha_0(1 + \delta \sin(w_2))x_2y_2 - (\sigma + \beta)y_2 + \mu_y, \\ \dot{z}_2 = \beta y_2 - \sigma z_2 + \mu_z, \\ \dot{w}_2 = \frac{2\pi}{T} + \mu_w. \end{cases} \quad (4.30)$$

Where μ_x, μ_y, μ_z , and μ_w stand for the active control laws that may be discovered. The error synchronization between systems (4.29) and (4.30) is defined by (4.31).

$$\begin{cases} e_x = x_2 - x_1, \\ e_y = y_2 - y_1, \\ e_z = z_2 - z_1, \\ e_w = w_2 - w_1, \end{cases}$$

Which implies

$$\begin{cases} \dot{e}_x &= -\sigma e_x - \alpha_0((1 + \delta \sin(w_2))x_2 y_2 - (1 + \delta \sin(w_1))x_1 y_1) + \mu_x, \\ \dot{e}_y &= -(\sigma + \beta)e_y + \alpha_0((1 + \delta \sin(w_2))x_2 y_2 - (1 + \delta \sin(w_1))x_1 y_1) + \mu_y, \\ \dot{e}_z &= \beta e_y - \sigma e_z + \mu_z, \\ \dot{e}_w &= \mu_w. \end{cases} \quad (4.31)$$

Adopting the same design framework used previously, the active control laws are derived for μ_x , μ_y , μ_z , and μ_w to achieve global asymptotic synchronization. The explicit expressions of these control inputs are given as follows:

$$\begin{cases} \mu_x &= \alpha_0((1 + \delta \sin(w_2))x_2 y_2 - (1 + \delta \sin(w_1))x_1 y_1) + v_x \\ \mu_y &= -\alpha_0((1 + \delta \sin(w_2))x_2 y_2 - (1 + \delta \sin(w_1))x_1 y_1) + v_y \\ \mu_z &= -\beta e_y + v_z \\ \mu_w &= -e_w + v_w. \end{cases} \quad (4.32)$$

After substituting these expressions into (4.31), the error dynamics simplify to the linear form:

$$\begin{cases} \dot{e}_x &= -\sigma e_x + v_x, \\ \dot{e}_y &= -(\sigma + \beta)e_y + v_y, \\ \dot{e}_z &= -\sigma e_z + v_z, \\ \dot{e}_w &= -e_w + v_w. \end{cases}$$

This system can be expressed in matrix form as:

$$[\dot{e}_x \ \dot{e}_y \ \dot{e}_z \ \dot{e}_w]^T = D[e_x \ e_y \ e_z \ e_w]^T,$$

and

$$[v_x \ v_y \ v_z \ v_w]^T = F[e_x \ e_y \ e_z \ e_w]^T,$$

where D is a constant matrix, and F is selected such that the error system remains asymptotically stable. The general form of F is:

$$F = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}.$$

Substituting F into the error dynamics gives:

$$\begin{pmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_z \\ \dot{e}_w \end{pmatrix} = \begin{pmatrix} -\sigma + a_1 & b_1 & c_1 & d_1 \\ a_2 & -(\beta + \sigma) + b_2 & c_2 & d_2 \\ a_3 & b_3 & -\sigma + c_3 & d_3 \\ a_4 & b_4 & c_4 & -1 + d_4 \end{pmatrix} \begin{pmatrix} e_x \\ e_y \\ e_z \\ e_w \end{pmatrix}. \quad (4.33)$$

Among the possible choices satisfying the stability conditions (e.g., Routh–Hurwitz criterion), one suitable example of F is given by:

$$F = \begin{pmatrix} -0.99 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.99 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, since the resulting system (4.33) is asymptotically stable, the master and slave SIR systems achieve complete synchronization under the designed active control law.

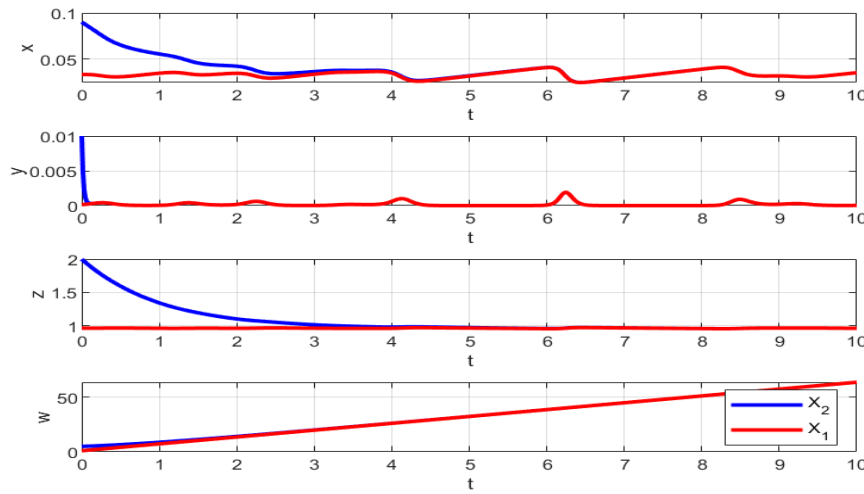


Figure 4.14: Synchronization of system (4.29) and (4.30) using active control

Sliding mode control

Since the ISMC method was already introduced, this section focuses only on its application to synchronization. Suitable sliding surfaces are chosen to stabilize the error dynamics and

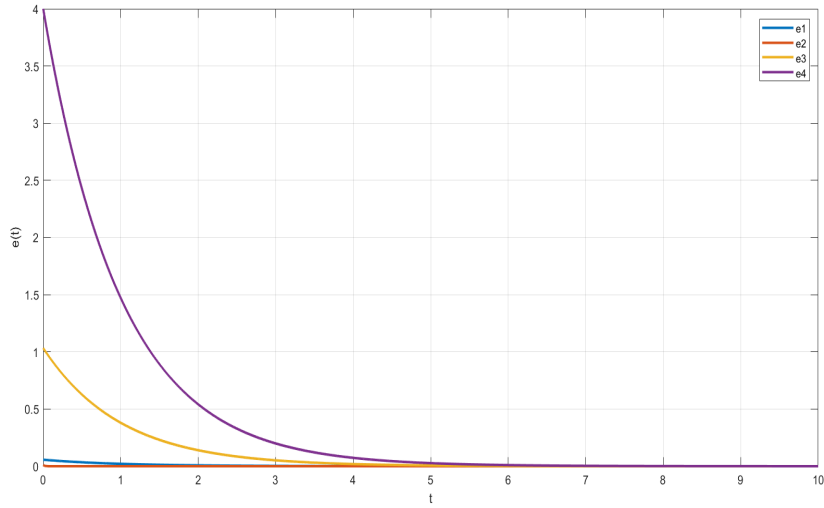


Figure 4.15: Time-history of synchronization errors using active control

drive the synchronization errors (4.34) to zero.

$$\begin{cases} \dot{e}_x = -\sigma e_x - \alpha_0((1 + \delta \sin(w_2))x_2 y_2 - (1 + \delta \sin(w_1))x_1 y_1) + \mu_x, \\ \dot{e}_y = -(\sigma + \beta)e_y + \alpha_0((1 + \delta \sin(w_2))x_2 y_2 - (1 + \delta \sin(w_1))x_1 y_1) + \mu_y, \\ \dot{e}_z = \beta e_y - \sigma e_z + \mu_z, \\ \dot{e}_w = \mu_w. \end{cases} \quad (4.34)$$

Assume the following for the sliding surfaces:

$$\begin{cases} s_x = e_x + h_1 \int_0^t e_x d\tau, \\ s_y = e_y + h_2 \int_0^t e_y d\tau, \\ s_z = e_z + h_3 \int_0^t e_z d\tau, \\ s_w = e_w + h_4 \int_0^t e_w d\tau. \end{cases}$$

Where the variables $h_i, i = 1, 2, 3, 4$ are all positive. So, we get:

$$\begin{cases} \dot{s}_x = \dot{e}_x + h_1 e_x, \\ \dot{s}_y = \dot{e}_y + h_2 e_y, \\ \dot{s}_z = \dot{e}_z + h_3 e_z, \\ \dot{s}_w = \dot{e}_w + h_4 e_w, \end{cases} \quad (4.35)$$

Next, we define the integral sliding mode controls by:

$$\left\{ \begin{array}{l} \mu_x = \alpha_0((1 + \delta \sin(w_2))x_2y_2 - (1 + \delta \sin(w_1))x_1y_1) + (\sigma - h_1)e_x \\ \quad - p.sign(s_x), \\ \mu_y = \alpha_0((1 + \delta \sin(w_2))x_2y_2 - (1 + \delta \sin(w_1))x_1y_1) + (\sigma + \beta - h_2)e_y \\ \quad - p.sign(s_y), \\ \mu_z = -\beta e_y + (\sigma - h_3)e_z - p.sign(s_z), \\ \mu_w = -h_4e_w - p.sign(s_w). \end{array} \right. \quad (4.36)$$

We replace the control laws defined in (4.36) in (4.34), we obtain:

$$\left\{ \begin{array}{l} \dot{e}_x = -h_1e_x - p.sign(s_x), \\ \dot{e}_y = -h_2e_y - p.sign(s_y), \\ \dot{e}_z = -h_3e_z - p.sign(s_z), \\ \dot{e}_w = -h_4e_w - p.sign(s_w). \end{array} \right.$$

Let the following Lyapunov function be:

$$P(s) = \frac{1}{2}(s_x^2 + s_y^2 + s_z^2 + s_w^2),$$

which implies that:

$$\dot{P} = s_x\dot{s}_x + s_y\dot{s}_y + s_z\dot{s}_z + s_w\dot{s}_w. \quad (4.37)$$

Substituting (4.35) into (4.37), we find:

$$\dot{P} = s_x(\dot{e}_x + h_1e_x) + s_y(\dot{e}_y + h_2e_y) + s_z(\dot{e}_z + h_3e_z) + s_w(\dot{e}_w + h_4e_w),$$

which implies that:

$$\begin{aligned} \dot{P} &= s_x(-p.sign(s_x)) + s_y(-p.sign(s_y)) + s_z(-p.sign(s_z)) + s_w(-p.sign(s_w)), \\ \dot{P} &\leq -p|s_x| - p|s_y| - p|s_z| - p|s_w|. \end{aligned}$$

Taking $p > 0$ ensures that $\dot{P}$ is negative definite. Hence, depending on the Lyapunov stability theorem, the sliding surface is guaranteed to converge to zero. As a result, it drives the error systems to be asymptotically stable and converge to zero.

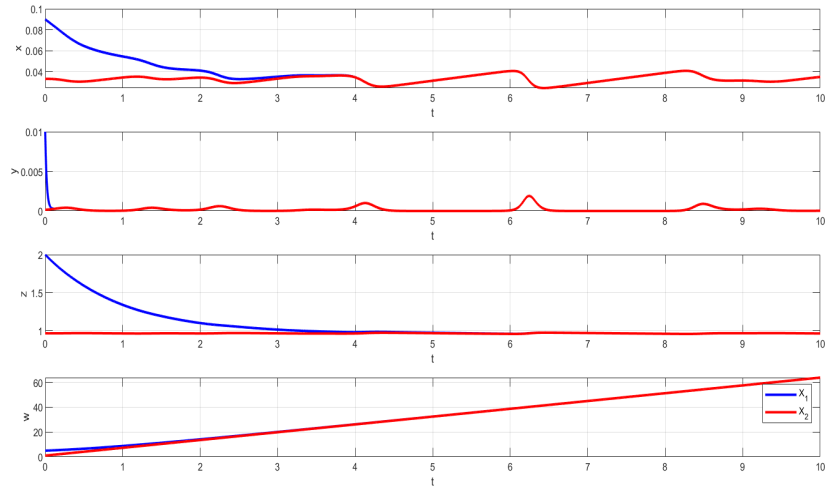


Figure 4.16: Synchronization of system (4.29) and (4.30) using sliding mode control.

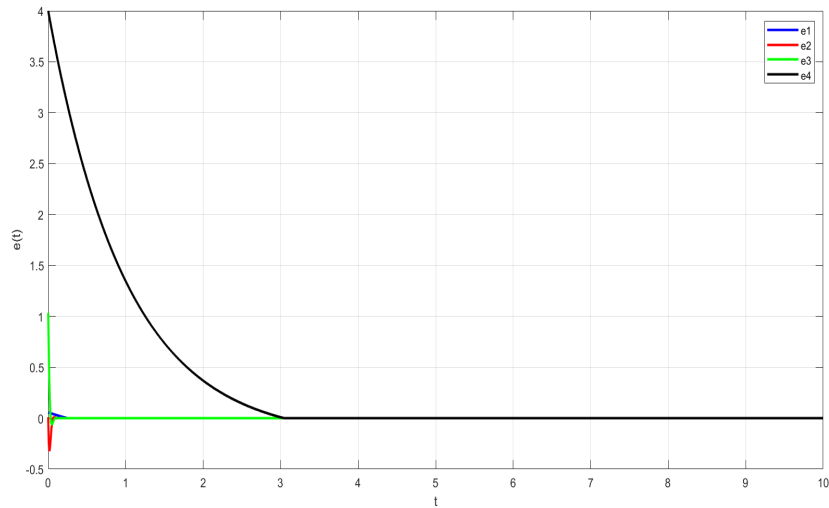


Figure 4.17: Time-history of synchronization errors using sliding mode control

Active backstepping control

Parameters $\mu, \sigma, \beta, \alpha, T$ are replaced by their values presented in (4.3) in systems (4.29) and (4.30). Applying a rearrangement of variables:

$$\begin{cases} x = w \\ y = z \\ z = y \\ w = x \end{cases} \Rightarrow \begin{cases} \dot{x} = \dot{w} \\ \dot{y} = \dot{z} \\ \dot{z} = \dot{y} \\ \dot{w} = \dot{x} \end{cases} \quad (4.38)$$

Thus, system (4.29) can be defined as:

$$\begin{cases} \dot{x}_1 &= 2\pi \\ \dot{y}_1 &= -0.01y_1 + 50z_1 \\ \dot{z}_1 &= -50.01z_1 + 1510(1 + \delta \sin(x_1))z_1w_1 \\ \dot{w}_1 &= 0.01 - 0.01w_1 - 1510(1 + \delta \sin(x_1))z_1w_1. \end{cases} \quad (4.39)$$

So system (4.30) becomes:

$$\begin{cases} \dot{x}_2 &= 2\pi + \mu_x \\ \dot{y}_2 &= -0.01y_2 + 50z_2 + \mu_y \\ \dot{z}_2 &= -50.01z_2 + 1510(1 + \delta \sin(x_2))z_2w_2 + \mu_z \\ \dot{w}_2 &= 0.01 - 0.01w_2 - 1510(1 + \delta \sin(x_2))z_2w_2 + \mu_w. \end{cases} \quad (4.40)$$

Where $\mu_x, \mu_y, \mu_z, \mu_w$ are the feedback control laws need to be found. The error system is:

$$\begin{cases} \dot{e}_x &= \mu_x \\ \dot{e}_y &= -0.01e_y + 50e_z + \mu_y \\ \dot{e}_z &= -50.01e_z + 1510((1 + \delta \sin(x_2))z_2w_2 - (1 + \delta \sin(x_1))z_1w_1) + \mu_z \\ \dot{e}_w &= -0.01e_w - 1510((1 + \delta \sin(x_2))z_2w_2 - (1 + \delta \sin(x_1))z_1w_1) + \mu_w. \end{cases} \quad (4.41)$$

First, feedback control restructures system (4.41) to facilitate the use of backstepping control.

The process starts with an examination of the feedback control law:

$$\begin{cases} \mu_x &= -e_x - e_y \\ \mu_y &= -0.99e_y \\ \mu_z &= 0.01e_z - 1510((1 + \delta \sin(x_2))z_2w_2 - (1 + \delta \sin(x_1))z_1w_1) + e_w \\ \mu_w &= -0.99e_w + 1510((1 + \delta \sin(x_2))z_2w_2 - (1 + \delta \sin(x_1))z_1w_1) + \mu. \end{cases} \quad (4.42)$$

Here, μ denotes the control command to be determined for feedback purposes. Replacing (4.42) in (4.31) leads to the updated system in the following form:

$$\begin{cases} \dot{e}_x &= -e_x - e_y \\ \dot{e}_y &= -e_y + 50e_z \\ \dot{e}_z &= -50e_z + e_w \\ \dot{e}_w &= -e_w + u. \end{cases}$$

We take the first Lyapunov function:

$$P_1(v_x) = \frac{1}{2}v_x^2,$$

where $v_x = e_x$, we obtain

$$\dot{P}_1(v_x) = v_x \dot{v}_x = v_x(-e_x - e_y) = -v_x^2 - v_x e_y, \quad (4.43)$$

next, we define

$$v_y = -e_y, \quad (4.44)$$

using (4.44), we can express (4.43) as

$$\dot{P}_1(v_x) = -v_x^2 + v_x v_y,$$

we define the second Lyapunov function:

$$P_2(v_x, v_y) = P_1(v_x) + \frac{1}{2}v_y^2,$$

we obtain

$$\dot{P}_2(v_x, v_y) = -v_x^2 + v_x v_y + v_y \dot{v}_y = -v_x^2 - v_y^2 + v_y(v_x + v_y + \dot{v}_y), \quad (4.45)$$

we define

$$v_z = v_x + v_y + \dot{v}_y = e_x - 50e_z,$$

thus, (4.45) becomes

$$\dot{P}_2(v_x, v_y) = -v_x^2 - v_y^2 + v_y v_z,$$

We define the third Lyapunov function:

$$P_3(v_x, v_y, v_z) = P_2(v_x, v_y) + \frac{1}{2}v_z^2,$$

we get

$$\dot{P}_3(v_x, v_y, v_z) = -v_x^2 - v_y^2 + v_y v_z + v_z \dot{v}_z = -v_x^2 - v_y^2 - v_z^2 + v_z(v_y + v_z + \dot{v}_z), \quad (4.46)$$

we define

$$\begin{aligned} v_w &= v_y + v_z + \dot{v}_z = -e_y + e_x - 50e_z + \dot{e}_x - 50\dot{e}_z = -e_y + e_x - 50e_z - e_x - e_y \\ &\quad + 2500e_z - 50e_w = -2e_y + 2450e_z - 50e_w. \end{aligned}$$

Hence, (4.46) becomes

$$\dot{P}_3(v_x, v_y, v_z) = -v_x^2 - v_y^2 - v_z^2 + v_z v_w,$$

we define the fourth Lyapunov function:

$$P_4(v_x, v_y, v_z, v_w) = P_3(v_x, v_y, v_z) + \frac{1}{2}v_w^2,$$

we obtain

$$\dot{P}_4(v_x, v_y, v_z, v_w) = -v_x^2 - v_y^2 - v_z^2 - v_w^2 + v_w(v_z + v_w + \dot{v}_w), \quad (4.47)$$

we set

$$\begin{aligned} m &= v_z + v_w + \dot{v}_w = e_x - 50e_z - 2e_y + 2450e_z - 50e_w - 2\dot{e}_y + 2450\dot{e}_z - 50\dot{e}_w \\ &= e_x - 50e_z - 2e_y + 2450e_z - 50e_w - 2(-e_y + 50e_z) + 2450(-50e_z + e_w) \\ &\quad - 50(-e_w + \mu) = e_x - 120200e_z + 2450e_w - 50\mu. \end{aligned} \quad (4.48)$$

We define μ as:

$$\mu = 0.02e_x - 2404e_z + 49e_w + lv_w. \quad (4.49)$$

By considering $l > 0$, we get $m = -50lv_w$. Therefore, equation (4.47) can be simplified by substituting (4.49) into (4.48) to:

$$\dot{P}_4(v_x, v_y, v_z, v_w) = -v_x^2 - v_y^2 - v_z^2 - (1 - 50l)v_w^2. \quad (4.50)$$

The obtained quadratic form is negative definite, which, by Lyapunov's direct method, ensures the global and exponential stability of system (1.6). Consequently, the control laws derived under the active backstepping framework are given as follows:

$$\left\{ \begin{array}{l} \mu_x = -e_x - e_y \\ \mu_y = -0.99e_y \\ \mu_z = 0.01e_z - 1510((1 + \delta \sin(x_2))z_2w_2 - (1 + \delta \sin(x_1))z_1w_1) + e_w \\ \mu_w = -0.99e_w + 1510((1 + \delta \sin(x_2))z_2w_2 - (1 + \delta \sin(x_1))z_1w_1) + 0.02e_x \\ \quad - 2404e_z + 49e_w - 50lv_w. \end{array} \right. \quad (4.51)$$

With $v_w = -2e_y + 2450e_z - 50e_w$,

Adaptive Control

The following is the method by which we rebuild systems (4.29) and (4.30) using undetermined parameters::

$$\left\{ \begin{array}{l} \dot{x} = j(x) + J(x)\Lambda \\ \dot{y} = j(y) + J(y)\Lambda + \mu. \end{array} \right. \quad (4.52)$$

Where

$$\dot{x} = \begin{pmatrix} \dot{x}_1 & \dot{y}_1 & \dot{z}_1 & \dot{w}_1 \end{pmatrix}^T \quad (4.53)$$

$$\dot{y} = \begin{pmatrix} \dot{x}_2 & \dot{y}_2 & \dot{z}_2 & \dot{w}_2 \end{pmatrix}^T, \quad (4.54)$$

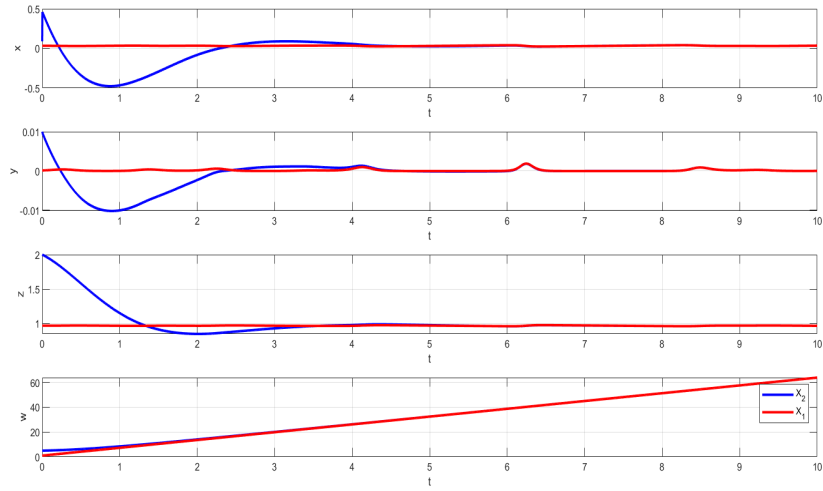


Figure 4.18: Synchronization of system (4.29) and (4.30) using active backstepping control.

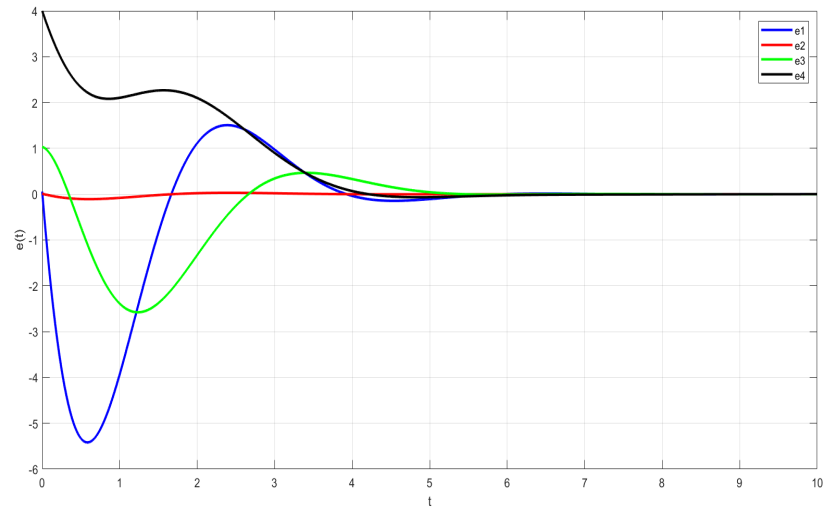


Figure 4.19: Time-history of synchronization errors using active backstepping control

and

$$j(x) = j(y) = \begin{pmatrix} 0 & 0 & 0 & 2\pi \end{pmatrix}^T \quad (4.55)$$

$$J(x) = \begin{pmatrix} 1 & -x_1 & -(1 + \delta \sin(w_1))x_1y_1 & -\alpha \sin(w_1)x_1y_1 & 0 \\ 0 & -y_1 & (1 + \delta \sin(w_1))x_1y_1 & \alpha \sin(w_1)x_1y_1 & -y_1 \\ 0 & -z_1 & 0 & 0 & y_1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.56)$$

$$J(y) = \begin{pmatrix} 1 & -x_2 & -(1 + \delta \sin(w_2))x_2y_2 & -\alpha \sin(w_2)x_2y_2 & 0 \\ 0 & -y_2 & (1 + \delta \sin(w_2))x_2y_2 & \alpha \sin(w_2)x_2y_2 & -y_2 \\ 0 & -z_2 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.57)$$

$$\Lambda = \begin{pmatrix} \gamma & \sigma & \alpha & \delta & \beta \end{pmatrix}^T; \mu = \begin{pmatrix} \mu_x & \mu_y & \mu_z & \mu_w \end{pmatrix}^T. \quad (4.58)$$

The error between systems (4.29) and (4.30) is expressed by:

$$\dot{e}_i = (J(y) - J(x))\Lambda + \mu, \quad (4.59)$$

we define adaptative control as:

$$\mu = (J(x) - J(y))\tilde{\Lambda} - pe, \quad (4.60)$$

where

$$\tilde{\Lambda} = \begin{pmatrix} \tilde{\gamma} & \tilde{\sigma} & \tilde{\alpha} & \tilde{\delta} & \tilde{\beta} \end{pmatrix}^T, e = \begin{pmatrix} e_x & e_y & e_z & e_w \end{pmatrix}^T, \quad (4.61)$$

and $p = \text{diag}(p_i) \in \mathbb{R}^{4 \times 4}, p_i > 0$. So

$$\begin{cases} \mu_x &= \tilde{\sigma}e_x + \tilde{\alpha}((1 + \delta \sin(w_2))x_2y_2 - (1 + \delta \sin(w_1))x_1y_1) + \alpha\tilde{\delta}(\sin(w_2)x_2y_2 \\ &\quad - \sin(w_1)x_1y_1) - p_1e_x \\ \mu_y &= \tilde{\sigma}e_y - \tilde{\alpha}((1 + \delta \sin(w_2))x_2y_2 - (1 + \delta \sin(w_1))x_1y_1) + \alpha\tilde{\delta}(\sin(w_2)x_2y_2 \\ &\quad - \sin(w_1)x_1y_1) + \tilde{\beta}e_y - p_2e_y \\ \mu_z &= \tilde{\sigma}e_z - \tilde{\beta}e_y - p_3e_z \\ \mu_w &= -p_4e_w \end{cases} \quad (4.62)$$

Based on the parameter estimate adjustment laws outlined below:

$$\dot{\tilde{\Lambda}} = [J(y) - J(x)]^T e + \epsilon(\Lambda - \tilde{\Lambda}),$$

hence

$$\begin{cases} \dot{\tilde{\gamma}} &= \epsilon(\gamma - \tilde{\gamma}) \\ \dot{\tilde{\sigma}} &= -(e_x^2 + e_y^2 + e_z^2) + \epsilon(\sigma - \tilde{\sigma}) \\ \dot{\tilde{\alpha}} &= (e_y - e_x)((1 + \delta \sin(w_2))x_2y_2(1 + \delta \sin(w_1))x_1y_1) + \epsilon(\alpha - \tilde{\alpha}) \\ \dot{\tilde{\delta}} &= \delta(e_y - e_x)(\sin(w_2)x_2y_2 - \sin(w_1)x_1y_1) + \epsilon(\delta - \tilde{\delta}) \\ \dot{\tilde{\beta}} &= -e_y^2 + e_ye_z + \epsilon(\beta - \tilde{\beta}) \end{cases} \quad (4.63)$$

Where

$$\epsilon = \text{diag}(\epsilon_i), \epsilon_i > 0, i = 1 \dots 5,$$

The Lyapunov function is taken as:

$$P(e) = e_x^2 + e_y^2 + e_z^2 + e_w^2 + e_\gamma^2 + e_\sigma^2 + e_\alpha^2 + e_\delta^2 + e_\beta^2,$$

when we take parameters as shown in(4.63), $\dot{P}$ is a function that is negative definite. Consequently, the error dynamics (4.59) exhibit global asymptotic stability in accordance with Lyapunov stability theory.

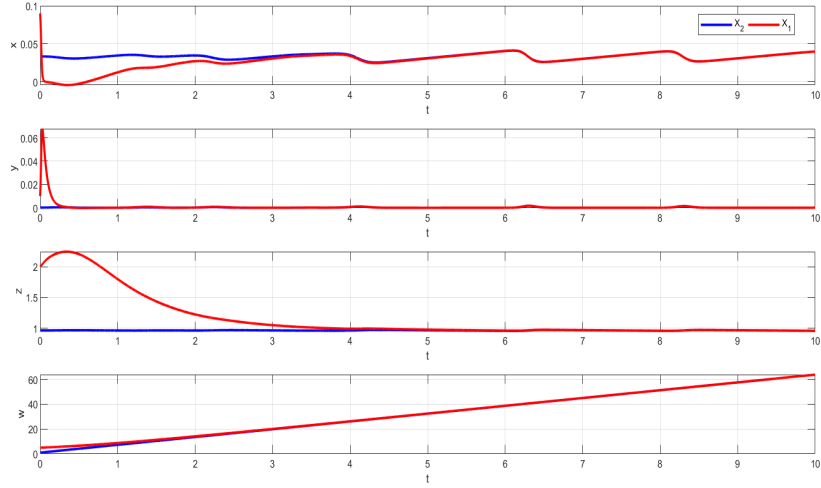


Figure 4.20: Synchronization of system (4.29) and (4.30) using adaptive control

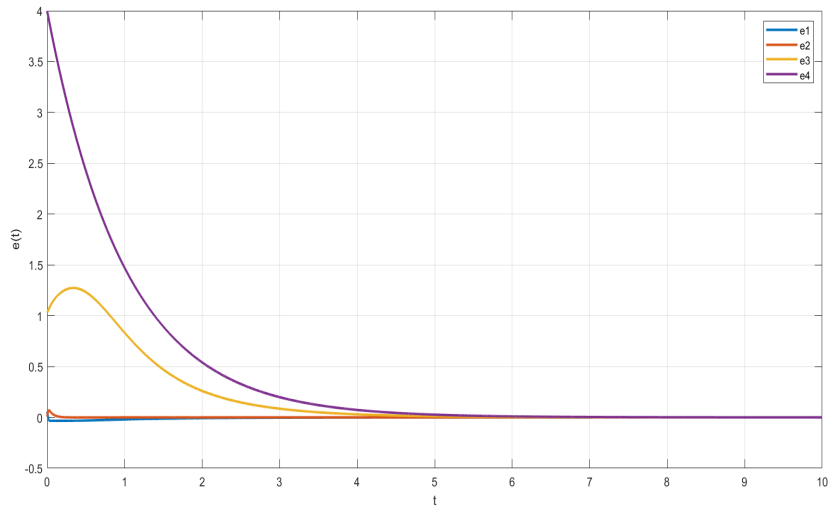


Figure 4.21: Time-history of synchronization errors using adaptive control

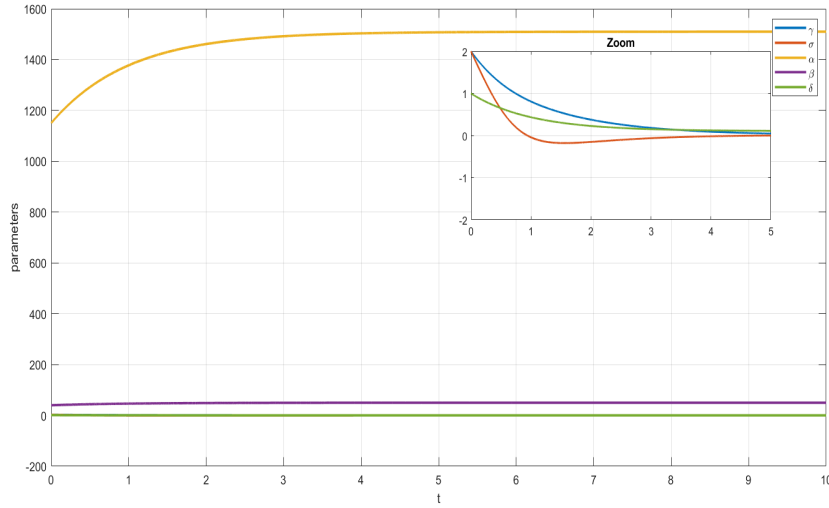


Figure 4.22: Parameters estimation in synchronization

4.4.4 A Comparison of synchronization control methods

Derived from the work of [66]. The efficiency of the Ac (active control), Sc (sliding mode control), Bc (active backstepping control), and Adc (adaptive control) techniques is compared in order to synchronize the forced SIR system. The integral square error IS_e and the control energy C_e are considered for comparison. This is due to the fact that the relationship between error dynamics and control input can differ significantly dependent on the specific system being analyzed. Consequently, it is crucial to meticulously evaluate all variables prior to formulating the most effective control strategy. Therefore, the performance index $I_d = h(IS_e, C_e)$ is expressed as a function of IS_e and C_e , which are defined as follows: $IS_e = \int_0^\infty e^2(t)dt$, $C_e = \int_{-\infty}^{+\infty} |\mu|^2 dt$, where e is the error vecteur and μ is the input for control. Three cases are taken, these are:

Case 1: IS_e and C_e possess equivalent weight.

$$I_d = h(IS_e, C_e) = \sum_{k=1}^n \left(\frac{1}{2} IS_{ek} + \frac{1}{2} C_{ek} \right).$$

Case 2: IS_e and C_e possess weights of 40% and 60%, respectively.

$$I_d = h(IS_e, C_e) = \sum_{k=1}^n \left(\frac{2}{5} IS_{ek} + \frac{3}{5} C_{ek} \right).$$

Case 3: IS_e possesses a 60% weight and C_e possesses a 40% weight.

$$I_d = h(IS_e, C_e) = \sum_{k=1}^n \left(\frac{3}{5} IS_{ek} + \frac{2}{5} C_{ek} \right),$$

Table 4.1: Performance indices compared for the first case

Initial conditions	Ac	Sc	Bc	Adc
0.0345 0.0002 1 5	4	8.05	15.75	8
0.0445 0.0102 1.01 5.01	4.02	8.17	1.2×10^3	8.66
0.0645 0.0302 1.03 5.03	4.09	8.72	8.9×10^3	12.82
0.0845 0.0502 1.05 5.05	4.26	10.25	2.5×10^4	19.84
0.1045 0.0702 1.07 5.07	4.62	14	5.7×10^4	32.26
0.1245 0.0902 1.09 5.09	5.31	22.3	1.2×10^5	54.76
0.1445 0.1102 1.11 5.11	6.50	39.69	2.2×10^5	92.1
0.1645 0.1302 1.13 5.13	8.4	75.16	3.9×10^5	194.25
0.1845 0.1502 1.15 5.15	11.24	142.12	6.5×10^5	375.82
0.2045 0.1702 1.17 5.17	15.32	304.79	10^6	545.6

Table 4.2: Performance indices compared second case

Initial conditions	Ac	Sc	Bc	Adc
0.0345 0.0002 1 5	3.20	8.07	17.31	8
0.0445 0.0102 1.01 5.01	3.22	8.20	1.4×10^3	8.79
0.0645 0.0302 1.03 5.03	3.29	8.85	1.1×10^4	13.76
0.0845 0.0502 1.05 5.05	3.47	10.67	3×10^4	22.16
0.1045 0.0702 1.07 5.07	3.89	15.15	6.8×10^4	37.05
0.1245 0.0902 1.09 5.09	4.70	25.1	1.4×10^5	64.03
0.1445 0.1102 1.11 5.11	6.11	45.94	2.6×10^5	108.82
0.1645 0.1302 1.13 5.13	8.37	88.44	4.6×10^5	231.38
0.1845 0.1502 1.15 5.15	11.76	176.03	7.8×10^5	427.66
0.2045 0.1702 1.17 5.17	16.64	364	1.2×10^6	652.98

where (IS_{ek}) and (C_{ek}) are calculated for $k = 1, \dots, 4$.

Discussion of Results

This section shows a comparative discussion of the performance of the four control techniques applied to synchronize the forced SIR system. The comparison is based on a performance index I_d , defined as a weighted function of the integral square error IS_e and the control

Table 4.3: Performance indices compared third case

Initial conditions	Ac	Sc	Bc	Adc
0.0345 0.0002 1 5	4.80	8.03	14.18	8
0.0445 0.0102 1.01 5.01	4.83	8.13	928.48	8.54
0.0645 0.0302 1.03 5.03	4.89	8.59	7049.7	11.88
0.0845 0.0502 1.05 5.05	5	9.83	2×10^4	17.51
0.1045 0.0702 1.07 5.07	5.22	12.85	4.5×10^4	27.47
0.1245 0.0902 1.09 5.09	5.66	19.51	9.2×10^4	45.48
0.1445 0.1102 1.11 5.11	6.89	33.43	1.7×10^5	75.37
0.1645 0.1302 1.13 5.13	8.43	61.83	3.1×10^5	157.11
0.1845 0.1502 1.15 5.15	10.72	120.21	5.2×10^5	287.99
0.2045 0.1702 1.17 5.17	14	245.56	8.3×10^5	438.23

energy C_e .

Across all weighting scenarios and initial conditions, the active control (Ac) method consistently outperforms the other three techniques. The active backstepping control(Bc) method yields the worst performance, with the performance index I_d escalating drastically, indicating instability or impracticality in synchronization. Sc and Adc demonstrate intermediate performance but remain less effective than Ac.

- Case 1: Equal weighting (50% IS_e , 50% C_e) In this balanced scenario, Ac exhibits the best compromise between synchronization accuracy and control effort. Its performance index values increase gradually with initial condition perturbations, reflecting both robustness and control efficiency.

Sc also shows moderate performance but suffers from high control energy, especially for larger deviations. Adc follows a similar trend but with higher cost. Bc is the least effective, with the performance index rapidly increasing to impractically high values.

- Case 2: Emphasis on control energy (40% IS_e , 60% C_e)

When more importance is assigned to minimizing control energy, Ac becomes even more favorable. Its energy-efficient nature leads to significantly lower I_d values.

Sc, known for high control effort, is heavily penalized under this criterion. Adc shows

moderate improvement, while Bc remains the least efficient, again showing explosive growth in the performance index, indicating infeasibility.

- Case 3: Emphasis on precision (60% IS_e , 40% C_e)

When prioritizing synchronization precision, Ac continues to deliver optimal results. Although Sc and AdC benefit slightly from the increased weight on accuracy, they remain inferior to Ac in all scenarios.

Once more, Bc fails to yield acceptable results, reinforcing its unsuitability for this synchronization problem.

Table 4.4: Summary comparison of control methods

Method	Robustness	Energy Efficiency	Accuracy	Overall Suitability
Ac	Excellent	Excellent	Excellent	Optimal choice
Sc	Good	Low	Moderate	Acceptable
Adc	Moderate	Moderate	Moderate	Moderate
Bc	Very Low	Very Poor	Poor	Not recommended

Overall, the active Control method offers the most balanced, efficient, and robust performance across all conditions. Therefore, it represents the most suitable technique for synchronizing the forced SIR model among those considered in this study.

4.5 Conclusion

This chapter has explored the fundamental concepts of synchronization and its diverse types, highlighting their significance in understanding complex dynamical systems. By applying anti-synchronization, and synchronization to the SIR model, we demonstrated how various control techniques can be employed to manage and predict the behavior of epidemic dynamics. The comparative analysis of these control techniques provided valuable insights into their effectiveness, efficiency, and applicability in different scenarios. This study not only underscores the importance of synchronization in epidemiological modeling but also lays the groundwork for future research aimed at optimizing control strategies for real-world public health challenges.

Conclusion

In this thesis, we investigated the control and synchronization of chaotic dynamical systems, with a particular focus on their application to epidemiological systems exhibiting chaotic behavior. The main objective was to analyze and compare different nonlinear control strategies in order to regulate chaotic dynamics and achieve reliable synchronization between coupled systems.

The results obtained throughout this work demonstrate that the considered control techniques are effective in suppressing chaotic behavior and ensuring synchronization under various conditions. Numerical simulations confirm the convergence of system trajectories toward desired states as well as the asymptotic decay of synchronization errors, thereby validating the theoretical analysis.

A detailed comparative study was carried out to evaluate the performance of the investigated control methods in terms of robustness, energy efficiency, accuracy, and overall suitability. The comparison results show that the active control method (Ac) outperforms the other approaches, exhibiting excellent robustness, high energy efficiency, and superior accuracy, which makes it the optimal choice for controlling chaotic dynamics in the studied systems. Sliding mode control (Sc) demonstrates good robustness but requires higher control energy and provides moderate accuracy, resulting in an acceptable but less efficient solution. Adaptive control (Adc) offers moderate performance across all evaluation criteria, representing a balanced yet less effective alternative. In contrast, backstepping control (Bc) shows very low robustness, poor accuracy, and inefficient energy consumption, and is therefore not recommended for practical implementation in this context.

In addition, synchronization and anti-synchronization of chaotic systems were successfully achieved and verified through numerical simulations. Beyond their theoretical significance, synchronization phenomena in chaotic epidemiological systems have important real-world implications. Synchronization can be interpreted as the coordination of epidemic dynamics

across interconnected regions or populations influenced by similar environmental conditions, mobility patterns, or public health interventions. Enforcing synchronization helps reduce discrepancies between epidemic trajectories, enhances predictability, and supports coherent large-scale epidemic monitoring and response strategies.

Moreover, advanced synchronization schemes, such as anti-synchronization, provide useful insights into contrasting epidemic behaviors between regions, enabling the identification of differences in intervention effectiveness or policy implementation. As a result, synchronization frameworks offer a powerful mathematical tool for understanding and coordinating complex epidemic dynamics in realistic, interconnected settings.

Overall, the significance of this work lies in the systematic application and comparative evaluation of chaos control and synchronization strategies within an epidemiological context. The findings contribute to a deeper understanding of how nonlinear control methods can be used to regulate chaotic dynamics and coordinate complex systems. This work provides a solid foundation for future research on more advanced chaotic frameworks, including stochastic epidemic models, network-based populations, and time-varying parameters.

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