الجمهوريـة الجزائـريـة الديمقراطيـة الشعبيـة
People's Democratic Republic of Algeria
وزارة التعليـم العالـي والبحـث العلمـي
Ministry of Higher Education and Scientific Research

University of Mila

جـــامعة ميلـة

Faculty of Mathematics and Computer Sciences
Order N°: …………………
Registration number: M237/2023

Department: Mathematics
Field: Mathematics
Specialty: Applied mathematics

Thesis

Submitted for the degree of
Doctorate LMD

Solving a class of multidimensional Volterra integral and integro-differential equations

Presented by: SAIDANE Sara

Jury Members:

HALIM Yacine	Pr	University of Mila	Chairman
LAIB Hafida	MCA	ENS of Kouba	Supervisor
CHELLOUF Yassamine	MCA	University of Mila	Co-supervisor
BELHANNACHE Farida	MCA	University of Jijel	Examiner
ROUIBAH Khaoula	MCA	University of Mila	Examiner
BOUOUDEN Rabah	MCA	University of Mila	Examiner
KAOUACHE Smail	MCA	University of Mila	Examiner

University year: 2025/2026

ACKNOWLEDGMENTS

I begin this work by expressing my sincere gratitude to Allah, the Almighty, for His countless blessings and for granting me the strength, patience, and perseverance to complete this academic journey. I also extend my heartfelt thanks to everyone who supported and guided me throughout the preparation of this thesis, including my professors, colleagues, friends, and family. Your encouragement and assistance have been essential in achieving this accomplishment.

I would like to express my deepest appreciation and respect to my supervisor, Dr. Hafida LAIB, who has been a true pillar of support throughout my academic journey. Her wise guidance, patience, and continuous support were invaluable in helping me overcome challenges and make meaningful progress in my research. Her insightful advice and dedication went far beyond the responsibilities of supervision, and I am profoundly grateful for her contribution to this work.

I also sincerely thank the members of the examination committee for their constructive feedback and valuable suggestions, which greatly contributed to broadening my perspective and enhancing the quality of this thesis.

I would also like to express my sincere gratitude to my colleagues and professors at the University of Mila, whose expertise and encouragement have significantly shaped my doctoral experience. I am particularly thankful to Prof. Mohammed Salah ABDELOUAHAB for his exceptional guidance and continuous support, and to Prof.

HAMRI Nasr Eddine for his valuable contributions.

I am deeply grateful to all the professors who guided me during the first year of my doctoral studies, as their knowledge and advice provided a solid foundation for my scientific and research development. I would also like to extend special thanks to my colleague HENNOUS Chahinaz for her support and assistance during the early stages of my work, as well as to my colleague Aida BELLOUT for her cooperation and kind spirit, which helped facilitate my research journey from the very beginning.

Finally, I extend my deepest appreciation to my husband for his constant understanding, patience, and encouragement throughout the years of this work. I am also profoundly grateful to my beloved father, who has been a constant source of support throughout my entire education. Despite the distance, I never felt alone, as his presence, guidance, and encouragement were always with me, greatly helping me overcome difficulties and achieve this accomplishment.

DEDICATION

To my beloved parents, Saliha and Omar, whose continuous support and great sacrifices have had a profound impact on my life. I express my deepest gratitude and appreciation to my father for his unwavering support, for sharing every effort and hardship with me, and for being my true pillar at every stage of my journey. To my mother, Saliha, who has given me her love, warmth, and constant patience, and who has always inspired me to continue and move forward.

To my beloved husband, Zakaria, who has been my pillar of support and encouragement throughout this journey and my partner in every success and challenge. I cannot express enough admiration for his wonderful character, patience, wisdom, and constant support, which have been a source of inspiration for me in accomplishing this work.

To my dear family, who have always been a source of warmth and inspiration, especially my sisters Amel, Malak, and Hajar, my brother Shams Eddine, my brother-in-law Hicham, and the little ones Matin and Watin, who have brought joy and happiness to my life and encouraged me to continue striving for excellence.

With sincere gratitude, I dedicate this work to everyone who has supported my growth and contributed to my success.

Sara Saidane

الملخـــــص

تتناول هذه الأطروحة دراسة منهج عددي لحل المعادلات التكاملية والتكاملية-التفاضلية الجزئية من نوع فولتيرا ثنائية البعد الخطية مع تأخير. تم تطوير طريقة التجميع (Collocation method) المعتمدة على كثيرات حدود تايلور لتقريب الحل على مجال زماني مكاني مُقسَّم إلى أجزاء. يتيح المخطط المقترح حساب معاملات التقريب بطريقة تكرارية دون الحاجة إلى حل أنظمة كبيرة من المعادلات الجبرية. كما يتم إثبات التقارب الشامل للطريقة نظريًا تحت افتراضات مناسبة لانتظام المعطيات. وتعرض أمثلة عددية لإبراز كفاءة الطريقة المقترحة ودقتها في تقريب الحلول.

الكلمات المفتاحية : معادلات فولتيرا التكاملية ذات التأخير, المعادلات التفاضلية التكاملية ثنائية البعد لفولتيرا ذات التأخير، طريقة التجميع، كثيرات حدود تايلور، تحليل الخطأ.

ABSTRACT

This thesis studies a numerical approach for solving linear two-dimensional delay partial Volterra integro-differential equations. A collocation method based on Taylor polynomials is developed to approximate the solution on a spatial domain defined piecewise. The proposed scheme computes the approximation coefficients iteratively without the need to solve large systems of algebraic equations. A theoretical analysis establishes the global convergence of the method under appropriate smoothness assumptions. Numerical examples are presented to demonstrate the efficiency and accuracy of the proposed method.

Key Words: Delay Volterra integral equations, two-dimensional delay Volterra integro-differential equations, collocation method, Taylor polynomials, error analysis.

RÉSUMÉ

Cette thèse étudie une approche numérique pour la résolution des équations intégro-différentielles partielles de Volterra bidimensionnelles linéaires avec retard. Une méthode de collocation basée sur les polynômes de Taylor est développée afin dapproximer la solution sur un domaine spatial défini par morceaux. Le schéma proposé permet de calculer les coefficients dapproximation de manière itérative sans avoir à résoudre de grands systèmes algébriques. Une analyse théorique établit la convergence globale de la méthode sous des hypothèses de régularité appropriées. Des exemples numériques illustrent lefficacité et la précision de la méthode proposée.

Mots-clés: Équations intégrales de Volterra à retard, équations intégrales-différentielles de Volterra à retard à deux dimensions, méthode de collocation, polynômes de Taylor, analyse des erreurs.

CONTENTS

GENERAL INTRODUCTION

Volterra integral equations have long been recognized as fundamental tools in the mathematical modeling of dynamic systems that depend on their past states. These equations provide a versatile framework for describing phenomena influenced by memory effects, making them indispensable in fields such as physics, biology, engineering, and economics [7, 15, 23]. By incorporating time delays and spatial variables, Volterra integral equations allow for a more realistic representation of complex systems, capturing both temporal and spatial dependencies.

which cannot be accurately described by conventional differential equations alone [7, 15]. For instance, in biological modeling, delay and memory effects are important in describing the evolution of populations and interactions among species, where the present state depends on previous states of the system. In engineering, Volterra-type equations are widely used to model materials and processes exhibiting hereditary behavior, such as viscoelastic systems, where the current response depends on the past history of the material [6, 11]. Moreover, in economic and financial modeling, memory-dependent models have been employed to describe systems in which current dynamics are influenced by previous trends and historical information [18].

Recent advances have extended the classical one-dimensional Volterra integral equations to **multi-dimensional and delayed forms** [12, 14, 20, 26]. Multi-dimensional Volterra integral equations (MDVIEs) generalize the classical theory to functions of multiple variables, enabling the modeling of systems that evolve simultaneously in space and time. When coupled with time delays, these equations effectively capture memory-dependent phenomena in complex systems, such as delayed responses in viscoelastic materials, ecological dispersal with memory effects, and delayed interactions in epidemic models [20, 24, 26].

Due to the complexity of delayed and multi-dimensional Volterra integral equations, analytical solutions are often unavailable. Therefore, efficient and reliable numerical methods are crucial. Among the widely used approaches are:

- **Taylor Series-based Collocation Methods:** This approach approximates the solution using Taylor polynomials within each subinterval and enforces the integral equation at selected collocation points [3, 5, 12, 20]. The main advantages include high accuracy, iterative computation of coefficients, and adaptability to higher-dimensional problems. These methods are particularly effective for systems with both spatial dependence and time delays, ensuring convergence and stability even in complex scenarios.

- **Galerkin Methods:** The Galerkin approach constructs approximate solutions as a linear combination of basis functions and minimizes the residual in a weighted manner [7, 8, 10]. It is widely applied in integral and integro-differential equations due to its strong theoretical foundation and ability to handle nonlinearity. Galerkin methods are particularly useful when the solution exhibits smooth behavior, and they can be combined with polynomial bases to improve accuracy.

- **Operational Matrices Methods:** These methods employ polynomial expansions, such as Bernstein or Chebyshev polynomials, and transform integral equations into systems of algebraic equations [13, 20, 23]. Operational matrices allow for efficient computation by directly approximating the integrals, providing a powerful tool for both linear and nonlinear Volterra integral equations. Their main benefit is reducing computational complexity while maintaining high accuracy.

- **Trapezoidal and Simpsons Rules:** Classical numerical integration techniques, like the trapezoidal and Simpsons rules, have been extended to multi-dimensional integrals [4, 5, 15]. These methods discretize the integration domain, providing simple yet effective approximations of the integral terms. They are particularly useful in heat conduction, fluid dynamics, and other engineering applications where accurate evaluation of multi-dimensional integrals is required.

These numerical techniques form the backbone of modern approaches to solving one-dimensional, two-dimensional, and even higher-dimensional Volterra integral equations with delays. Their development has enabled the analysis of systems previously intractable due to their complexity and dimensionality.

Delayed and multi-dimensional Volterra integral equations have proven essential in several scientific and engineering applications:

- **Ecological modeling:** Multi-dimensional equations are employed to describe ecosystems where species interactions depend on both spatial location and temporal history, including predator-prey dynamics, resource competition, and nutrient dispersal [1, 23, 26]. For example, a spatio-temporal predator-prey system can be modeled as:

$$u(x,t) = \int_0^t \int_\Omega K(x,y,t-s) f(u(y,s))\, dy\, ds,$$

  where $u(x,t)$ represents population density at location x and time t, and $K(x,y,t-s)$ is a kernel capturing delayed spatial interactions.

- **Engineering and materials science:** These equations model viscoelastic and composite materials, capturing how historical stress or strain affects current material behavior [6, 11, 21]. For instance, the stress in a material with memory can be expressed as:

$$\sigma(t) = E\epsilon(t) + \int_0^t K(t-s)\epsilon(s)\, ds,$$

where $\sigma(t)$ is the stress, $\epsilon(t)$ is the strain, E is the elastic modulus, and $K(t-s)$ is the memory kernel. They are also used in thermal analysis for materials with memory-dependent heat conduction [21, 24], such as:

$$u(x,t) = \int_0^t K(t-s)\frac{\partial^2 u}{\partial x^2}(x,s)\,ds + f(x,t),$$

where $u(x,t)$ is the temperature distribution and $f(x,t)$ an external heat source.

- **Economics and control systems:** In industrial processes and economic modeling, delayed Volterra equations describe systems where present outcomes depend on past performance, environmental factors, or delayed feedback in control systems [18, 23, 24]. A simple example is:

$$y(t) = \int_0^t K(t-s)u(s)\,ds + \alpha y(t-\tau),$$

where $y(t)$ is the economic indicator or system output at time t, $u(t)$ is input, $K(t-s)$ models delayed impact, and τ represents a time delay in feedback.

- **Population dynamics and epidemiology:** Delays in reproduction or disease incubation periods can be accurately captured using delay Volterra integral equations, improving forecasting and policy planning [1, 21, 26]. For example, the number of infected individuals in a population with incubation period τ can be modeled as:

$$I(t) = \int_0^{t-\tau} K(t-s)S(s)I(s)\,ds,$$

where $I(t)$ is the number of infected, $S(t)$ the susceptible population, and $K(t-s)$ a kernel representing the rate of infection spread over time.

The main objectives of this thesis are:

1. To develop a comprehensive numerical framework for solving **delayed Volterra integral equations with spatial dependence** using Taylor Collocation methods [3, 5, 12].

2. To establish **convergence and stability** properties of the proposed numerical method [14, 20, 26].

3. To provide numerical examples validating the efficiency and accuracy of the method for both one-dimensional and two-dimensional cases [14, 20, 26].

4. To extend the proposed framework for potential applications to multi-dimensional integral and integro-differential equations in science and engineering.

This thesis is structured into three main chapters, each focusing on a key aspect of delayed and multi-dimensional Volterra integral equations:

- **Chapter 1:** This chapter establishes the mathematical foundation required for the study of Volterra integral equations with delays. It introduces essential definitions, including function spaces and Lipschitz-type conditions, which are fundamental for ensuring the well-posedness of the problems considered [10, 22, 23]. The chapter also presents Taylor series expansions in one and multiple variables, providing the theoretical tools necessary for constructing polynomial approximations of unknown functions. Integral equation formulations are rigorously discussed, including first- and second-kind Volterra equations, and their relationships with differential and integro-differential equations. Furthermore, comparison techniques and relevant theorems are provided to analyze the convergence, stability, and error bounds of numerical methods. This chapter serves as the foundation upon which the numerical schemes in the following chapters are built, ensuring that the approximations preserve the essential properties of the exact solutions.

- **Chapter 2:** In this chapter, the focus shifts to the numerical solution of two-dimensional Volterra integral equations with time delays. The concept of delayed dependence is formally introduced, emphasizing how the solution at a given time depends not only on the current state but also on past states through integrals with shifted arguments [3, 5, 20, 26]. The Taylor Collocation method is developed in detail, including the construction of collocation points, the expansion of the unknown function using Taylor polynomials, and the iterative computation of coefficients without the need to solve large systems of algebraic equations. Convergence proofs are provided to demonstrate that the method yields increasingly accurate approximations as the number of collocation points grows. The chapter also includes illustrative numerical examples, showing the practical application of the method to delay Volterra integral equations arising in population dynamics and viscoelastic materials. These examples highlight the ability of the method to handle both nonlinearities and time delays effectively, while maintaining computational efficiency.

- **Chapter 3:** This chapter extends the study to more complex two-dimensional integro-differential equations with delays in both spatial and temporal variables [12, 14, 20, 26]. The mathematical formulation of 2D-DPVIDEs is presented, along with the conditions for existence and uniqueness of solutions. The Taylor Collocation method is adapted to handle two-dimensional spatial domains, requiring a careful partitioning of the domain and construction of two-dimensional Taylor expansions. The chapter provides detailed convergence and stability analyses, proving that the numerical approximations converge to the exact solution under appropriate smoothness and Lipschitz-type assumptions. Several numerical examples are included to demonstrate the method's accuracy and efficiency in solving representative two-dimensional delay integro-differential problems. The chapter emphasizes the generality and scalability of the method, illustrating how it can be extended to higher-dimensional problems and more complex integro-differential equations encountered in physics, engineering, and applied sciences.

Two-dimensional delay Volterra integro-differential equations form a versatile class of integral equations, widely applied in engineering, physics, and biological modeling [14, 20, 26]. The combination of time delays and multi-dimensional spatial variables allows for the modeling of complex systems whose current states depend intricately on past spatial-temporal interactions. The proposed methodology provides a robust and flexible numerical framework suitable for analyzing high-dimensional systems with delays, ensuring convergence, stability, and high accuracy. It establishes a solid foundation for future research into higher-dimensional and more complex Volterra-type integral and integro-differential equations.

CHAPTER 1

BASIC AND ESSENTIAL CONCEPTS

The purpose of this chapter is to establish the mathematical framework underlying our study. It gathers essential definitions and theoretical results, including Taylor expansion representations, integral equation formulations, and comparison techniques, which are instrumental in proving the convergence properties of the numerical approximations considered.

1.1 Taylor Expansion

The Taylor expansion is a fundamental analytical tool that allows a smooth function to be locally approximated by a polynomial constructed from its successive derivatives at a given point. This representation plays a crucial role in both theoretical and numerical analysis, particularly when dealing with functions whose exact evaluation is difficult or impractical. In a neighborhood of the expansion point, the Taylor expansion provides an accurate approximation of the function and, under suitable regularity conditions, coincides with it. When the expansion point is chosen to be zero, the resulting expansion is referred to as the Maclaurin expansion.

In numerical analysis, Taylor expansions serve as the theoretical foundation for many approximation techniques, especially those based on polynomial interpolation and collocation methods. They enable the transformation of differential, integral, and integro-differential equations into algebraic systems by replacing unknown functions with truncated polynomial representations. This feature makes Taylor expansion particularly suitable for constructing efficient numerical algorithms.

Definition 1.1.1 *Consider a function φ that is differentiable up to order $(\eta + 1)$ in the vicinity of a point τ_0. The Taylor theorem asserts that $\varphi(\tau)$ can be approximated by a polynomial composed of its derivatives evaluated at τ_0, together with a remainder term that quantifies the* *approximation error. Formally, one can write*

$$\varphi(\tau) = \sum_{k=0}^{\eta} \frac{\varphi^{(k)}(\tau_0)}{k!}(\tau - \tau_0)^k + \underbrace{\frac{\varphi^{(\eta+1)}(\xi)}{(\eta+1)!}(\tau - \tau_0)^{\eta+1}}_{\text{remainder term } R_\eta(\tau)},$$

where ξ is some point between τ_0 and τ. This formulation emphasizes that the Taylor polynomial provides a local approximation of order η, while the remainder term $R_\eta(\tau)$ offers a precise characterization of the truncation error, ensuring controlled convergence in a neighborhood of τ_0.

The remainder term plays a central role in error estimation and convergence analysis of numerical methods. In particular, it allows one to derive explicit bounds on the difference between the exact function and its polynomial approximation, which is essential for assessing the accuracy and stability of numerical schemes based on Taylor expansions.

Lemma 1.1.1 *(Taylor's theorem for functions of two variables, [22]) Let φ be a function continuously differentiable up to order p on the rectangle $\mathcal{R} = [A_0, A_1] \times [B_0, B_1]$, and let $(t_0, s_0) \in \mathcal{R}$. Then, for any $(t, s) \in \mathcal{R}$, the function $\varphi(t, s)$ can be expressed as*

$$\varphi(t, s) = \sum_{i+j=0}^{p-1} \frac{1}{i!\, j!} \partial_t^i \partial_s^j \varphi(t_0, s_0)(t - t_0)^i (s - s_0)^j + R_p(t, s),$$

where the remainder term $R_p(t, s)$ is given by

$$R_p(t, s) = \sum_{i+j=p} \frac{1}{i!\, j!} \partial_t^i \partial_s^j \varphi(\tilde{t}, \tilde{s})(t - t_0)^i (s - s_0)^j,$$

for some point $(\tilde{t}, \tilde{s})$ lying on the line segment joining (t_0, s_0) and (t, s). This expansion shows that $\varphi(t, s)$ can be locally approximated by a polynomial in $(t - t_0)$ and $(s - s_0)$ of total degree $p - 1$, while the remainder term $R_p(t, s)$ provides a rigorous measure of the approximation error.

Remark 1.1.1 *The multidimensional Taylor expansion presented in Lemma 1.1.1 is a key* *analytical ingredient in the numerical treatment of problems involving functions of several variables. It enables the construction of local polynomial approximations that are essential in higher-dimensional numerical schemes, particularly those based on collocation techniques. Moreover, the explicit form of the remainder term allows one to control truncation errors and plays a crucial role in the theoretical analysis of numerical stability and convergence.*

Such Taylor expansions are especially relevant in the numerical solution of Volterra integral and integro-differential equations with delays. In these problems, the unknown function often depends on multiple variables as well as on delayed arguments. By applying Taylor series expansions, delayed terms can be systematically approximated using local derivative information, which leads to finite-dimensional representations of the original problem. This approach constitutes the theoretical foundation of Taylor collocation methods employed for solving one-, two-, and three-dimensional delay Volterra equations.

Furthermore, the structure of the remainder terms in both the one-dimensional and multidimensional Taylor theorems provides the mathematical framework required for convergence analysis. These terms allow the derivation of explicit error bounds for the approximate solutions and will be extensively used in subsequent chapters to establish the consistency, stability, and convergence of the numerical algorithms developed in this thesis.

1.2 Integral Equations

An integral equation is a mathematical expression in which the unknown function, typically denoted by $\omega(x)$, appears under an integral sign. These equations serve as a fundamental tool in both theoretical and applied mathematics, providing a framework for formulating and solving diverse problems in physics and engineering. Many problems that originate as initial or boundary value problems for ordinary or partial differential equations can be equivalently reformulated as integral equations.

Definition 1.2.1 *An integral equation is an equation in which the unknown function appears inside an integral operator, possibly together with the function itself. The solution of the equation consists in determining a function that satisfies the relation for all values of the independent variable in a prescribed domain.*

A general linear form of an integral equation can be written as

$$\varphi(\tau)\omega(\tau) = u(\tau) + \lambda \int_{\alpha(\tau)}^{\beta(\tau)} \kappa(\tau, s)\omega(s)\, ds,$$

where λ is a constant parameter, $\kappa(\tau, s)$ is the kernel function, and the unknown function $\omega(\tau)$ may appear inside the integral, outside the integral, or both. The functions $\varphi(\tau)$, $u(\tau)$, and $\kappa(\tau, s)$ are assumed known, while the limits of integration $\alpha(\tau)$ and $\beta(\tau)$ can be either constants, variables, or a combination thereof.

1.2.1 Linearity and Nonlinearity

integral equations may also be classified according to their linearity. The equation is said to be *linear* if the unknown function ω appears only linearly, that is, it is not multiplied by itself nor composed with nonlinear operators. Otherwise, the equation is referred to as a *nonlinear integral equation*. Typical nonlinear integral equations involve kernels or integrands depending nonlinearly on ω, such as $\kappa(\tau, s, \omega(s))$.

1.2.2 Classification by Limits of Integration

integral equations can be classified according to the limits of integration:

- **Fredholm integral equations (FIEs)**: If both integration limits are constants, the equation takes the form

$$\varphi(\tau)\omega(\tau) = u(\tau) + \lambda \int_{a}^{b} \kappa(\tau, s)\omega(s)\, ds,$$

where a and b are fixed constants.

- **Volterra integral equations (VIEs)**: If at least one limit of integration is variable, the equation is expressed as

$$\varphi(\tau)\omega(\tau) = u(\tau) + \lambda \int_a^\tau \kappa(\tau, s)\omega(s)\, ds.$$

Both FIEs and VIEs may be homogeneous or inhomogeneous. The equation is homogeneous if $u(\tau) = 0$, and inhomogeneous otherwise.

1.2.3 First, Second, and Third Kind

Another common classification is based on the function $\varphi(\tau)$:

- If $\varphi(\tau) = 0$, the equation is called a *first-kind integral equation* (FIE or VIE).
- If $\varphi(\tau) = 1$, it is referred to as a *second-kind integral equation* (FIE or VIE).
- If $\varphi(\tau) \neq 0$ and $\varphi(\tau) \neq 1$, it is classified as a *third-kind integral equation* (FIE or VIE).

1.2.4 Order of Volterra Equations

A Volterra integral equation of the first kind involves only the integral term, whereas a Volterra integral equation of the second kind contains both the unknown function and its integral representation. Equations of the second kind are generally better posed and more suitable for numerical treatment [23].

1.2.5 Delay Volterra Integral Equations

In many practical applications, integral equations involve delay arguments, where the unknown function is evaluated at shifted variables. A delay Volterra integral equation typically takes the form

$$\omega(\tau) = u(\tau) + \int_a^\tau \kappa(\tau, s)\, \omega(s - \delta)\, ds,$$

where $\delta > 0$ denotes a constant or variable delay. Such equations naturally arise in systems with memory effects and hereditary phenomena [3, 14].

1.2.6 Higher-Dimensional Volterra Equations

integral equations can also be defined in higher-dimensional domains. For instance, a two-dimensional Volterra integral equation involves an unknown function depending on two independent variables and integrals taken over planar regions. These equations appear frequently in models describing spatialtemporal processes and pose additional analytical and numerical challenges compared to the one-dimensional case [12].

1.2.7 Connection to Numerical Methods

The classification and structural properties of integral equations play a crucial role in the development of numerical methods. In particular, Volterra integral equations of the second kind, including their delayed and multidimensional forms, are well suited for collocation-based techniques. This observation motivates the use of Taylor collocation methods in the subsequent chapters for constructing accurate and efficient numerical approximations.

1.3 Piecewise Polynomial Spaces

Piecewise polynomial spaces provide a flexible framework for approximating functions over an interval by partitioning the domain into smaller subintervals and defining polynomial functions locally on each subinterval. These spaces are widely used in numerical analysis, finite element methods, computer-aided geometric design, data interpolation, and in the numerical solution of integral and integro-differential equations, offering a localized and adaptable approach to capture complex behaviors within a domain [7, 10].

Consider a grid (or mesh) on the interval $J = [0, b]$ given by

$$J_k = \{y_m : 0 = y_0 < y_1 < \cdots < y_M = b\}, \quad k = \frac{b}{M},$$

where k is the step size. Each subinterval is denoted by

$$\delta_m = [y_m, y_{m+1}], \quad m = 0, \ldots, M-1.$$

Definition 1.3.1 (Piecewise Polynomial Spaces, adapted from [7]) *For given integers $\eta \geq 0$ and $-1 \leq d \leq \eta$, the piecewise polynomial space $\mathcal{S}_\eta^{(d)}(J_k)$ associated with the mesh J_k is defined as*

$$\mathcal{S}_\eta^{(d)}(J_k) = \{u \in C^d(J) : u|_{\delta_m} \in \pi_\eta,\ 0 \leq m \leq M-1\},$$

where π_η denotes the space of real polynomials of degree at most η.

This space forms a real linear vector space, and its dimension is given by

$$\dim \mathcal{S}_\eta^{(d)}(J_k) = M(\eta - d) + d + 1.$$

Remark 1.3.1 *A commonly used space for collocation approximations is $\mathcal{S}_{p+d}^{(d)}(J_k)$ with $p \geq 1$ and $d \geq -1$, which has dimension $Mp + d + 1$. This space is well-suited for approximating solutions to initial-value problems for ordinary differential equations or Volterra integral equations [7, 14].*

The choice of the regularity parameter d depends on the number of initial conditions:

- *For Volterra integral equations with no initial conditions, set $d = -1$ [5].*
- *For first-order ODEs or Volterra integro-differential equations with one initial condition, set $d = 0$ [4, 26].*

- *For higher-order ODEs or Volterra integro-differential equations of order $k \geq 2$ with k initial conditions, set $d = k - 1$ [5, 26].*

For bivariate spline approximations, the tensor-product piecewise polynomial space

$$\mathcal{S}^{(-1)}_{p-1,q-1}(\Pi_{N,M})$$

is constructed from the univariate spline spaces $\mathcal{S}^{(-1)}_{p}(\Pi_N)$ in t and $\mathcal{S}^{(-1)}_{q}(\Pi_M)$ in s [8, 17]. Functions in this space are allowed to have discontinuities along the interior grid lines $t = t_n$ ($n = 1, \ldots, N-1$) and $s = s_m$ ($m = 1, \ldots, M-1$), which is particularly useful for solving two-dimensional Volterra integral equations of the first and second kind numerically [17, 12].

Remark 1.3.2 *Piecewise polynomial spaces, especially those based on splines, provide the foundation for collocation methods, spectral methods, and Galerkin-type approaches. They allow local error control, high-order convergence, and the flexibility to handle nonuniform grids or singularities in the solution [7, 14]. In the context of delay Volterra equations, tensor-product piecewise polynomial spaces facilitate the efficient approximation of functions depending on multiple variables and delays [3, 5].*

1.4 Collocation Method

The collocation method is a prominent numerical technique used to approximate solutions of differential and integral equations. The central idea is to select a finite-dimensional space of candidate functions, often polynomials of a specified degree, and a discrete set of points within the domain, referred to as *collocation points*. The approximate solution is then constructed such that the governing equation is satisfied exactly at these points [7, 14].

This approach demonstrates considerable flexibility and can be adapted to a wide variety of equations. In the context of ordinary differential equations (ODEs), several specialized collocation techniques have been developed, including:

- **Chebyshev collocation:** representing solutions via Chebyshev polynomial expansions [23];
- **Legendre-Gauss collocation:** using Legendre-Gauss nodes to efficiently solve initial-value problems for second-order ODEs [9];
- **Bernstein collocation:** employing Bernstein polynomials to handle nonlinear ODEs ;
- **Block hybrid collocation:** designed specifically for direct solution of third-order ODEs [13].

For partial differential equations (PDEs), the collocation methodology extends to more complex frameworks, such as sparse grid stochastic collocation methods based on Smolyak grids for statistical approximations, wavelet collocation methods utilizing Daubechies wavelet autocorrelations, and spline collocation employing B-spline functions for accurate numerical approximation [10, 12].

When applied to integral equations, the collocation method provides effective strategies tailored to the problem type. For instance, the Galerkin collocation method is well-suited for first-kind integral equations, whereas iterated collocation methods efficiently address nonlinear Volterra integral equations of the second kind [4, 14]. In particular, Taylor collocation methods have proven effective for both one-dimensional and two-dimensional Volterra integral equations, including those with delays [3, 5, 12].

The reliability and precision of the collocation method depend critically on the appropriate choice of collocation points, the selection of basis functions (e.g., piecewise polynomials, splines, or orthogonal polynomials), and the inherent properties of the equation under study. Careful attention to these aspects ensures both the convergence and the accuracy of the approximated solution [4, 14].

An additional advantage of this approach is its versatility: in certain implementations, mesh generation can be avoided entirely through node-based schemes, reducing computational cost significantly. Combined with its capacity to handle irregular domains, multidimensional problems, and complex boundary conditions, the collocation method serves as a robust and adaptable tool in modern numerical analysis, capable of delivering highly accurate approximations across a wide spectrum of problems [4, 14, 12].

1.5 Taylor Collocation Method

The Taylor collocation method is a numerical approach for approximating solutions of differential, integral, and integro-differential equations by representing the unknown function as a truncated Taylor expansion. In this method, a set of collocation points is selected within the domain, and the coefficients of the expansion are determined such that the governing equation is satisfied exactly at these points [3, 14, 12, 13].

Initially developed for Volterra integral equations, the method has been progressively refined and applied to more complex problems:

- Karamete and Sezer [13] introduced a spectral Taylor matrix collocation technique for solving linear integro-differential equations, emphasizing spectral convergence properties.
- Bellour and Bousselsal [3] proposed a framework using Taylor polynomials to approximate solutions of delay integral and integro-differential equations, demonstrating accuracy for single- and multi-delay problems.
- Laib et al. [14] applied the method to systems of nonlinear Volterra delay integro-differential equations, demonstrating its efficiency for both linear and nonlinear forms in two-dimensional cases.
- Khennaoui et al. [12] extended the Taylor collocation approach to two-dimensional partial Volterra integro-differential equations, illustrating its versatility in multi-dimensional domains and compatibility with tensor-product piecewise polynomial spaces [7, 10].

By carefully incorporating delayed terms, multidimensional extensions, and suitable basis functions from piecewise polynomial spaces, the Taylor collocation method has proven to be a robust and adaptable tool for numerically solving Volterra-type equations, providing accurate approximations while maintaining computational efficiency [4, 14, 12].

1.5.1 Application to Delayed Volterra Equations

The collocation method, and in particular the Taylor collocation approach, can be effectively adapted to solve Volterra integral and integro-differential equations with delay. In these equations, the unknown function depends not only on the current time variable but also on past values, introducing a delay term.

To handle this, the method is modified as follows:

- The collocation points are carefully selected within the domain to account for the maximum delay, ensuring that all delayed terms are evaluated correctly [3, 14].
- The approximate solution is expressed using a truncated Taylor expansion in each subinterval, capturing both the current and delayed contributions of the unknown function [4, 14]
- The coefficients of the Taylor expansion are determined such that the equation, including all delay terms, is satisfied exactly at the collocation points [3, 14, 12].

This adaptation allows the Taylor collocation method to provide accurate and efficient numerical approximations for both linear and nonlinear Volterra equations with delay, extending the flexibility of the general collocation framework to a class of problems that are otherwise challenging to solve analytically [3, 5, 14, 12].

Building upon these fundamental studies, this thesis develops and refines the Taylor collocation method specifically for delay Volterra integral and integro-differential equations, addressing both two-dimensional and three-dimensional cases. By systematically incorporating the effects of delayed terms and leveraging piecewise polynomial basis functions, the proposed approach targets both linear and nonlinear forms of the equations. This work aims to extend the practical applicability of the Taylor collocation technique, offering robust and accurate numerical approximations for a broad range of multidimensional delayed Volterra problems that are otherwise difficult to solve analytically [3, 14, 12, 13].

1.6 Comparison Theorems

In this section, we focus on several fundamental theorems related to both discrete and continuous inequalities. These results play a central role in establishing the theoretical foundations required to verify the convergence of numerical approximations toward the true solution. The applications and proofs of these theorems will be systematically developed in the following chapters to support the analysis of the proposed methods.

1.6.1 Discrete inequalities

Lemma 1.6.1 (Discrete Gronwall-type inequality [7]) Let $\{k_j\}_{j=0}^{n}$ be a given non-negative sequence, and suppose that the sequence $\{\varepsilon_n\}$ satisfies $\varepsilon_0 \le p_0$ and

$$\varepsilon_n \le p_0 + \sum_{i=0}^{n-1} k_i \varepsilon_i, \quad n \ge 1,$$

with $p_0 \ge 0$. Then ε_n is bounded by

$$\varepsilon_n \le p_0 \exp\left(\sum_{j=0}^{n-1} k_j\right), \quad n \ge 1.$$

Lemma 1.6.2 (Bellman's inequality [2]) Let h and f be continuous and non-negative functions defined on $J = [\alpha, \beta]$, and let c be a non-negative constant. Then the inequality

$$h(t) \leq c + \int_{\alpha}^{t} f(s)h(s)ds, \quad t \in J,$$

implies that

$$h(t) \leq c \exp\left(\int_{\alpha}^{t} f(s)ds\right), \quad t \in J.$$

Lemma 1.6.3 (Taylor's theorem for functions of two independent variables) *Let ω be a p-times continuously differentiable function on $D = [a, b] \times [c, d]$ and let $(x_0, y_0) \in D$. Then for all $(x, y) \in D$ we have*

$$\begin{aligned}\omega(x, y) = &\sum_{i+j=0}^{p-1} \frac{1}{i!\, j!} \frac{\partial^{i+j}\omega(x_0, y_0)}{\partial x^i \partial y^j}(x - x_0)^i (y - y_0)^j \\ &+ \sum_{i+j=p} \frac{1}{i!\, j!} \frac{\partial^{i+j}\omega(x_1, y_1)}{\partial x^i \partial y^j}(x - x_0)^i (y - y_0)^j,\end{aligned}$$

where

$$\begin{cases} x_1 = \varepsilon x + (1 - \varepsilon)x_0 \in [a, b], \\ y_1 = \varepsilon y + (1 - \varepsilon)y_0 \in [c, d], \end{cases} \qquad \varepsilon \in (0, 1).$$

Lemma 1.6.4 (Discrete Gronwall inequality) *Assume $\omega(n, m)$, $b(n, m)$ and $e(n, m)$ are nonnegative sequences, with $b(n, m)$ being nondecreasing in each variable n and m. Suppose that*

$$\omega(n, m) \leq b(n, m) + \sum_{s=0}^{n-1} \sum_{t=0}^{m-1} e(s, t)\, \omega(s, t),$$

for all $n, m \in \mathbb{N}$. Then

$$\omega(n, m) \leq b(n, m) \prod_{s=0}^{n-1} \left[1 + \sum_{t=0}^{m-1} e(s, t)\right], \qquad n, m \in \mathbb{N}.$$

CHAPTER 2

NUMERICAL SOLUTION OF DELAY VOLTERRA INTEGRAL EQUATION

2.1 Introduction

In this chapter, we consider the numerical solution of delay Volterra integral equations with time delay $\tau > 0$:

$$u(t,x) = \begin{cases} g(t,x) + A(t,x)u(t-\tau,x) + \int_0^t K_1(t,x,s)u(s,x)\,ds \\ \quad + \int_0^{t-\tau} K_2(t,x,s)u(s,x)\,ds, & t \in [0,T],\ x \in [0,X], \\ \Phi(t,x), & t \in [-\tau,0],\ x \in [0,X], \end{cases} \tag{2.1}$$

where the functions g, Φ, A, K_1, and K_2 are given smooth real-valued functions defined on $D := [0,T] \times [0,X] \subset \mathbb{R}^2$ and $S := \{(t,x,s) : 0 \le s \le t \le T,\ 0 \le x \le X\}$. According to the classical theory of Volterra integral equations (VIEs), equation (2.1) admits a unique solution $u \in C(D)$ [7].

Delay Volterra integral equations (DVIEs) play a crucial role in modeling nonlinear dynamical systems in which memory effects and delays significantly influence system behavior. Such systems arise in various disciplines, including population dynamics, epidemic models, viscoelastic materials, control systems, and financial modeling, where the future state of the system depends not only on its current state but also on its past states. These delay effects often introduce additional complexity, making analytical solutions difficult or impossible to obtain. Therefore, efficient and accurate numerical methods are essential for their analysis.

Several researchers have studied numerical methods for solving one-dimensional Volterra integral equations with time delays (see, for example, [1, 3, 5, 9, 26]). Ali et al. [1] proposed a spectral method for pantograph-type delay integral equations using the Legendre collocation method. Bellour and Bousselsal [3] constructed a collocation solution to approximate the solution of delay integral equations. Brunner and Yatsenko [9] investigated polynomial spline collocation methods for systems of VIEs with unknown delays. Zheng and Chen [26] developed spectral methods using Chebyshev bases and spectral collocation techniques to analyze VIEs with two types of delays.

While numerous studies have focused on two-dimensional Volterra integral equations [4, 8, 16, 17], fewer works have addressed delay Volterra integral equations with a spatial variable [20]. The primary challenge in solving such problems lies in handling both the spatial dependence and the time delay in the upper integral limits, which distinguishes these equations from conventional two-dimensional Volterra integral equations.

To address this problem, we propose a Taylor collocation method that extends the approach in [3] to approximate the solution of equation (2.1). The main contributions of this work are as follows:

1. The adaptation of the Taylor collocation method to DVIEs with spatial dependence, which remains a relatively unexplored area in numerical analysis.

2. The development of a computationally efficient framework in which the approximation coefficients are computed iteratively without requiring the solution of large systems of algebraic equations.

3. The formulation of a general numerical approach that can be extended to higher-dimensional delay integral equations encountered in physics, control theory, and engineering applications.

This chapter is organized as follows. In Section 2.2, we partition the domain $[0, T] \times [0, X]$ into subintervals and approximate the solution of (2.1) in each subinterval using Taylor polynomials. Section 2.3 establishes the global convergence of the method. Section 2.4 presents numerical examples illustrating the efficiency and accuracy of the proposed approach. Finally, the conclusion summarizes the main findings and outlines possible directions for future work.

2.2 Description of the Method

We assume, without loss of generality, that $T = r\tau$, where $r \in \mathbf{N}^*$. Let

$$\Pi_N^\epsilon = \{t_n^\epsilon = \epsilon\tau + nh,\ n = 0, 1, \dots, N,\ \epsilon = 0, 1, \dots, r\}$$

and

$$\Pi_M = \{x_m = mk,\ m = 0, 1, \dots, M\}$$

denote uniform partitions of the intervals $[0, T]$ and $[0, X]$, respectively, with step sizes $h = \frac{\tau}{N}$ and $k = \frac{X}{M}$. These partitions define a grid for D:

$$\Pi_{N,M}^\epsilon = \Pi_N^\epsilon \times \Pi_M = \{(t_n^\epsilon, x_m) \mid 0 \le n \le N,\ 0 \le m \le M,\ \epsilon = 0, 1, \dots, r\}.$$

Define the subintervals as follows:

$$\sigma_n^\epsilon = [t_n^\epsilon, t_{n+1}^\epsilon), \quad \sigma_{N-1}^\epsilon = [t_{N-1}^\epsilon, t_N^\epsilon], \quad n = 0, 1, \dots, N-2, \quad \epsilon = 0, 1, \dots, r-1.$$

$$\delta_m = [x_m, x_{m+1}), \quad m = 0, 1, \dots, M-2, \quad \delta_{M-1} = [x_{M-1}, x_M].$$

The rectangular subdomains are defined by

$$D_{n,m}^\epsilon := \sigma_n^\epsilon \times \delta_m, \quad n = 0, 1, \dots, N-1, \quad m = 0, 1, \dots, M-1, \quad \epsilon = 0, 1, \dots, r-1.$$

Moreover, let π_{p-1} denote the set of all real polynomials of degree at most $p-1$ in the variables t and x. We define the real polynomial spline space of degree $p-1$ in t and x as follows:

$$\begin{aligned} S_{p-1}^{(-1)}(\Pi_{N,M}^\epsilon) = \{ & v \in C^{(-1)} : v_{n,m}^\epsilon = v|_{D_{n,m}^\epsilon} \in \pi_{p-1}, \\ & n = 0, \dots, N-1;\ m = 0, 1, \dots, M-1;\ \epsilon = 0, 1, \dots, r-1\}. \end{aligned}$$

This represents the space of bivariate polynomial spline functions of degree at most $p-1$ in t and x. Its dimension is $rNMp^2$, which corresponds to the total number of coefficients in the polynomials $v^{\epsilon}_{n,m}$, where $n = 0,\dots,N-1$, $m = 0,\dots,M-1$, and $\epsilon = 0,\dots,r-1$.

To determine these coefficients, we apply Taylor polynomials on each rectangle.

2.2.1 Approximate solution in $D^0_{0,m}$

The polynomial $v^0_{0,m}(t,x)$ is used to approximate $u(t,x)$ within the rectangle $D^0_{0,m}$, where $m = 0,\dots,M-1$, such that

$$v^0_{0,m}(t,x) = \sum_{i+j=0}^{p-1} \frac{1}{i!j!}\partial_t^{(i)}\partial_x^{(j)}u(0,x_m)\, t^i(x-x_m)^j. \tag{2.2}$$

Here, $\partial_t^{(i)}\partial_x^{(j)}u(0,x_m)$ denotes the exact value of $\partial_t^{(i)}\partial_x^{(j)}u$ at the point $(0,x_m)$.

To determine $\partial_t^{(i)}\partial_x^{(j)}u(t,x)$, we differentiate equation (2.1) j times with respect to x and i times with respect to t, obtaining

$$\begin{aligned}
\partial_t^{(i)}\partial_x^{(j)}u(t,x) &= \partial_t^{(i)}\partial_x^{(j)}g(t,x) + \partial_t^{(i)}\partial_x^{(j)}\left(A(t,x)\Phi(t-\tau,x)\right) \\
&\quad + \sum_{q=0}^{i-1}\sum_{l=0}^{j}\sum_{\eta=0}^{q}\binom{j}{l}\binom{q}{\eta}\partial_t^{(q-\eta)}\left(\partial_t^{(i-1-q)}\partial_x^{(j-l)}K_1(t,x,s)\Big|_{s=t}\right)\partial_t^{(\eta)}\partial_x^{(l)}u(t,x) \\
&\quad + \int_0^t \sum_{l=0}^{j}\binom{j}{l}\partial_t^{(i)}\partial_x^{(j-l)}K_1(t,x,s)\,\partial_x^{(l)}u(s,x)\,ds \\
&\quad - \partial_t^{(i)}\partial_x^{(j)}\left(\int_{t-\tau}^{0} K_2(t,x,s)\Phi(s,x)\,ds\right).
\end{aligned} \tag{2.3}$$

Hence,

$$\begin{aligned}\partial_t^{(i)}\partial_x^{(j)}u(0,x_m) &= \partial_t^{(i)}\partial_x^{(j)}g(0,x_m) + \partial_t^{(i)}\partial_x^{(j)}\left(A(0,x_m)\Phi(-\tau,x_m)\right)\\ &+ \sum_{q=0}^{i-1}\sum_{l=0}^{j}\sum_{\eta=0}^{q}\binom{j}{l}\binom{q}{\eta}\partial_t^{(q-\eta)}\left(\partial_t^{(i-1-q)}\partial_x^{(j-l)}K_1(t,x,s)\Big|_{s=t}\right)_{t=0,x=x_m}\partial_t^{(\eta)}\partial_x^{(l)}u(0,x_m)\\ &- \partial_t^{(i)}\partial_x^{(j)}\left(\int_{-\tau}^{0}K_2(t,x,s)\Phi(s,x)\,ds\right)\Big|_{t=0,x=x_m}.\end{aligned}$$

2.2.2 Approximate solution in $D^0_{n,m}$

The polynomial $v^0_{n,m}(t,x)$ is used to approximate $u(t,x)$ within the rectangle $D^0_{n,m}$, where $n=1,\ldots,N-1$ and $m=0,\ldots,M-1$, such that

$$v^0_{n,m}(t,x) = \sum_{i+j=0}^{p-1}\frac{1}{i!j!}\partial_t^{(i)}\partial_x^{(j)}\hat{v}^0_{n,m}(t^0_n,x_m)(t-t^0_n)^i(x-x_m)^j. \tag{2.4}$$

Here, $\hat{v}^0_{n,m}(t,x)$ denotes the exact solution of the integral equation

$$\begin{aligned}\hat{v}^0_{n,m}(t,x) &= g(t,x) + A(t,x)\Phi(t-\tau,x) + \sum_{\eta=0}^{n-1}\int_{t^0_\eta}^{t^0_{\eta+1}}K_1(t,x,s)v^0_{\eta,m}(s,x)\,ds\\ &+ \int_{t^0_n}^{t}K_1(t,x,s)\hat{v}^0_{n,m}(s,x)\,ds - \int_{t-\tau}^{0}K_2(t,x,s)\Phi(s,x)\,ds.\end{aligned} \tag{2.5}$$

To compute $\partial_t^{(i)}\partial_x^{(j)}\hat{v}^0_{n,m}(t,x)$, we differentiate equation (2.5) j times with respect to x

and i times with respect to t, obtaining

$$\begin{aligned}
\partial_t^{(i)}\partial_x^{(j)}\hat{v}^0_{n,m}(t,x) &= \partial_t^{(i)}\partial_x^{(j)}g(t,x) + \partial_t^{(i)}\partial_x^{(j)}\left(A(t,x)\Phi(t-\tau,x)\right) \\
&+ \sum_{\eta=0}^{n-1}\int_{t^0_\eta}^{t^0_{\eta+1}}\sum_{l=0}^{j}\binom{j}{l}\partial_t^{(i)}\left(\partial_x^{(j-l)}K_1(t,x,s)\right)\partial_x^{(l)}v^0_{\eta,m}(s,x)\,ds \\
&+ \sum_{q=0}^{i-1}\sum_{l=0}^{j}\sum_{\eta=0}^{q}\binom{j}{l}\binom{q}{\eta}\partial_t^{(q-\eta)}\left(\partial_t^{(i-1-q)}\partial_x^{(j-l)}K_1(t,x,s)\right)\Big|_{s=t}\partial_t^{(\eta)}\partial_x^{(l)}\hat{v}^0_{n,m}(t,x) \\
&+ \int_{t^0_n}^{t}\sum_{l=0}^{j}\binom{j}{l}\partial_t^{(i)}\left(\partial_x^{(j-l)}K_1(t,x,s)\right)\partial_x^{(l)}\hat{v}^0_{n,m}(s,x)\,ds \\
&- \partial_t^{(i)}\partial_x^{(j)}\left(\int_{t-\tau}^{0}K_2(t,x,s)\Phi(s,x)\,ds\right).
\end{aligned} \tag{2.6}$$

2.2.3 Approximate solution in $D^\epsilon_{n,m}$

The polynomial $v^\epsilon_{n,m}(t,x)$ is employed to approximate $u(t,x)$ within the rectangle $D^\epsilon_{n,m}$,where $n=0,...,N-1$, $\epsilon=1,...,r$, and $m=0,1,...,M-1$, such that

$$v^\epsilon_{n,m}(t,x) = \sum_{j+i=0}^{p-1}\frac{1}{i!j!}\partial_t^{(i)}\partial_x^{(j)}\hat{v}^\epsilon_{n,m}(t^\epsilon_n,x_m)(t-t^\epsilon_n)^i(x-x_m)^j. \tag{2.7}$$

Here, $\hat{v}^\epsilon_{n,m}(t,x)$ is the exact solution of the integral equation

$$\begin{aligned}
\hat{v}^\epsilon_{n,m}(t,x) &= g(t,x) + A(t,x)v^{\epsilon-1}_{n,m}(t-\tau,x) + \sum_{e=0}^{\epsilon-1}\sum_{\eta=0}^{N-1}\int_{t^e_\eta}^{t^e_{\eta+1}}K_1(t,x,s)v^e_{\eta,m}(s,x)ds \\
&+ \sum_{\eta=0}^{n-1}\int_{t^\epsilon_\eta}^{t^\epsilon_{\eta+1}}K_1(t,x,s)v^\epsilon_{\eta,m}(s,x)ds + \int_{t^\epsilon_n}^{t}K_1(t,x,s)\hat{v}^\epsilon_{n,m}(s,x)ds \\
&+ \sum_{e=0}^{\epsilon-2}\sum_{\eta=0}^{N-1}\int_{t^e_\eta}^{t^e_{\eta+1}}K_2(t,x,s)v^e_{\eta,m}(s,x)ds + \sum_{\eta=0}^{n-1}\int_{t^{\epsilon-1}_\eta}^{t^{\epsilon-1}_{\eta+1}}K_2(t,x,s)v^{\epsilon-1}_{\eta,m}(s,x)ds \\
&+ \int_{t^{\epsilon-1}_n}^{t-\tau}K_2(t,x,s)v^{\epsilon-1}_{n,m}(s,x)ds.
\end{aligned} \tag{2.8}$$

To compute $\partial_t^{(i)}\partial_x^{(j)}\hat{v}^{\epsilon}_{n,m}(t,x)$, we differentiate equation (2.8) j- times with respect to x and i- times with respect to t, obtaining

$$\begin{aligned}
\partial_t^{(i)}\partial_x^{(j)}\hat{v}^{\epsilon}_{n,m}(t,x) = \quad & \partial_t^{(i)}\partial_x^{(j)}g(t,x) + \sum_{l=0}^{j}\sum_{\eta=0}^{i}\binom{j}{l}\binom{i}{\eta}\partial_t^{(i-\eta)}\left(\partial_x^{(j-l)}A(t,x)\right)\partial_t^{(\eta)}\partial_x^{(l)}v^{\epsilon-1}_{n,m}(t-\tau,x) \\
&+ \sum_{e=0}^{\epsilon-1}\sum_{\eta=0}^{N-1}\int_{t^e_\eta}^{t^e_{\eta+1}}\sum_{l=0}^{j}\binom{j}{l}\partial_t^{(i)}\left(\partial_x^{(j-l)}K_1(t,x,s)\right)\partial_x^{(l)}v^{e}_{\eta,m}(s,x)\,ds \\
&+ \sum_{\eta=0}^{n-1}\int_{t^\epsilon_\eta}^{t^\epsilon_{\eta+1}}\sum_{l=0}^{j}\binom{j}{l}\partial_t^{(i)}\left(\partial_x^{(j-l)}K_1(t,x,s)\right)\partial_x^{(l)}v^{\epsilon}_{\eta,m}(s,x)\,ds \\
&+ \sum_{q=0}^{i-1}\sum_{l=0}^{j}\sum_{\eta=0}^{q}\binom{j}{l}\binom{q}{\eta}\partial_t^{(q-\eta)}\left[\partial_t^{(i-1-q)}\Big|_{s=t}\left(\partial_x^{(j-l)}K_1(t,x,s)\right)\right]\partial_t^{(\eta)}\partial_x^{(l)}\hat{v}^{\epsilon}_{n,m}(t,x)
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
&+ \int_{t^\epsilon_n}^{t}\sum_{l=0}^{j}\binom{j}{l}\partial_t^{(i)}\left(\partial_x^{(j-l)}K_1(t,x,s)\right)\partial_x^{(l)}\hat{v}^{\epsilon}_{n,m}(s,x)ds, \\
&+ \sum_{e=0}^{\epsilon-2}\sum_{\eta=0}^{N-1}\int_{t^e_\eta}^{t^e_{\eta+1}}\sum_{l=0}^{j}\binom{j}{l}\partial_t^{(i)}\left(\partial_x^{(j-l)}K_2(t,x,s)\right)\partial_x^{(l)}v^{e}_{\eta,m}(s,x)\,ds \\
&+ \sum_{\eta=0}^{n-1}\int_{t^{\epsilon-1}_\eta}^{t^{\epsilon-1}_{\eta+1}}\sum_{l=0}^{j}\binom{j}{l}\partial_t^{(i)}\left(\partial_x^{(j-l)}K_2(t,x,s)\right)\partial_x^{(l)}v^{\epsilon-1}_{\eta,m}(s,x)\,ds \\
&+ \sum_{q=0}^{i-1}\sum_{l=0}^{j}\sum_{\eta=0}^{q}\binom{j}{l}\binom{q}{\eta}\partial_t^{(q-\eta)}\left[\partial_t^{(i-1-q)}\Big|_{s=t-\tau}\left(\partial_x^{(j-l)}K_2(t,x,s)\right)\right]\partial_t^{(\eta)}\partial_x^{(l)}v^{\epsilon-1}_{n,m}(t-\tau,x)
\end{aligned}$$

Hence,

$$
\begin{aligned}
\partial_t^{(i)}\partial_x^{(j)}\hat{v}_{n,m}^{\epsilon}(t_n^{\epsilon},x_m) = \quad & \partial_t^{(i)}\partial_x^{(j)} g(t_n^{\epsilon},x_m) \\
+\sum_{l=0}^{j}\sum_{\eta=0}^{i}\binom{j}{l}\binom{i}{\eta}&\partial_t^{(i-\eta)}\left(\partial_x^{(j-l)}A(t_n^{\epsilon},x_m)\right)\partial_t^{(\eta)}\partial_x^{(l)}v_{n,m}^{\epsilon-1}(t_n^{\epsilon}-\tau,x_m) \\
+\sum_{e=0}^{\epsilon-1}\sum_{\eta=0}^{N-1}\int_{t_\eta^e}^{t_{\eta+1}^e}\sum_{l=0}^{j}&\binom{j}{l}\partial_t^{(i)}\left(\partial_x^{(j-l)}K_1(t_n^{\epsilon},x_m,s)\right)\partial_x^{(l)}v_{\eta,m}^{e}(s,x_m)\,ds \\
+\sum_{\eta=0}^{n-1}\int_{t_\eta^{\epsilon}}^{t_{\eta+1}^{\epsilon}}\sum_{l=0}^{j}&\binom{j}{l}\partial_t^{(i)}\left(\partial_x^{(j-l)}K_1(t_n^{\epsilon},x_m,s)\right)\partial_x^{(l)}v_{\eta,m}^{\epsilon}(s,x_m)\,ds \\
+\sum_{q=0}^{i-1}\sum_{l=0}^{j}\sum_{\eta=0}^{q}&\binom{j}{l}\binom{q}{\eta}\partial_t^{(q-\eta)}\left[\partial_t^{(i-1-q)}\Big|_{s=t}\left(\partial_x^{(j-l)}K_1(t_n^{\epsilon},x_m,s)\right)\right]_{t=t_n^{\epsilon},x=x_m}\partial_t^{(\eta)}\partial_x^{(l)}\hat{v}_{n,m}^{\epsilon}(t_n^{\epsilon},x_m) \\
+\sum_{e=0}^{\epsilon-2}\sum_{\eta=0}^{N-1}\int_{t_\eta^e}^{t_{\eta+1}^e}\sum_{l=0}^{j}&\binom{j}{l}\partial_t^{(i)}\left(\partial_x^{(j-l)}K_2(t_n^{\epsilon},x_m,s)\right)\partial_x^{(l)}v_{\eta,m}^{e}(s,x_m)\,ds \\
+\sum_{\eta=0}^{n-1}\int_{t_\eta^{\epsilon-1}}^{t_{\eta+1}^{\epsilon-1}}\sum_{l=0}^{j}&\binom{j}{l}\partial_t^{(i)}\left(\partial_x^{(j-l)}K_2(t_n^{\epsilon},x_m,s)\right)\partial_x^{(l)}v_{\eta,m}^{\epsilon-1}(s,x_m)\,ds \\
+\sum_{q=0}^{i-1}\sum_{l=0}^{j}\sum_{\eta=0}^{q}&\binom{j}{l}\binom{q}{\eta}\partial_t^{(q-\eta)}\left[\partial_t^{(i-1-q)}\Big|_{s=t-\tau}\left(\partial_x^{(j-l)}K_2(t_n^{\epsilon},x_m,s)\right)\right]\partial_t^{(\eta)}\partial_x^{(l)}v_{n,m}^{\epsilon-1}(t_n^{\epsilon}-\tau,x_m)
\end{aligned}
$$

2.3 Error Analysis

We require the following lemma to establish the convergence of the proposed method. We will work within the space $L^\infty(D)$ equipped with the norm

$$\left\|\varphi\right\|_{L^\infty(D)} = \inf\left\{C\in\mathbb{R} : \left|\varphi(t,x)\right| \le C \text{ for a.e. } (t,x)\in D\right\} < \infty.$$

Lemma 2.3.1 Let g, Φ, A, K_1, and K_2 be p-times continuously differentiable on their respective domains. Then there exists a positive constant $\alpha(p)$ such that for all $n = 0,\ldots,N-1$, $m = 0,\ldots,M-1$, $\epsilon = 0,\ldots,r-1$, and $i+j = 0,\ldots,p$, we have

$$\left\|\frac{\partial^{i+j}\hat{v}_{n,m}^{\epsilon}}{\partial t^i\partial x^j}\right\|_{L^\infty(D_{n,m}^{\epsilon})} \le \alpha(p).$$

Proof. Let $a_{\epsilon,n}^{i+j} = \|\frac{\partial^{i+j}\hat{v}_{n,m}^{\epsilon}}{\partial t^i \partial x^j}\|_{L^\infty(D_{n,m}^{\epsilon})}$, the proof is organized into three steps

Step 1. we have for all $i+j = 0,1,...,p$,

$$a_{0,0}^{i+j} \le \max\left\{\left\|\frac{\partial^{i+j}u}{\partial t^i \partial x^j}\right\|_{L^\infty(D_{0,m}^0)}, i+j = 0,1,...,p, \quad m = 0,1,...,M-1\right\} = \alpha_1(p). \tag{2.10}$$

Step 2. from (2.6), we have for all $n = 1,\dots,N-1$, and $i+j = 0,1,...,p$.

$$\begin{aligned} a_{0,n}^{i+j} \le c_1 &+ c_2 \sum_{\eta=0}^{n-1}\sum_{l=0}^{j}\sum_{\alpha+\beta=l}^{p-1}\int_{t_\eta^0}^{t_{\eta+1}^0} a_{0,\eta}^{\alpha+\beta}\,ds \\ &+ c_3\sum_{q=0}^{i-1}\sum_{l=0}^{j}\sum_{s=0}^{q} a_{0,n}^{s+l} + c_2\sum_{l=0}^{j}\int_{t_n^0}^{t} a_{0,n}^{l}\,ds \end{aligned}$$

where, for all $i+j = 0,1,\dots,p$, and

$$K = \max\left\{K_{1L^\infty(D)}, K_{2L^\infty(D)}\right\},$$

$$c_1 = \max\left\{\left\|\frac{\partial^{i+j}g(t,x)}{\partial t^i \partial x^j}\right\|_{L^\infty(D)} + \left\|\frac{\partial^i}{\partial t^i}\frac{\partial^j}{\partial x^j}\left[\int_{t-\tau}^{t} K(t,x,s)\Phi(s,x)\,ds\right]\right\|_{L^\infty(D)} + \left\|\partial_t^{(i)}\partial_x^{(j)}(A(t,x)\Phi(t-\tau,x))\right\|_{L^\infty(D)}\right\}.$$

$$c_2 = \max\left\{\left\|\binom{j}{l}\frac{\partial^i}{\partial t^i}\left(\frac{\partial^{j-l}}{\partial x^{j-l}}\left[K(t,x,s)\right]\right)\frac{1}{\alpha!\beta!}(t-t_\eta^0)(x-x_m)\right\|_{L^\infty(D)}; l = 0,...,j.\right\},$$

$$c_3 = \max\left\{\begin{array}{c}\binom{j}{l}\binom{q}{s}\left\|\frac{\partial^{q-s}}{\partial t^{q-s}}\left[\frac{\partial^{i-1-q}}{\partial t^{i-1-q}}\Big|_{s=t}\left(\frac{\partial^{j-l}}{\partial x^{j-l}}\left[K(t,x,s)\right]\right)\right]\right\|_{L^\infty(D)} \\ l = 0,...,j; q = 0,...,i-1; s = 0,...,q\end{array}\right\},$$

The constants $c_i, i = \{1,2,3\}$, are positive and independent of N and M, hence,

$$
\begin{aligned}
a_{0,n}^{i+j} &\leq c_1 + c_2 h \sum_{\eta=0}^{n-1} \sum_{l=0}^{j} \sum_{\alpha+\beta=l}^{p-1} a_{0,\eta}^{\alpha+\beta} + c_3 p \sum_{s=0}^{i-1} \sum_{l=0}^{j} a_{0,n}^{s+l} + c_2 h \sum_{l=0}^{j} a_{0,n}^{0+l} \\
&\leq c_1 + pc_2 h \sum_{\eta=0}^{n-1} \sum_{\alpha+\beta=0}^{p-1} a_{0,\eta}^{\alpha+\beta} + c_4 \sum_{s+l=0}^{i+j-1} a_{0,n}^{s+l},
\end{aligned}
$$

where $c_4 = c_3 p + c_2 \tau$.

$$
a_{0,n}^{i+j} \leq \underbrace{c_1 + pc_2 h \sum_{\eta=0}^{n-1} \sum_{\alpha+\beta=0}^{p-1} a_{0,\eta}^{\alpha+\beta}}_{\text{denoted as } P_0} + c_4 \sum_{s+l=0}^{i+j-1} a_{0,n}^{s+l}.
$$

Consider the sequence $\varepsilon_{i+j} = a_{0,n}^{i+j}$,

$$
\varepsilon_{i+j} \leq P_0 + c_4 \sum_{s+l=0}^{i+j-1} \varepsilon_{l+s}.
$$

By applying Lemma 1.6.1, we get

$$
\begin{aligned}
\varepsilon_{i+j} &\leq P_0 \exp\left(\sum_{s+l=0}^{i+j-1} c_4 \right) \\
&\leq P_0 \underbrace{\exp(c_4 (p-1)^2)}_{c_5}.
\end{aligned}
$$

$$
\begin{aligned}
a_{0,n}^{i+j} &\leq \underbrace{c_1 c_5}_{c_6} + \underbrace{pc_2 c_5}_{c_7} h \sum_{\eta=0}^{n-1} \sum_{\alpha+\beta=0}^{p-1} a_{0,\eta}^{\alpha+\beta} \\
&\leq c_6 + c_7 h \sum_{\eta=0}^{n-1} \sum_{\alpha+\beta=0}^{p-1} a_{0,\eta}^{\alpha+\beta}.
\end{aligned}
\tag{2.11}
$$

Consider $y_n = \sum_{i+j=0}^{p-1} a_{0,n}^{i+j}$. From (2.11), we find:

$$y_n \leq pc_6 + pc_7 h \sum_{\eta=0}^{n-1} y_\eta.$$

Using Lemma 1.6.1 we obtain

$$a_{0,n}^{i+j} \leq y_n \leq \underbrace{pc_6 exp(pc_7\tau)}_{\alpha_2(p)} \tag{2.12}$$

Step 3. from (2.9), for all $n = 0, \dots N-1, m = 0, \dots, M-1$, and $\epsilon = 1, \dots, r-1, i+j = 0, \dots, p$.

$$\begin{aligned} a_{\epsilon,n}^{i+j} \leq c_1 + c_9 \sum_{l=0}^{j}\sum_{s=0}^{i}\sum_{\alpha+\beta=l+s}^{p-1} a_{\epsilon-1,n}^{\alpha+\beta} + c_2 \sum_{e=0}^{\epsilon-1}\sum_{\eta=0}^{N-1}\int_{t_\eta^e}^{t_{\eta+1}^e}\sum_{l=0}^{j}\sum_{\alpha+\beta=l}^{p-1} a_{e,\eta}^{\alpha+\beta} ds + c_2 \sum_{\eta=0}^{n-1}\int_{t_\eta^\epsilon}^{t_{\eta+1}^\epsilon}\sum_{l=0}^{j}\sum_{\alpha+\beta=l}^{p-1} a_{\epsilon,\eta}^{\alpha+\beta} ds \\ + c_3 \sum_{q=0}^{i-1}\sum_{l=0}^{j}\sum_{s=0}^{q} a_{\epsilon,n}^{s+l} + c_2 \sum_{l=0}^{j}\int_{t_n^\epsilon}^{t} a_{\epsilon,n}^{0+l} ds + c_2 \sum_{e=0}^{\epsilon-2}\sum_{\eta=0}^{N-1}\int_{t_\eta^e}^{t_{\eta+1}^e}\sum_{l=0}^{j}\sum_{\alpha+\beta=l}^{p-1} a_{e,\eta}^{\alpha+\beta} ds \\ + c_2 \sum_{\eta=0}^{n-1}\int_{t_\eta^{\epsilon-1}}^{t_{\eta+1}^{\epsilon-1}}\sum_{l=0}^{j}\sum_{\alpha+\beta=l}^{p-1} a_{\epsilon-1,\eta}^{\alpha+\beta} ds + c_8 \sum_{q=0}^{i-1}\sum_{l=0}^{j}\sum_{s=0}^{q}\sum_{\alpha+\beta=l+s}^{p-1} a_{\epsilon-1,n}^{\alpha+\beta} + c_2 \int_{t_n^{\epsilon-1}}^{t-\tau}\sum_{l=0}^{j}\sum_{\alpha+\beta=l}^{p-1} a_{\epsilon-1,n}^{\alpha+\beta} ds \end{aligned}$$

where, for all $i+j = 0, 1, ..., p$,

$$c_8 = \max\left\{ \begin{array}{c} \binom{j}{l}\binom{q}{s}\left\| \frac{\partial^{q-s}}{\partial t^{q-s}}\left[\frac{\partial^{i-1-q}}{\partial t^{i-1-q}}\bigg|_{s=t-\tau}\left(\frac{\partial^{j-l}}{\partial x^{j-l}}\left[K(t,x,s)\right]\right)\right]\right\|_{L^\infty(D)} \\ l = 0, ..., j; q = 0, ..., i-1; s = 0, ..., q \end{array} \right\},$$

$$c_9 = \max\left\{ \left\| \binom{j}{l}\binom{i}{s} \partial_t^{(i-\eta)}\left(\partial_x^{(j-l)} A(t,x)\right)\right\|_{L^\infty(D)}; l = 0, ..., j; s = 0, ..., i. \right\},$$

the constants $c_i, i = \{8, 9\}$, is positive and independent of N and M, consider $\xi_\epsilon = \max\{a_{\epsilon,\eta}^{\alpha+\beta}, \alpha+\beta = 0, \dots p, \eta = 0 \dots N-1\}$

hence,

$$a^{i+j}_{\epsilon,n} \leq c_1 + c_9 p^2 \sum_{\alpha+\beta=0}^{p-1} a^{\alpha+\beta}_{\epsilon-1,n} + p^3 c_2 \tau \sum_{e=0}^{\epsilon-1} \xi_e + pc_2 h \sum_{\eta=0}^{n-1} \sum_{\alpha+\beta=0}^{p-1} a^{\alpha+\beta}_{\epsilon,\eta} + c_3 p^2 \sum_{s+l=0}^{i+j-1} a^{s+l}_{\epsilon,n} + c_2 h \sum_{l=0}^{j} a^{0+l}_{\epsilon,n}$$
$$+ p^3 c_2 \tau \sum_{e=0}^{\epsilon-2} \xi_e + pc_2 \tau \sum_{\alpha+\beta=0}^{p-1} \xi_{\epsilon-1} + c_8 p^5 \xi_{\epsilon-1} + p^3 c_2 h \xi_{\epsilon-1}$$
$$a^{i+j}_{\epsilon,n} \leq c_{10} + c_{11} \sum_{e=0}^{\epsilon-1} \xi_e + pc_2 h \sum_{\eta=0}^{n-1} \sum_{\alpha+\beta=0}^{p-1} a^{\alpha+\beta}_{\epsilon,\eta} + (c_2\tau + c_3 p) \sum_{\eta+l=0}^{i+j-1} a^{\eta+l}_{\epsilon,n}$$

where $c_{10} = c_8 p^5 + 2pc_2 h + c_9 p^2$, $c_{11} = c_3 p^2 + pc_2 h$, and $c_{12} = c_2\tau + pc_3$,

$$a^{i+j}_{\epsilon,n} \leq \underbrace{c_{10} + c_{11} \sum_{e=0}^{\epsilon-1} \xi_e + pc_2 h \sum_{\eta=0}^{n-1} \sum_{\alpha+\beta=0}^{p-1} a^{\alpha+\beta}_{\epsilon,\eta}}_{P_0} + c_{12} \sum_{\eta+l=0}^{i+j-1} a^{\eta+l}_{\epsilon,n}$$

consider the sequence $\varepsilon_{i+j} = a^{i+j}_{\epsilon,n}$,

$$\varepsilon_{i+j} \leq P_0 + c_{12} \sum_{\eta+l=0}^{i+j-1} \varepsilon_{\eta+l},$$

by applying Lemma 1.6.1 we get

$$\varepsilon_{i+j} \leq P_0 exp \sum_{\eta+l=0}^{i+j-1} c_{12},$$
$$\leq P_0 \underbrace{exp(c_{12}.(p-2))}_{c_{13}}$$

$$a^{i+j}_{\epsilon,n} \leq c_{13}.c_{10} + c_{13}.c_{11} \sum_{e=0}^{\epsilon-1} \xi_e + c_{13} pc_2 h \sum_{\eta=0}^{n-1} \sum_{\alpha+\beta=0}^{p-1} a^{\alpha+\beta}_{\epsilon,\eta} \tag{2.13}$$

consider $\quad y_n = \sum_{\alpha+\beta=0}^{p-1} a_{\epsilon,n}^{\alpha+\beta}$ From (2.13) we find:

$$y_n \leq \underbrace{p^2 c_{13}.c_{10} + c_{13}p^3 c_2 h \sum_{\eta=0}^{n-1} y_\eta}_{c_{14}} + \underbrace{p^2 c_{13}.c_{11}}_{c_{15}} \sum_{e=0}^{\epsilon-1} \xi_e$$

$$a_{\epsilon,n}^{i+j} \leq y_n \leq c_{14} + c_{15} \sum_{e=0}^{\epsilon-1} \xi_e$$

we deduce for all $\epsilon \in \{0,\ldots,r-1\}$

$$\xi_\epsilon \leq c_{14} + c_{15} \sum_{e=0}^{\epsilon-1} \xi_e$$

Using Lemma 1.6.1 and $\forall\epsilon, \forall n, \forall i+j$ we obtain :

$$a_{\epsilon,n}^{i+j} \leq \xi_\epsilon \leq \underbrace{c_{14}.exp(rc_{15})}_{\alpha_3(p)} \tag{2.14}$$

Hence, from (2.10), (2.12) and (2.14), by setting :

$$\alpha(p) = \max\{\alpha_1(p), \alpha_2(p), \alpha_3(p)\}$$

The proof of Lemma 2.3.1 is completed. ■

The convergence of the suggested method is established by the following theorem.

Theorem 2.3.1 Suppose that g, Φ, A, K_1, and K_2 are p-times continuously differentiable on their respective domains. Then the equations (3.4), (2.4), (2.7) uniquely define an approximation $v \in S_{p-1}^{(-1)}(\Pi_{N,M}^{\epsilon})$. Furthermore, the associated error function $E = u - v$ satisfies the following bound:

$$||E||_{L^\infty(D)} \leq C(h+k)^p,$$

where C is a finite constant independent of h and k.

Proof. Define the error function E on $D^{\epsilon}_{n,m}$ as

$$E^{\epsilon}_{n,m} = u - v^{\epsilon}_{n,m}, \quad \forall n \in \{0, \dots, N-1\},\ m \in \{0, \dots, M-1\},\ \epsilon \in \{0, \dots, r-1\}.$$

The proof is split into three steps.

Step 1. There exists a constant C_1 independent of h and k such that,

$$||E^0_{0,m}||_{L^\infty(D^0_{0,m})} \leq C_1(h+k)^p.$$

Let $(t, x) \in D^{\epsilon}_{n,m}$, $\epsilon = 0$ and $n = 0, m \in \{0, \dots, M-1\}$
by using Theorem 1.6.3 and equation (3.4), we write

$$|E^0_{0,m}(t,x)| \leq \sum_{i+j=p} \frac{1}{i!j!} \left\| \partial^{(i)}_t \partial^{(j)}_x u(0, x_m) \right\| h^i (t - t^0_m)^j.$$

According to Lemma 2.3.1, this implies

$$|E^0_{0,m}(t,x)| \leq \alpha(p) \sum_{i+j=p} \frac{1}{i!j!} h^i k^j = \frac{\alpha(p)}{p!}(h+k)^p.$$

Define $C_1 = \frac{\alpha(p)}{p!}$. Thus,

$$||E^0_{0,m}|| \leq C_1(h+k)^p.$$

Step 2. There exists a constant C_2 independent of h and k such that,

$$||E^0_{n,m}||_{L^\infty(D^0_{n,m})} \leq C_2(h+k)^p,$$

For all $\epsilon = 0$ and $n \in \{1, \dots, N-1\}, m \in \{0, \dots, M-1\}$.

Let $(t,x) \in D^0_{n,m}$, we have from (2.5)

$$u(t,x) - \hat{v}^0_{n,m}(t,x) = \sum_{\eta=0}^{n-1} \int_{t^0_\eta}^{t^0_{\eta+1}} K_1(t,x,s)(u(s,x) - v^0_{\eta,m}(s,x))\,ds$$
$$+ \int_{t^0_n}^{t} K_1(t,x,s)(u(s,x) - \hat{v}^0_{n,m}(s,x))\,ds.$$

Taking the norm, we get

$$\|u - \hat{v}^0_{n,m}\| \le \sum_{\eta=0}^{n-1} hK\|E^0_{\eta,m}\| + K\int_{t^0_n}^{t} \|u - \hat{v}^0_{n,m}\|\,dt,$$

where $K = \max\{\|K_1\|, \|K_2\|\}$. by Lemma 1.6.2,

$$\|u(t,x) - \hat{v}^0_{n,m}(t,x)\| \le \sum_{\eta=0}^{n-1} hK\|E^0_{\eta,m}\|_{L^\infty(D^0_{\eta,m})} \exp{(K\tau)}$$

$$\|u - \hat{v}^0_{n,m}\| \le \sum_{\eta=0}^{n-1} hd_1\|E^0_{\eta,m}\|, \quad \text{where } d_1 = K\exp(K\tau).$$

On the other hand,

$$\|E^0_{n,m}\| \le \|u - \hat{v}^0_{n,m}\| + \|\hat{v}^0_{n,m} - v^0_{n,m}\|.$$
$$\le \sum_{\eta=0}^{n-1} hd_1\|E^0_{\eta,m}\|_{L^\infty(D^0_{\eta,m})} + \sum_{i+j=p} \frac{1}{i!j!}\left\|\frac{\partial^{i+j}\hat{v}^0_{n,m}}{\partial t^i \partial x^j}\right\| h^i k^j.$$

Substituting the bounds and applying Lemma 2.3.1 and Lemma 1.6.1,we get

$$\|E^0_{n,m}\|_{L^\infty(D^0_{n,m})} \le \sum_{\eta=0}^{n-1} hd_1\|E^0_{\eta,m}\|_{L^\infty(D^0_{\eta,m})} + \frac{\alpha(p)}{p!}(h+k)^p.$$

$$\|E^0_{n,m}\| \le \frac{\alpha(p)}{p!}(h+k)^p \exp(\tau d_1).$$

Define $C_2 = \frac{\alpha(p)}{p!}\exp(\tau d_1)$.

Step 3. There exists a constant C_3 independent of h and k such that,

$$\|E^{\epsilon}_{n,m}\|_{L^{\infty}(D^{\epsilon}_{n,m})} \leq C_3(h+k)^p,$$

$$\forall n \in \{0,\dots,N-1\},\ m \in \{0,\dots,M-1\},\ \epsilon \in \{0,\dots,r-1\}.$$

Let $(t,x) \in D^{\epsilon}_{n,m}$, we have from (2.7)

$$\begin{aligned}
u(t,x) - \hat{v}^{\epsilon}_{n,m}(t,x) &\leq A(t,x)(u(t,x) - v^{\epsilon-1}_{n,m}(t-\tau,x)) \\
&\quad + \sum_{e=0}^{\epsilon-1}\sum_{\eta=0}^{N-1}\int_{t^e_\eta}^{t^e_{\eta+1}} K_1(t,x,s)(u(t,x) - v^{e}_{\eta,m}(s,x))ds + \sum_{\eta=0}^{n-1}\int_{t^\epsilon_\eta}^{t^\epsilon_{\eta+1}} K_1(t,x,s)(u(t,x) - v^{\epsilon}_{\eta,m}(s,x))ds \\
&\quad + \int_{t^\epsilon_n}^{t} K_1(t,x,s)\hat{(}u(t,x) - v^{\epsilon}_{n,m}(s,x))ds \\
&\quad + \sum_{e=0}^{\epsilon-2}\sum_{\eta=0}^{N-1}\int_{t^e_\eta}^{t^e_{\eta+1}} K_2(t,x,s)(u(t,x) - v^{e}_{\eta,m}(s,x))ds + \sum_{\eta=0}^{n-1}\int_{t^{\epsilon-1}_\eta}^{t^{\epsilon-1}_{\eta+1}} K_2(t,x,s)(u(t,x) - v^{\epsilon-1}_{\eta,m}(s,x))ds \\
&\quad + \int_{t^{\epsilon-1}_n}^{t-\tau} K_2(t,x,s)(u(t,x) - v^{\epsilon-1}_{n,m}(s,x))ds. \\
&\leq \quad A(t,x)E^{\epsilon-1}_{n,m} \\
&\quad + \sum_{e=0}^{\epsilon-1}\sum_{\eta=0}^{N-1}\int_{t^e_\eta}^{t^e_{\eta+1}} K_1(t,x,s)E^{e}_{\eta,m}(s,x)ds + \sum_{\eta=0}^{n-1}\int_{t^\epsilon_\eta}^{t^\epsilon_{\eta+1}} K_1(t,x,s)(u(t,x) - v^{\epsilon}_{\eta,m}(s,x))ds \\
&\quad + \int_{t^\epsilon_n}^{t} K_1(t,x,s)\hat{(}u(t,x) - v^{\epsilon}_{n,m}(s,x))ds + \sum_{e=0}^{\epsilon-2}\sum_{\eta=0}^{N-1}\int_{t^e_\eta}^{t^e_{\eta+1}} K_2(t,x,s)E^{e}_{\eta,m}(s,x)ds \\
&\quad + \int_{t^{\epsilon-1}_n}^{t-\tau} K_2(t,x,s)E^{\epsilon-1}_{n,m}(s,x)ds
\end{aligned}$$

hence

$$\begin{aligned}\|u(t,x)-\hat{v}^{\epsilon}_{n,m}(t,x)\| \quad &\leq A(t,x)E^{\epsilon-1}_{n,m}+Kh\sum_{e=0}^{\epsilon-1}\sum_{\eta=0}^{N-1}\|E^{e}_{\eta,m}\|_{L^\infty(D^{e}_{\eta,m})}\\ &\quad+K\sum_{\eta=0}^{n-1}h\left\|u-v^{\epsilon}_{\eta,m}\right\|_{L^\infty(D^{\epsilon}_{\eta,m})}+K\int_{t^{\epsilon}_{n}}^{t}\left\|u(t,x)-\hat{v}^{\epsilon}_{n,m}(s,x)\right\|ds\\ &\leq \|A(t,x)\|\|E^{\epsilon-1}_{n,m}\|+Kh\sum_{e=0}^{\epsilon-1}\sum_{\eta=0}^{N-1}\|E^{e}_{\eta,m}\|_{L^\infty(D^{e}_{\eta,m})}\\ &\quad+K\sum_{\eta=0}^{n-1}h\left\|u-\hat{v}^{\epsilon}_{\eta,m}\right\|_{L^\infty(D^{\epsilon}_{\eta,m})}+\tau K\sum_{i+j=p}\frac{1}{i!j!}\left\|\frac{\partial^{i+j}\hat{v}^{\epsilon}_{n,m}}{\partial t^{i}\partial x^{j}}\right\|\\ &\quad+K\int_{t^{\epsilon}_{n}}^{t}\left\|u(t,x)-\hat{v}^{\epsilon}_{n,m}(s,x)\right\|ds\end{aligned}$$

$$\|u-\hat{v}^{\epsilon}_{n,m}\|\leq\|A\|\sum_{e=0}^{\epsilon-1}\|E^{e}\|+Kh\sum_{\eta=0}^{n-1}\|E^{\epsilon}_{\eta,m}\|+K\int_{t^{\epsilon}_{n}}^{t}\|u-\hat{v}^{\epsilon}_{n,m}\|\,ds.$$

According to Lemma 1.6.2, this implies

$$\|u-\hat{v}^{\epsilon}_{n,m}\|\leq\widehat{A}\sum_{e=0}^{\epsilon-1}\|E^{e}\|+\widehat{K}h\sum_{\eta=0}^{n-1}\|E^{\epsilon}_{\eta,m}\|,$$

where $\widehat{A}=\|A\|\exp(K\tau)$ and $\widehat{K}=K\exp(K\tau)$.

On the other hand,

$$\begin{aligned}\|E^{\epsilon}_{n,m}\|_{L^\infty(D^{\epsilon}_{n,m})}\leq&\|u-\hat{v}^{\epsilon}_{n,m}\|_{L^\infty(D^{\epsilon}_{n,m})}+\|\hat{v}^{\epsilon}_{n,m}-v^{\epsilon}_{n,m}\|_{L^\infty(D^{\epsilon}_{n,m})}\\ \leq&\widehat{A}\sum_{e=0}^{\epsilon-1}\|E^{e}\|+\widehat{K}h\sum_{\eta=0}^{n-1}\|E^{\epsilon}_{\eta,m}\|+\sum_{i+j=p}\frac{1}{i!j!}\left\|\frac{\partial^{i+j}\hat{v}^{\epsilon}_{n,m}}{\partial t^{i}\partial x^{j}}\right\|.\end{aligned}$$

Applying Lemma 2.3.1, we have

$$\|E^{\epsilon}_{n,m}\|\leq\widehat{A}\sum_{e=0}^{\epsilon-1}\|E^{e}\|+\widehat{K}h\sum_{\eta=0}^{n-1}\|E^{\epsilon}_{\eta,m}\|+\frac{\alpha(p)}{p!}(h+k)^{p}.$$

consider

$$||E^{\epsilon}|| = \max ||E^{\epsilon}_{n,m}||$$

Applying Lemma 1.6.1, we have

$$||E^{\epsilon}|| \leq \exp(\widehat{K}\tau)\frac{\alpha(p)}{p!}(h+k)^p + \exp(\widehat{K}\tau)\widehat{A}\sum_{e=0}^{\epsilon-1}||E^{e}||.$$

Using Lemma 1.6.1 again, we obtain

$$||E^{\epsilon}|| \leq \underbrace{\exp(\widehat{K}\tau)\frac{\alpha(p)}{p!}\exp\Big((r-1)\exp(\widehat{K}\tau)\widehat{A}\Big)}_{C_3(p)}(h+k)^p.$$

Finally, setting $C = \max\{C_1(p), C_2(p), C_3(p)\}$, we conclude

$$||E^{\epsilon}_{n,m}|| \leq C(h+k)^p, \quad \forall n \in \{0,\dots,N-1\},\ m \in \{0,\dots,M-1\},\ \epsilon \in \{0,\dots,r-1\}.$$

Thus, the proof is complete. ■

2.4 Experimental Results

In this section, we validate the theoretical results derived in the previous section through numerical examples. The exact solutions for all examples are known in advance. For each example, we compute the error between the exact solution u and the Taylor collocation solution v.

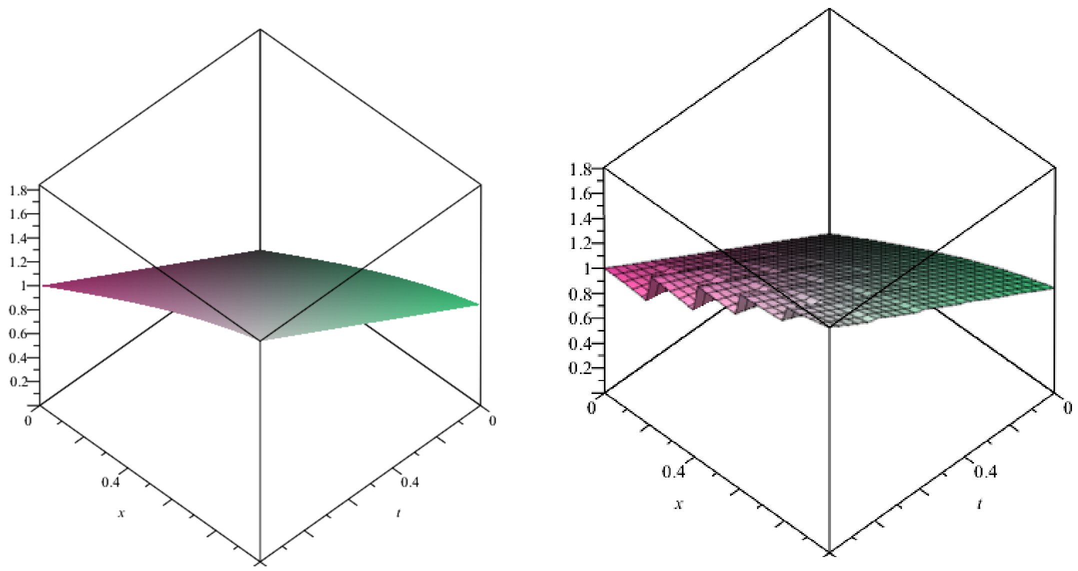
Example 2.4.1 Consider the DVIEs with a time delay $\tau = \frac{1}{5}$ of the form

$$u(t,x) = \begin{cases} g(t,x) + (t+x)u(t-\frac{1}{5},x) + \displaystyle\int_0^t s\cos(xt)u(s,x)ds \\ + \displaystyle\int_0^{t-\frac{1}{5}} (2\sin(t+x)+s)u(s,x)ds, \quad t \in [0,1], \quad x \in [0,1], \\ \Phi(t,x), \quad t \in [-\frac{1}{5},0]. \end{cases}$$

The exact solution of this problem is given by $u(t,x) = t + \sin(x)$. The function $g(t,x)$ is then computed using this exact solution. Numerical results were obtained by applying the proposed Taylor collocation method for different numbers of collocation points. The absolute error values are reported in Table 2.1 and Fig. 2.2, showing that the absolute errors decrease as the number of collocation points increases. The exact and approximate solutions for $N = M = 5$, $p = 3$ are shown in Fig. 2.1.

$x \backslash t$	0	0.2	0.4	0.6	0.8	1
0	0	1.32×10^{-12}	1.15×10^{-11}	2.36×10^{-11}	2.92×10^{-10}	8.23×10^{-10}
0.2	3.8×10^{-14}	6.54×10^{-11}	2.30×10^{-11}	1.50×10^{-10}	3.45×10^{-10}	9.21×10^{-10}
0.4	2.66×10^{-11}	7.81×10^{-12}	7.20×10^{-11}	4.01×10^{-10}	1.24×10^{-9}	1.18×10^{-10}
0.6	4.38×10^{-11}	3.29×10^{-11}	1.99×10^{-10}	1.48×10^{-10}	3.99×10^{-10}	2.79×10^{-10}
0.8	8.73×10^{-11}	3.45×10^{-10}	2.61×10^{-10}	2.59×10^{-10}	1.00×10^{-9}	5.41×10^{-9}

Table 2.1 Numerical results for Example 2.4.1.



(a) Exact solution (b) Approximate solution

Figure 2.1: Exact and approximate solution of Example 2.4.1 for $N = M = 5$, $p = 3$.

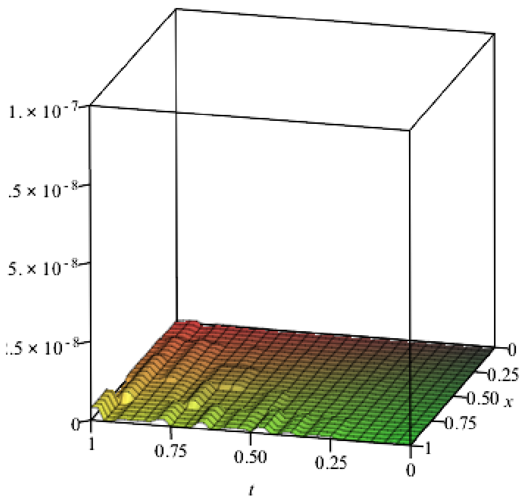


Figure 2.2: Absolute errors function of Example 2.4.1 for $N = M = 5$, $p = 3$.

Example 2.4.2 Let us dedicate the first example to the case that the desired equation is of form

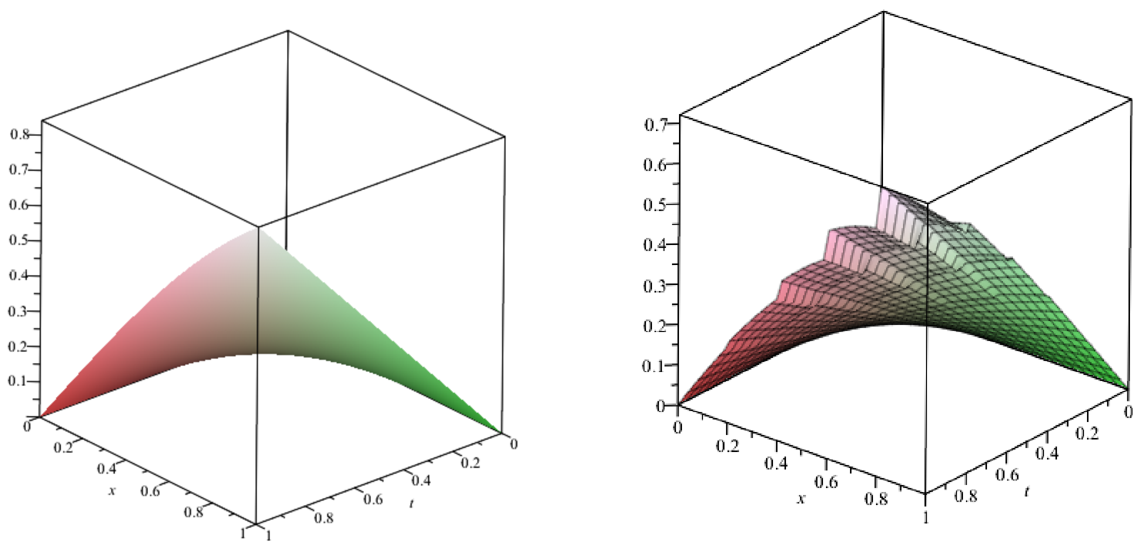
$$u(t,x) = \begin{cases} g(t,x) + (t.x)u(t-\frac{1}{5},x) + \displaystyle\int_0^t x.t.\cos(s).u(s,x)ds \\ + \displaystyle\int_0^{t-\frac{1}{5}} x.e^{(s-t)}(s,x)ds, \quad t \in [0,1], \quad x \in [0,1] \\ \qquad \Phi(t,x) \qquad\qquad t \ \in \ [-\frac{1}{5},0] \end{cases}$$

The exact solution of this problem is given by $u(t,x) = t\sin(x)$. The function $g(t,x)$ is then computed using this exact solution. Numerical results are obtained by applying the proposed Taylor collocation method for different numbers of collocation points. The absolute errors are reported in Table 2.2. The results show that the absolute errors decrease as the number of collocation points increases.

To illustrate the behavior of the approximate solution, the absolute error function is depicted in Fig. 2.4 for $0 \leq t \leq 1$ and $0 \leq x \leq 1$ with $N = M = 5$. The exact and approximate solutions for $N = M = 5$ are shown in Fig. 2.3.

$x \backslash t$	0	0.2	0.4	0.6	0.8	1
0	0	0	0	0	0	0
0.2	0	1.90×10^{-11}	7.72×10^{-12}	3.28×10^{-11}	8.86×10^{-11}	5.00×10^{-11}
0.4	0	1.05×10^{-12}	8.22×10^{-11}	3.79×10^{-11}	3.43×10^{-11}	1.67×10^{-10}
0.6	0	4.17×10^{-11}	1.45×10^{-10}	7.09×10^{-11}	4.69×10^{-11}	1.64×10^{-10}
0.8	0	8.48×10^{-12}	2.69×10^{-11}	3.68×10^{-11}	1.75×10^{-10}	9.19×10^{-11}

Table 2.2: Numerical results for Example 2.4.2.



(a) Exact solution (b) Approximate solution

Figure 2.3: Exact and approximate solution of Example 2.4.2 for $N = M = 5$.

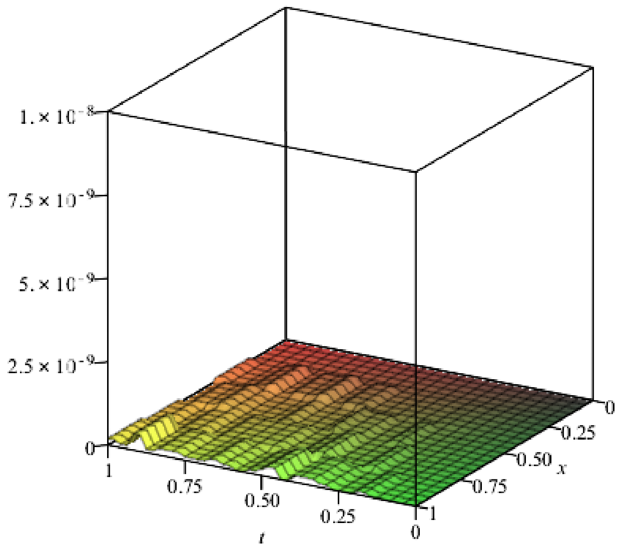


Figure 2.4: Absolute errors function of Example 2.4.2 for N=M=5.

Example 2.4.3 *Consider the linear two-dimensional Volterra integral equation (VIE) from [17]*

$$u(x,t) = x^2\Big(-1 + e^{-t} + x^2 + e^t - x^2e^t\Big) + \int_0^t (x^2 + e^{-2s})\, u(x,s)\, ds,$$

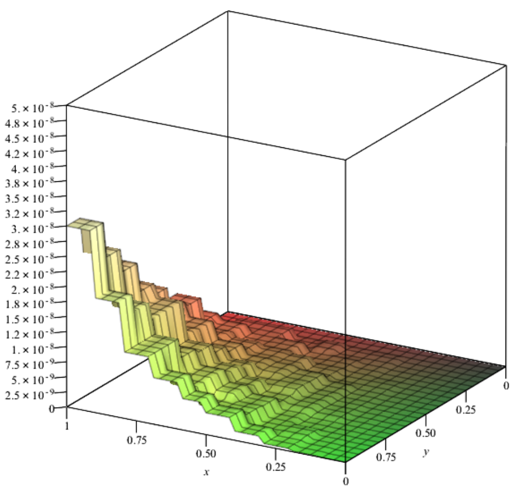
for $(x, t) \in [0, 1] \times [0, 1]$, *which has the exact solution*

$$u(x, t) = x^2 e^t.$$

The numerical results for $p - 1 = 2, 4, 6$ *with* $h = k = 0.1$, *obtained using the Taylor collocation method (TCM), are compared with the numerical results obtained using a method based on expanding the solution in terms of bivariate shifted Legendre polynomials [17]. The comparison is presented in Table 2.3.*

(x,t)	Present method (TCM) $p-1=2$	$p-1=4$	$p-1=6$	Method in Ref. [17] $M=2$	$M=4$	$M=6$
$(0,0)$	0	0	0	0	0	0
$(0.1,0.1)$	3.6×10^{-8}	7.3×10^{-12}	1.4×10^{-11}	2.9×10^{-5}	2.0×10^{-7}	1.3×10^{-9}
$(0.2,0.2)$	3.0×10^{-7}	9.8×10^{-11}	1.3×10^{-11}	1.6×10^{-4}	4.7×10^{-7}	5.8×10^{-9}
$(0.3,0.3)$	1.0×10^{-6}	2.5×10^{-10}	3.4×10^{-10}	4.9×10^{-4}	9.5×10^{-7}	1.3×10^{-8}
$(0.4,0.4)$	2.8×10^{-6}	1.1×10^{-9}	3.2×10^{-10}	5.8×10^{-4}	2.4×10^{-6}	2.3×10^{-8}
$(0.5,0.5)$	6.3×10^{-6}	2.1×10^{-9}	1.0×10^{-10}	1.5×10^{-5}	3.4×10^{-7}	3.6×10^{-8}
$(0.6,0.6)$	1.3×10^{-5}	3.9×10^{-9}	3.0×10^{-10}	1.3×10^{-3}	4.9×10^{-6}	5.2×10^{-8}
$(0.7,0.7)$	2.5×10^{-5}	7.8×10^{-9}	2.8×10^{-9}	3.0×10^{-3}	5.6×10^{-6}	7.0×10^{-8}
$(0.8,0.8)$	4.8×10^{-5}	1.5×10^{-8}	1.5×10^{-9}	3.4×10^{-3}	6.9×10^{-6}	9.1×10^{-8}
$(0.9,0.9)$	9.1×10^{-5}	3.0×10^{-8}	1.3×10^{-9}	6.2×10^{-4}	1.7×10^{-5}	1.1×10^{-7}
CPU time (sec)	56.17	253.82	997.32	/	/	/

Table 2.3 Comparison of absolute errors for Example 2.4.3.



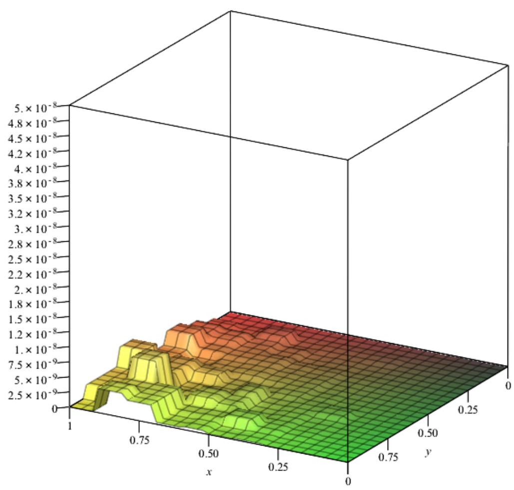


Figure 2.5: Plot of the absolute error function for Example 2.4.3 (method In Ref.[17]); left: $p = 5$, right: $p = 7$.

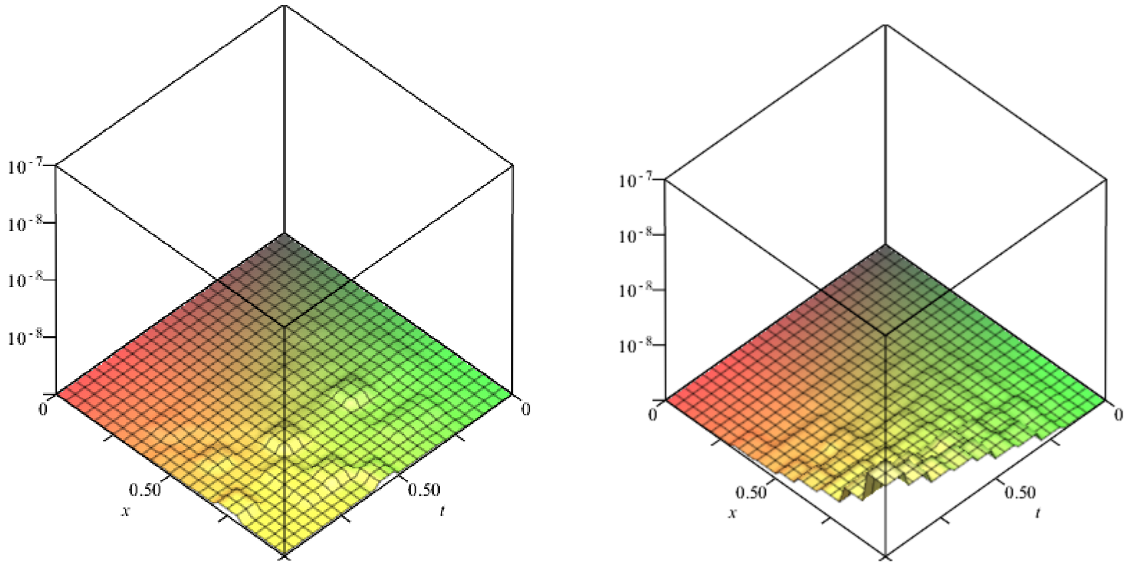


Figure 2.6: Plot of the absolute error function for Example 2.4.3 (Present method) ;$p = 7$,$p = 5$.

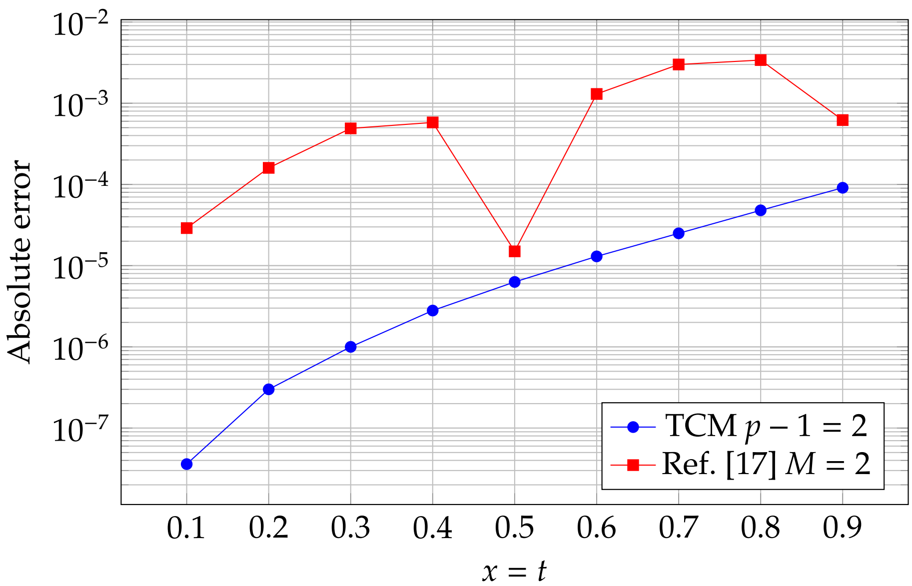


Figure 2.7: Comparison of absolute errors for $p - 1 = 2$.

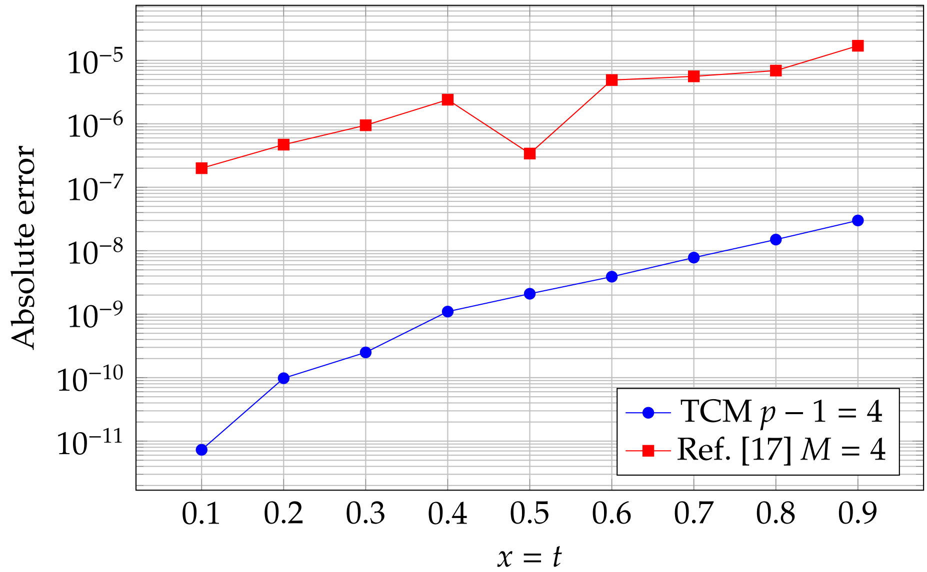


Figure 2.8: Comparison of absolute errors for $p - 1 = 4$.

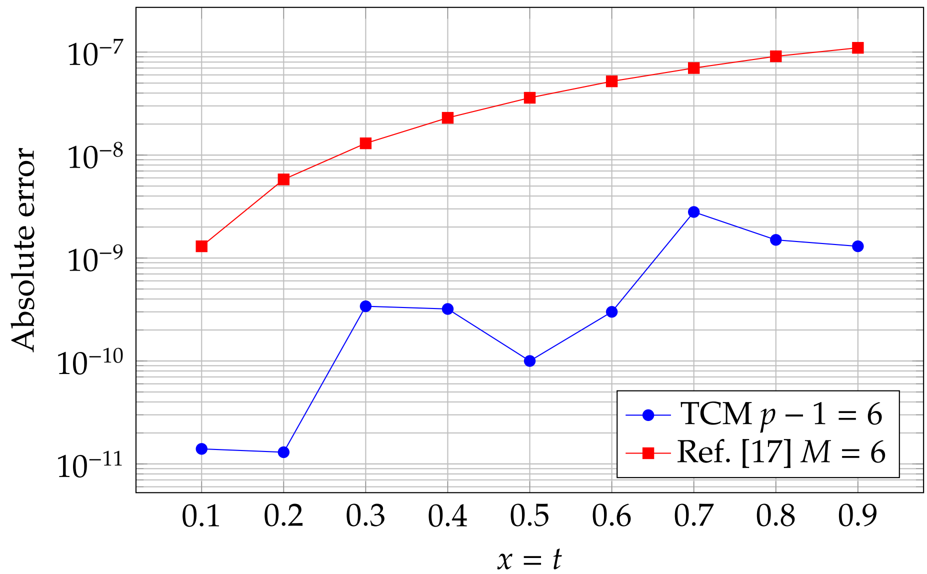


Figure 2.9: Comparison of absolute errors for $p - 1 = 6$.

2.5 Conclusion

In this chapter, we proposed a new numerical method based on Taylor polynomials to construct a collocation solution for approximating the solution of delay Volterra integral equations (DVIEs) with a spatial variable. Unlike existing approaches, our method efficiently handles both time delay and spatial dependence in the integral equation,

which is a key challenge in solving such problems
A rigorous convergence analysis confirms that the proposed method is accurate and stable, with an $O(h + k)^p$ error bound. Numerical examples validate its effectiveness, demonstrating that it provides high-precision approximations while maintaining computational efficiency.
The novelty of our approach lies in the extension of the Taylor Collocation Method to DVIEs with spatial dependence,an area that has received limited attention in previous studies. Additionally, our method is computationally efficient as it computes the approximation coefficients iteratively without requiring the solution of any system of algebraic equations. Moreover, it can be adapted to various delay integral equations encountered in applied sciences and engineering.
These contributions make our method a powerful and efficient alternative for solving DVIEs with spatial variables,laying the groundwork for further extensions to higherdimensional problems.

CHAPTER 3

TAYLOR COLLOCATION ALGORITHMS FOR 2D DELAY VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

3.1 Introduction

In this chapter, we investigate a numerical method for the linear two-dimensional delay partial Volterra integrodifferential equation (2D-DPVIDE):

$$\begin{aligned} \frac{\partial u}{\partial x}(x,y) &= g(x,y) + L(x,y)u(x,y) + M(x,y)u(x-\tau_1, y-\tau_2) \\ &\quad + \int_0^x \int_0^y K(x,y,t,s)u(t-\tau_1, s-\tau_2)\,ds\,dt, \qquad (x,y) \in D, \end{aligned} \tag{3.1}$$

where $u(x,y) = \phi(x,y)$ for $x \in [-\tau_1, 0]$ or $y \in [-\tau_2, 0]$, and the functions g, L, M, and K are given real-valued smooth functions defined, respectively, on

$$D := [0,a] \times [0,b] \subset \mathbb{R}^2,$$

and

$$S := \{(x,y,t,s) : 0 \le t \le x \le a,\ 0 \le s \le y \le b\}.$$

It follows, by analogy with the results in [12], that the linear twodimensional delay Volterra integrodifferential equation (3.1) admits a unique solution $u \in C(D)$ under sufficiently smooth data and appropriate Lipschitz-type conditions.

Twodimensional delay Volterra integrodifferential equations play an important role in modeling linear and semi-linear dynamical systems in which delays in both variables significantly affect the system behavior. Such systems arise in various applied fields, including engineering processes, control systems, and economic or physical modeling, where the future state depends not only on the current state but also on past states of the system. The presence of delays introduces additional mathematical complexity and often prevents the derivation of analytical solutions. Therefore, accurate and efficient numerical methods, such as the Taylor collocation method, are essential for approximating the solutions of these equations.

Several researchers have investigated numerical methods for solving twodimensional delay Volterra integrodifferential equations (see, for example, [4, 8, 10, 16, 2, 14, 12]). Brunner [7] presented collocation methods for Volterra integral and related functional differential equations. Brunner [8] studied the numerical solution of twodimensional Volterra integral equations using collocation and iterated collocation methods. Ghorpade and Limaye [10] provided theoretical foundations in multivariable calculus and analysis applicable to numerical methods for integral equations. McKee et al. [16] developed an Euler-type method for twodimensional Volterra integral equations of the first kind. Bellman [2] investigated the stability of solutions of linear differential equations, which supports uniqueness results for linear integral equations. Laib et al. [14] proposed a Taylor collocation method for twodimensional linear and nonlinear Volterra integral equations. Khennaoui et al. [12] extended the Taylor collocation method to twodimensional partial Volterra integrodifferential equations.

While numerous studies have focused on twodimensional Volterra integral equations [4, 8, 16, 17], relatively few works have addressed twodimensional delay Volterra integrodifferential equations with spatial variables [20, 14, 12]. The main difficulty in solving such equations lies in simultaneously handling the spatial dependence and the delays appearing in the arguments of the unknown function inside the integral terms, which distinguishes these problems from conventional twodimensional Volterra integral equations.

To address this problem, we propose a Taylor collocation method that extends the approach presented in [3, 14, 12] to approximate the solution of the linear twodimensional delay Volterra integrodifferential equation (3.1). The main contributions of this work can be summarized as follows:

1. The adaptation of the Taylor collocation method to twodimensional delay Volterra integrodifferential equations with spatial variables, which remains a relatively unexplored area in numerical analysis.

2. The development of a computationally efficient scheme in which the approximation coefficients are computed iteratively without solving a large system of algebraic equations.

3. The formulation of a general numerical framework that can be extended to higherdimensional delay integral and integrodifferential equations arising in physics, control systems, and engineering applications.

This chapter is organized as follows. In Section 3.2, we partition the domain $[0, a] \times [0, b]$ into subdomains and approximate the solution of (3.1) in each subdomain using Taylor polynomials. Section 3.3 establishes the global convergence of the proposed method. Section 2.4 presents numerical examples illustrating the accuracy and effectiveness of the method. Finally, the conclusion summarizes the main findings and outlines possible directions for future work.

3.2 Description of the Method

we suppose without loss of generality that $a = r_1\tau_1$,$b = r_2\tau_2$ where $\{r_1, r_2\} \in \mathbf{N}^*$
Let $\Pi_N^\epsilon = \{x_n^\epsilon = \epsilon\tau_1 + nh,\ n = 0, 1, ..., N,\ \epsilon = 0, 1, ..., r_1 - 1\}$
and $\Pi_M^\rho = \{y_m^\rho = \rho\tau_2 + mk,\ m = 0, 1, ..., M,\ \rho = 0, 1, 2, ...r_2 - 1\}$ denote, respectively, uniform partitions of the intervals $[0, a]$ and $[0, b]$ with the stepsizes are given by $h = \frac{\tau_1}{N}$ and $k = \frac{\tau_2}{M}$. These partitions defined a grid for D

$$\Pi_{N,M}^{\epsilon+\rho} = \Pi_N^\epsilon \times \Pi_M^\rho = \{(x_n^\epsilon, y_m^\rho),\ 0 \le n \le N,\ 0 \le m \le M,\ \epsilon = 0, 1, 2, ...r_1 - 1,\ \rho = 0, 1, 2, ...r_2 - 1\}$$

Set the subintervals

$$\sigma_n^\epsilon = [x_n^\epsilon; x_{n+1}^\epsilon],\ n = 0, 1, ..., N - 1;\quad \sigma_{N-1}^\epsilon = [x_{N-1}^\epsilon, x_N^\epsilon],\ \epsilon = 0, 1, ..., r_1 - 1$$

$$\delta_m^\rho = [y_m^\rho; y_{m+1}^\rho)],\ m = 0, 1, ..., M - 1;\quad \delta_{M-1}^\rho = [y_{M-1}^\rho, y_M^\rho],\ \rho = 0, 1, ...r_2 - 1.$$

and $D_{n,m}^{\epsilon,\rho} := \sigma_n^\epsilon \times \delta_m^\rho$ for all $n = 0, 1, ..., N - 1,\ m = 0, 1, ..., M - 1,\ \epsilon = 0, 1, ..., r_1 - 1,$ $\rho = 0, 1, ..., r_2 - 1.$
Moreover, denote by $\pi_{p,p}$ the set of all real polynomials of degree not exceeding p in x and y. We define the real polynomial spline space of degree p in x and y as follows:

$$S_{p,p}^{(0)}(\Pi_{N,M}^{\epsilon,\rho}) = \{v \in C(D) : v_{n,m}^{\epsilon,\rho} = v|_{D_{n,m}^{\epsilon,\rho}} \in \pi_{p,p}, \tag{3.2}$$

$$n = 0, ..., N-1; \;\; m = 0, 1, ..., M-1; \;\; \epsilon = 0, 1, ..., r_1 - 1; \;\; \rho = 0, 1, ..., r_2 - 1; \}. \tag{3.3}$$

This is the space of bivariate polynomial spline functions of degree (at most) p in x and y. Its dimension is $r_1 r_2 NM(p+1)^2$,i.e., the same as the total number of the coefficients of the polynomials $v_{n,m}^{\epsilon,\rho}$, $n = 0, ..., N-1$, $m = 0, 1, ..., M-1$, $\epsilon = 0, 1, ..., r_1-1$, $\rho = 0, 1, ..., r_2-1$, .To find these coefficients, we use Taylor polynomial on each rectangle.

3.2.1 Approximate solution in $D_{0,0}^{0,0}$

The polynomial $v_{0,0}^{0,0}(x, y)$ is employed to approximate $u(x, y)$ within the rectangle $D_{0,0}^{0,0}$ such that

$$v_{0,0}^{0,0}(x, y) = \sum_{i+j=0}^{p} \frac{1}{i!\,j!} \frac{\partial^{i+j} u(0, 0)}{\partial x^i \partial y^j} x^i y^j \;; \quad (x, y) \in D_{0,0}^{0,0}, \tag{3.4}$$

where $\frac{\partial^{i+j} u(0, 0)}{\partial x^i \partial y^j}$ is the exact value of $\frac{\partial^{i+j} u}{\partial x^i \partial y^j}$ at point $(0, 0)$.
To compute $\frac{\partial^{i+j} u(x, y)}{\partial x^i \partial y^j}$, we differentiate equation (3.1) j- times with respect to y and

$(i-1)$- times with respect to x, obtaining

$$
\begin{aligned}
\frac{\partial^{i+j}u(x,y)}{\partial x^i \partial y^j} &= \partial_1^{(i-1)}\partial_2^{(j)}g(x,y) + \sum_{l=0}^{j}\sum_{\eta=0}^{i-1}\binom{j}{l}\binom{i-1}{\eta}\frac{\partial^{i-1-\eta}}{\partial x^{i-1-\eta}}\left[\frac{\partial^{j-l}}{\partial y^{j-l}}L(x,y)\right]\frac{\partial^{l+\eta}u(x,y)}{\partial x^\eta \partial y^l} \\
&+ \sum_{l=0}^{j}\sum_{\eta=0}^{i-1}\binom{j}{l}\binom{i-1}{\eta}\frac{\partial^{i-1-\eta}}{\partial x^{i-1-\eta}}\left[\frac{\partial^{j-l}}{\partial y^{j-l}}M(x,y)\right]\frac{\partial^{l+\eta}\phi(x-\tau_1,y-\tau_2)}{\partial x^\eta \partial y^l} \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-2}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-2-q}}{\partial x^{i-2-q}}\bigg|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}K(x,y,t,y)\right]\right)\right]\frac{\partial^{l+\eta}\phi(x-\tau_1,y-\tau_2)}{\partial x^\eta \partial y^l} \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_0^x \frac{\partial^{i-1}}{\partial x^{i-1}}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\right]\frac{\partial^l\phi(t-\tau_1,y-\tau_2)}{\partial y^l}dt \\
&+ \sum_{q=0}^{i-2}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_0^y \frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-2-q)}\partial_2^{(j)}K(x,y,x,s)\right]\frac{\partial^\eta\phi(x-\tau_1,s-\tau_2)}{\partial x^\eta}ds \\
&+ \int_0^x\int_0^y \partial_1^{(i-1)}\partial_2^{(j)}K(x,y,t,s)\phi(t-\tau_1,s-\tau_2)dsdt
\end{aligned}
\tag{3.5}
$$

Hence,

$$
\begin{aligned}
\frac{\partial^{i+j}u(0,0)}{\partial x^i \partial y^j} &= \partial_1^{(i-1)}\partial_2^{(j)}g(0,0) + \sum_{l=0}^{j}\sum_{\eta=0}^{i-1}\binom{j}{l}\binom{i-1}{\eta}\frac{\partial^{i-1-\eta}}{\partial x^{i-1-\eta}}\left[\frac{\partial^{j-l}}{\partial y^{j-l}}L(0,0)\right]\frac{\partial^{l+\eta}u(0,0)}{\partial x^\eta \partial y^l} \\
&+ \sum_{l=0}^{j}\sum_{\eta=0}^{i-1}\binom{j}{l}\binom{i-1}{\eta}\frac{\partial^{i-1-\eta}}{\partial x^{i-1-\eta}}\left[\frac{\partial^{j-l}}{\partial y^{j-l}}M(0,0)\right]\frac{\partial^{l+\eta}\phi(-\tau_1,-\tau_2)}{\partial x^\eta \partial y^l} \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-2}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-2-q}}{\partial x^{i-2-q}}\bigg|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}K(x,y,t,y)\right]\right)\right]_{x=y=0}\frac{\partial^{l+\eta}\phi(-\tau_1,-\tau_2)}{\partial x^\eta \partial y^l}.
\end{aligned}
$$

Moreover, by integrating both sides of Equation (3.1) with respect to x we get integral equation

$$
\begin{aligned}
u(x,y) = u(0,y) + \int_0^x \bigg(& g(z,y) + L(z,y)\,u(z,y) + M(z,y)\,u(z-\tau_1,y-\tau_2) \\
& + \int_0^z\int_0^y K(z,y,t,s)\,u(t-\tau_1,s-\tau_2)\,ds\,dt\bigg)dz.
\end{aligned}
\tag{3.6}
$$

Then

$$\begin{aligned}\frac{\partial^j u}{\partial y^j}(x,y) = \frac{\partial^j u}{\partial y^j}(0,y) + \int_0^x \frac{\partial^j}{\partial y^j}\Bigg(& g(z,y) + L(z,y)\,u(z,y) + M(z,y)\,u(z-\tau_1, y-\tau_2) \\ &+ \int_0^z \int_0^y K(z,y,t,s)\,u(t-\tau_1, s-\tau_2)\,ds\,dt\Bigg)dz,\end{aligned} \tag{3.7}$$

such that $u(0,0) = \phi(0,0)$ and $\frac{\partial^j u}{\partial y^j}(0,0) = \frac{d^j}{dy^j}\,u(0,y)\Big|_{y=0} = \frac{d^j}{dy^j}\,\phi(0,y)\Big|_{y=0}$.

3.2.2 Approximate solution in $D^{\epsilon,0}_{n,m}$

The polynomial $v^{\epsilon,0}_{n,m}(x,y)$ is employed to approximate $u(x,y)$ within the rectangle $D^{\epsilon,0}_{n,m}$, where $n = 0,\ldots,N-1$, $m = 0,\ldots,M-1$, and $\epsilon = 1,\ldots,r_1-1$, such that

$$v^{\epsilon,0}_{n,m}(x,y) = \sum_{j+i=0}^{p} \frac{1}{i!\,j!}\frac{\partial^{i+j}\hat{v}^{\epsilon,0}_{n,m}(x^{\epsilon}_n, y^0_m)}{\partial x^i \partial y^j}(x-x^{\epsilon}_n)^i (y-y^0_m)^j; \quad (x,y) \in D^{\epsilon,0}_{n,m}, \tag{3.8}$$

where $\hat{v}^{\epsilon,0}_{n,m}(x,y)$ is the exact solution of the integral equation:

$$\begin{aligned}\frac{\partial \hat{v}^{\epsilon,0}_{n,m}(x,y)}{\partial x} &= g(x,y) + L(x,y)\hat{v}^{\epsilon,0}_{n,m}(x,y) + M(x,y)\phi(x-\tau_1, y-\tau_2) \\ &\quad + \int_0^x \int_0^y K(x,y,t,s)\phi(t-\tau_1, s-\tau_2)dsdt.\end{aligned} \tag{3.9}$$

To compute $\dfrac{\partial^{i+j}\hat{v}^{\epsilon,0}_{n,m}(x,y)}{\partial x^i \partial y^j}$, we differentiate equation (3.9) j- times with respect to y and $(i-1)$- times with respect to x, obtaining

$$\begin{aligned}
\frac{\partial^{i+j}\hat{v}^{\epsilon,0}_{n,m}(x,y)}{\partial x^i\partial y^j} &= \partial_1^{(i-1)}\partial_2^{(j)}g(x,y) + \sum_{l=0}^{j}\sum_{\eta=0}^{i-1}\binom{j}{l}\binom{i-1}{\eta}\frac{\partial^{i-1-\eta}}{\partial x^{i-1-\eta}}\left[\frac{\partial^{j-l}}{\partial y^{j-l}}L(x,y)\right]\frac{\partial^{l+\eta}\hat{v}^{\epsilon,0}_{n,m}(x,y)}{\partial x^{\eta}\partial y^{l}}\\
&+ \sum_{l=0}^{j}\sum_{\eta=0}^{i-1}\binom{j}{l}\binom{i-1}{\eta}\frac{\partial^{i-1-\eta}}{\partial x^{i-1-\eta}}\left[\frac{\partial^{j-l}}{\partial y^{j-l}}M(x,y)\right]\frac{\partial^{l+\eta}\phi(x-\tau_1,y-\tau_2)}{\partial x^{\eta}\partial y^{l}}\\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-2}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-2-q}}{\partial x^{i-2-q}}\bigg|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}K(x,y,t,y)\right]\right)\right]\frac{\partial^{l+\eta}\phi(x-\tau_1,y-\tau_2)}{\partial x^{\eta}\partial y^{l}}
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_0^x \frac{\partial^{i-1}}{\partial x^{i-1}}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\right]\frac{\partial^{l}\phi(t-\tau_1,y-\tau_2)}{\partial y^{l}}dt\\
&+ \sum_{q=0}^{i-2}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_0^y \frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-2-q)}\partial_2^{(j)}K(x,y,x,s)\right]\frac{\partial^{\eta}\phi(x-\tau_1,s-\tau_2)}{\partial x^{\eta}}ds\\
&+ \int_0^x\int_0^y \partial_1^{(i-1)}\partial_2^{(j)}K(x,y,t,s)\phi(t-\tau_1,s-\tau_2)dsdt
\end{aligned}$$

Moreover, when $(x,y)\in D^{\epsilon,0}_{n,m}$ from Equation (3.6) we get

$$\begin{aligned}
\hat{v}^{\epsilon,0}_{n,m}(x,y) &= \phi(0,y) + \sum_{e=0}^{\epsilon-1}\sum_{\eta=0}^{N-1}\int_{x^e_\eta}^{x^e_{\eta+1}} L(z,y)\, v^{e,0}_{\eta,m}(z,y)\,dz + \sum_{\eta=0}^{n-1}\int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}} L(z,y)\, v^{\epsilon,0}_{\eta,m}(z,y)\,dz\\
&+ \int_{x^\epsilon_n}^{x} L(z,y)\,\hat{v}^{\epsilon,0}_{n,m}(z,y)\,dz + \int_0^x\bigg(g(z,y) + M(z,y)\,\phi(z-\tau_1,y-\tau_2)\\
&+ \int_0^z\int_0^y K(z,y,t,s)\,\phi(t-\tau_1,s-\tau_2)\,ds\,dt\bigg)dz.
\end{aligned} \tag{3.11}$$

Then

$$\begin{aligned}
\frac{\partial^j \hat{v}^{\epsilon,0}_{n,m}}{\partial y^j}(x,y) &= \frac{\partial^j \phi}{\partial y^j}(0,y) \\
&+ \sum_{e=0}^{\epsilon-1}\sum_{\eta=0}^{N-1}\int_{x^e_\eta}^{x^e_{\eta+1}} \sum_{k=0}^{j}\binom{j}{k}\frac{\partial^k L}{\partial y^k}(z,y)\,\frac{\partial^{\,j-k} v^{e,0}_{\eta,m}}{\partial y^{\,j-k}}(z,y)\,dz \\
&+ \sum_{\eta=0}^{n-1}\int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}} \sum_{k=0}^{j}\binom{j}{k}\frac{\partial^k L}{\partial y^k}(z,y)\,\frac{\partial^{\,j-k} v^{\epsilon,0}_{\eta,m}}{\partial y^{\,j-k}}(z,y)\,dz \\
&+ \int_{x^\epsilon_n}^{x} \sum_{k=0}^{j}\binom{j}{k}\frac{\partial^k L}{\partial y^k}(z,y)\,\frac{\partial^{\,j-k} \hat{v}^{\epsilon,0}_{n,m}}{\partial y^{\,j-k}}(z,y)\,dz \\
&+ \int_0^x \frac{\partial^j}{\partial y^j}\Bigg(g(z,y)+M(z,y)\,\phi(z-\tau_1,y-\tau_2) \\
&+ \int_0^z\int_0^y K(z,y,t,s)\,\phi(t-\tau_1,s-\tau_2)\,ds\,dt\Bigg)dz.
\end{aligned} \tag{3.12}$$

Now, in the subdomain $\sigma^{0,0}_{n,0}, n = 1, ..., N-1$, we impose the continuity condition

$$\hat{v}^{0,0}_{n,0}(x^0_n, 0) = \phi(x^0_n, 0).$$

Hence,for Equation (3.10) for $i \geq 1$,

$$\begin{aligned}
\frac{\partial^{i+j}\hat{v}^{0,0}_{n,0}(x^0_n,0)}{\partial x^i \partial y^j} &= \partial_1^{(i-1)}\partial_2^{(j)} g(x^0_n,0) \\
&+ \sum_{l=0}^{j}\sum_{\eta=0}^{i-1}\binom{j}{l}\binom{i-1}{\eta}\frac{\partial^{i-1-\eta}}{\partial x^{\,i-1-\eta}}\left[\frac{\partial^{j-l}}{\partial y^{\,j-l}}L(x^0_n,0)\right]\frac{\partial^{l+\eta}\hat{v}^{0,0}_{n,0}(x^0_n,0)}{\partial x^\eta \partial y^l} \\
&+ \sum_{l=0}^{j}\sum_{\eta=0}^{i-1}\binom{j}{l}\binom{i-1}{\eta}\frac{\partial^{i-1-\eta}}{\partial x^{\,i-1-\eta}}\left[\frac{\partial^{j-l}}{\partial y^{\,j-l}}M(x^0_n,0)\right]\frac{\partial^{l+\eta}\phi(x^0_n-\tau_1,-\tau_2)}{\partial x^\eta \partial y^l} \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-2}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-2-q}}{\partial x^{\,i-2-q}}\left(\frac{\partial^{r-l}}{\partial y^{\,r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\right)\Bigg|_{t=x}\right]_{\substack{x=x^0_n\\ y=0}}\frac{\partial^{l+\eta}\phi(x^0_n-\tau_1,-\tau_2)}{\partial x^\eta \partial y^l} \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_0^{x^0_n}\frac{\partial^{i-1}}{\partial x^{\,i-1}}\left[\frac{\partial^{r-l}}{\partial y^{\,r-l}}[\partial_2^{(j-1-r)}K(x^0_n,0,t,0)]\right]\frac{\partial^l\phi(t-\tau_1,-\tau_2)}{\partial y^l}\,dt.
\end{aligned}$$

and for Equation (3.22)

$$\begin{aligned}\frac{\partial^j \hat{v}^{0,0}_{n,0}}{\partial y^j}\Big(x^0_n,0\Big) &= \frac{\partial^j \phi}{\partial y^j}(0,0)\\ &+ \sum_{\eta=0}^{n-1}\int_{x^0_\eta}^{x^0_{\eta+1}}\sum_{k=0}^{j}\binom{j}{k}\frac{\partial^k L}{\partial y^k}(z,0)\ \frac{\partial^{\,j-k} v^{0,0}_{\eta,0}}{\partial y^{\,j-k}}(z,0)\ dz\\ &+ \int_0^{x^0_n}\frac{\partial^j}{\partial y^j}\Big(g(z,0)+M(z,0)\,\phi(z-\tau_1,-\tau_2)\Big)\,dz.\end{aligned} \tag{3.13}$$

Now, in the subdomain $\sigma^{0,0}_{n,m}, n = 0,...,N-1;\ \ m = 1,...,M-1$, we impose the continuity condition

$$\hat{v}^{0,0}_{n,m}(x^0_n, y^0_m) = v^{0,0}_{n,m-1}(x^0_n, y^0_m).$$

Hence,for Equation (3.10), for $i \geq 1$,

$$\begin{aligned}\frac{\partial^{i+j}\hat{v}^{0,0}_{n,m}(x^0_n,y^0_m)}{\partial x^i \partial y^j} &= \partial_1^{(i-1)}\partial_2^{(j)} g(x^0_n,y^0_m)\\ &+ \sum_{l=0}^{j}\sum_{\eta=0}^{i-1}\binom{j}{l}\binom{i-1}{\eta}\frac{\partial^{i-1-\eta}}{\partial x^{\,i-1-\eta}}\Big[\frac{\partial^{j-l}}{\partial y^{\,j-l}}L(x^0_n,y^0_m)\Big]\frac{\partial^{l+\eta}\hat{v}^{0,0}_{n,m}(x^0_n,y^0_m)}{\partial x^\eta \partial y^l}\\ &+ \sum_{l=0}^{j}\sum_{\eta=0}^{i-1}\binom{j}{l}\binom{i-1}{\eta}\frac{\partial^{i-1-\eta}}{\partial x^{\,i-1-\eta}}\Big[\frac{\partial^{j-l}}{\partial y^{\,j-l}}M(x^0_n,y^0_m)\Big]\frac{\partial^{l+\eta}\phi(x^0_n-\tau_1,y^0_m-\tau_2)}{\partial x^\eta \partial y^l}\\ &+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-2}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial x^{\,q-\eta}}\Big[\ \frac{\partial^{i-2-q}}{\partial x^{\,i-2-q}}\Big|_{t=x}\Big(\frac{\partial^{r-l}}{\partial y^{\,r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\Big)\Big]_{\substack{x=x^0_n\\ y=y^0_m}}\frac{\partial^{l+\eta}\phi(x^0_n-\tau_1,y^0_m-\tau_2)}{\partial x^\eta \partial y^l}\\ &+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_0^{x^0_n}\frac{\partial^{i-1}}{\partial x^{\,i-1}}\Big[\frac{\partial^{r-l}}{\partial y^{\,r-l}}[\partial_2^{(j-1-r)}K(x^0_n,y^0_m,t,y^0_m)]\Big]\frac{\partial^l\phi(t-\tau_1,y^0_m-\tau_2)}{\partial y^l}\,dt\\ &+ \sum_{q=0}^{i-2}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_0^{y^0_m}\frac{\partial^{q-\eta}}{\partial x^{\,q-\eta}}\Big[\partial_1^{(i-2-q)}\partial_2^{(j)}K(x^0_n,y^0_m,x^0_n,s)\Big]\frac{\partial^\eta\phi(x^0_n-\tau_1,s-\tau_2)}{\partial x^\eta}\,ds\\ &+ \int_0^{x^0_n}\int_0^{y^0_m}\partial_1^{(i-1)}\partial_2^{(j)}K(x^0_n,y^0_m,t,s)\,\phi(t-\tau_1,s-\tau_2)\,ds\,dt.\end{aligned}$$

and for Equation (3.22)

$$\frac{\partial^j \hat{v}^{0,0}_{n,m}}{\partial y^j}\Big(x^0_n, y^0_m\Big) = \frac{\partial^j \phi}{\partial y^j}\Big(0, y^0_m\Big) + \sum_{\eta=0}^{n-1} \int_{x^0_\eta}^{x^0_{\eta+1}} \sum_{k=0}^{j} \binom{j}{k} \frac{\partial^k L}{\partial y^k}\Big(z, y^0_m\Big) \frac{\partial^{\,j-k} v^{0,0}_{\eta,m}}{\partial y^{\,j-k}}\Big(z, y^0_m\Big)\, dz$$

$$+ \int_0^{x^0_n} \frac{\partial^j}{\partial y^j}\Big(g\Big(z, y^0_m\Big) + M\Big(z, y^0_m\Big)\, \phi\Big(z - \tau_1, y^0_m - \tau_2\Big) \tag{3.14}$$

$$+ \int_0^z \int_0^{y^0_m} K\Big(z, y^0_m, t, s\Big)\, \phi(t - \tau_1, s - \tau_2)\, ds\, dt\Big) dz.$$

Now in $\sigma^{\epsilon,0}_{0,m}$, where $m = 0, \ldots, M-1$, and $\epsilon = 1, \ldots, r_1 - 1$, such that $\hat{v}^{\epsilon,0}_{0,m}(x^\epsilon_0, y^0_m) = v^{\epsilon-1,0}_{N-1,m}(x^\epsilon_0, y^0_m)$,

Hence,for Equation (3.10), for $i \geq 1$,

$$\frac{\partial^{i+j} \hat{v}^{\epsilon,0}_{0,m}(x^\epsilon_0, y^0_m)}{\partial x^i \partial y^j} = \partial_1^{(i-1)} \partial_2^{(j)} g(x^\epsilon_0, y^0_m) + \sum_{l=0}^{j} \sum_{\eta=0}^{i-1} \binom{j}{l} \binom{i-1}{\eta} \frac{\partial^{i-1-\eta}}{\partial x^{i-1-\eta}} \left[\frac{\partial^{j-l}}{\partial y^{j-l}} L(x^\epsilon_0, y^0_m) \right] \frac{\partial^{l+\eta} \hat{v}^{\epsilon,0}_{0,m}(x^\epsilon_0, y^0_m)}{\partial x^\eta \partial y^l}$$

$$+ \sum_{l=0}^{j} \sum_{\eta=0}^{i-1} \binom{j}{l} \binom{i-1}{\eta} \frac{\partial^{i-1-\eta}}{\partial x^{i-1-\eta}} \left[\frac{\partial^{j-l}}{\partial y^{j-l}} M(x^\epsilon_0, y^0_m) \right] \frac{\partial^{l+\eta} \phi(x^\epsilon_0 - \tau_1, y^0_m - \tau_2)}{\partial x^\eta \partial y^l}$$

$$+ \sum_{r=0}^{j-1} \sum_{l=0}^{r} \sum_{q=0}^{i-2} \sum_{\eta=0}^{q} \binom{r}{l} \binom{q}{\eta} \frac{\partial^{q-\eta}}{\partial x^{q-\eta}} \left[\frac{\partial^{i-2-q}}{\partial x^{i-2-q}}\bigg|_{t=x} \left(\frac{\partial^{r-l}}{\partial y^{r-l}} \left[\partial_2^{(j-1-r)} K(x, y, t, y) \right] \right) \right]_{x=x^\epsilon_0, y=y^0_m} \frac{\partial^{l+\eta} \phi(x^\epsilon_0 - \tau_1, y^0_m - \tau_2)}{\partial x^\eta \partial y^l}$$

$$+ \sum_{r=0}^{j-1} \sum_{l=0}^{r} \binom{r}{l} \int_0^{x^\epsilon_0} \frac{\partial^{i-1}}{\partial x^{i-1}} \left[\frac{\partial^{r-l}}{\partial y^{r-l}} [\partial_2^{(j-1-r)} K(x^\epsilon_0, y^0_m, t, y^0_m)] \right] \frac{\partial^l \phi(t - \tau_1, y^0_m - \tau_2)}{\partial y^l} dt$$

$$+ \sum_{q=0}^{i-2} \sum_{\eta=0}^{q} \binom{q}{\eta} \int_0^{y^0_m} \frac{\partial^{q-\eta}}{\partial x^{q-\eta}} \left[\partial_1^{(i-2-q)} \partial_2^{(j)} K(x^\epsilon_0, y^0_m, x^\epsilon_0, s) \right] \frac{\partial^\eta \phi(x^\epsilon_0 - \tau_1, s - \tau_2)}{\partial x^\eta} ds$$

$$+ \int_0^{x^\epsilon_0} \int_0^{y^0_m} \partial_1^{(i-1)} \partial_2^{(j)} K(x^\epsilon_0, y^0_m, t, s) \phi(t - \tau_1, s - \tau_2) ds dt$$

and for Equation (3.22)

$$\frac{\partial^j \hat{v}^{\epsilon,0}_{0,m}}{\partial y^j}\Big(x^{\epsilon}_0, y^0_m\Big) = \frac{\partial^j \phi}{\partial y^j}\Big(0, y^0_m\Big) + \sum_{e=0}^{\epsilon-1}\sum_{\eta=0}^{N-1}\int_{x^e_\eta}^{x^e_{\eta+1}}\sum_{k=0}^{j}\binom{j}{k}\frac{\partial^k L}{\partial y^k}\Big(z, y^0_m\Big)\ \frac{\partial^{\,j-k} v^{e,0}_{0,m}}{\partial y^{\,j-k}}\Big(z, y^0_m\Big)\ dz$$
$$+ \int_0^{x^{\epsilon}_0}\frac{\partial^j}{\partial y^j}\Big(g\Big(z, y^0_m\Big) + M\Big(z, y^0_m\Big)\ \phi\Big(z-\tau_1, y^0_m-\tau_2\Big) \tag{3.15}$$
$$+ \int_0^{z}\int_0^{y^0_m} K\Big(z, y^0_m, t, s\Big)\ \phi(t-\tau_1, s-\tau_2)\ ds\, dt\Big)dz.$$

Now in $\sigma^{\epsilon,0}_{n,m}$, where $n = 1, \dots, N-1$, $m = 0, \dots, M-1$, and $\epsilon = 1, \dots, r_1 - 1$, such that $\hat{v}^{\epsilon,0}_{n,m}(x^{\epsilon}_n, y^0_m) = v^{\epsilon,0}_{n-1,m}(x^{\epsilon}_n, y^0_m)$,

Hence, for Equation (3.10), for $i \geq 1$,

$$\frac{\partial^{i+j}\hat{v}^{\epsilon,0}_{n,m}(x^{\epsilon}_n, y^0_m)}{\partial x^i \partial y^j} = \partial_1^{(i-1)}\partial_2^{(j)} g(x^{\epsilon}_n, y^0_m) + \sum_{l=0}^{j}\sum_{\eta=0}^{i-1}\binom{j}{l}\binom{i-1}{\eta}\frac{\partial^{i-1-\eta}}{\partial x^{i-1-\eta}}\left[\frac{\partial^{j-l}}{\partial y^{j-l}}L(x^{\epsilon}_n, y^0_m)\right]\frac{\partial^{l+\eta}\hat{v}^{\epsilon,0}_{n,m}(x^{\epsilon}_n, y^0_m)}{\partial x^\eta \partial y^l}$$
$$+ \sum_{l=0}^{j}\sum_{\eta=0}^{i-1}\binom{j}{l}\binom{i-1}{\eta}\frac{\partial^{i-1-\eta}}{\partial x^{i-1-\eta}}\left[\frac{\partial^{j-l}}{\partial y^{j-l}}M(x^{\epsilon}_n, y^0_m)\right]\frac{\partial^{l+\eta}\phi(x^{\epsilon}_n-\tau_1, y^0_m-\tau_2)}{\partial x^\eta \partial y^l}$$
$$+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-2}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-2-q}}{\partial x^{i-2-q}}\bigg|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}K(x,y,t,y)\right]\right)\right]_{x=x^{\epsilon}_n, y=y^0_m}\frac{\partial^{l+\eta}\phi(x^{\epsilon}_n-\tau_1, y^0_m-\tau_2)}{\partial x^\eta \partial y^l}$$
$$+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_0^{x^{\epsilon}_n}\frac{\partial^{i-1}}{\partial x^{i-1}}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x^{\epsilon}_n, y^0_m, t, y^0_m)]\right]\frac{\partial^l \phi(t-\tau_1, y^0_m-\tau_2)}{\partial y^l}dt$$
$$+ \sum_{q=0}^{i-2}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_0^{y^0_m}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-2-q)}\partial_2^{(j)}K(x^{\epsilon}_n, y^0_m, x^{\epsilon}_n, s)\right]\frac{\partial^\eta \phi(x^{\epsilon}_n-\tau_1, s-\tau_2)}{\partial x^\eta}ds$$
$$+ \int_0^{x^{\epsilon}_n}\int_0^{y^0_m}\partial_1^{(i-1)}\partial_2^{(j)}K(x^{\epsilon}_n, y^0_m, t, s)\phi(t-\tau_1, s-\tau_2)dsdt$$

and for Equation (3.22)

$$
\begin{aligned}
\frac{\partial^j \hat{v}^{\epsilon,0}_{n,m}}{\partial y^j}\left(x^\epsilon_n, y^0_m\right) &= \frac{\partial^j \phi}{\partial y^j}\left(0, y^0_m\right) \\
&+ \sum_{e=0}^{\epsilon-1} \sum_{\eta=0}^{N-1} \int_{x^e_\eta}^{x^e_{\eta+1}} \sum_{k=0}^{j} \binom{j}{k} \frac{\partial^k L}{\partial y^k}\left(z, y^0_m\right) \frac{\partial^{\,j-k} v^{e,0}_{\eta,m}}{\partial y^{\,j-k}}\left(z, y^0_m\right) dz \\
&+ \sum_{\eta=0}^{n-1} \int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}} \sum_{k=0}^{j} \binom{j}{k} \frac{\partial^k L}{\partial y^k}\left(z, y^0_m\right) \frac{\partial^{\,j-k} v^{\epsilon,0}_{\eta,m}}{\partial y^{\,j-k}}\left(z, y^0_m\right) dz \\
&+ \int_0^{x^\epsilon_n} \frac{\partial^j}{\partial y^j}\Bigg(g\left(z, y^0_m\right) + M\left(z, y^0_m\right) \phi\left(z - \tau_1, y^0_m - \tau_2\right) \\
&+ \int_0^{z} \int_0^{y^0_m} K\left(z, y^0_m, t, s\right) \phi(t - \tau_1, s - \tau_2)\; ds\, dt\Bigg) dz.
\end{aligned}
\tag{3.16}
$$

3.2.3 Approximate solution in $D^{0,\rho}_{n,m}$

The polynomial $v^{0,\rho}_{n,m}(x, y)$ is employed to approximate $u(x, y)$ within the rectangle $D^{0,\rho}_{n,m}$, $n = 0, ..., N-1$, $m = 0, ..., M-1$, $\rho = 1, ..., r_2 - 1$, such that

$$
v^{0,\rho}_{n,m}(x, y) = \sum_{i+j=0}^{p} \frac{1}{i!j!} \frac{\partial^{i+j} \hat{v}^{0,\rho}_{n,m}(x^0_n, y^\rho_m)}{\partial x^i \partial y^j}(x - x^0_n)^i (y - y^\rho_m)^j \; ; \quad (x, y) \in D^{0,\rho}_{n,m}, \tag{3.17}
$$

where $\dfrac{\partial \hat{v}^{0,\rho}_{n,m}(x, y)}{\partial x}$ is the exact solution of the integral equation:

$$
\begin{aligned}
\frac{\partial \hat{v}^{0,\rho}_{n,m}(x, y)}{\partial x} &= g(x, y) + L(x, y)\hat{v}^{0,\rho}_{n,m}(x, y) + M(x, y)\phi(x - \tau_1, y - \tau_2) \\
&+ \int_0^x \int_0^y K(x, y, t, s)\phi(t - \tau_1, s - \tau_2) ds dt
\end{aligned}
\tag{3.18}
$$

To compute $\dfrac{\partial^{i+j} \hat{v}^{0,\rho}_{n,m}(x, y)}{\partial x^i \partial y^j}$, we differentiate equation (3.18) j- times with respect to y and

$(i-1)$- times with respect to x, obtaining

$$
\begin{aligned}
\frac{\partial^{i+j}\hat{v}^{0,\rho}_{n,m}(x,y)}{\partial x^i \partial y^j} &= \partial_1^{(i-1)}\partial_2^{(j)} g(x,y) + \sum_{l=0}^{j}\sum_{\eta=0}^{i-1}\binom{j}{l}\binom{i-1}{\eta}\frac{\partial^{i-1-\eta}}{\partial x^{i-1-\eta}}\left[\frac{\partial^{j-l}}{\partial y^{j-l}}\left[L(x,y)\right]\right]\frac{\partial^{l+\eta}\hat{v}^{0,\rho}_{n,m}(x,y)}{\partial x^{\eta}\partial y^{l}} \\
&+ \sum_{l=0}^{j}\sum_{\eta=0}^{i-1}\binom{j}{l}\binom{i-1}{\eta}\frac{\partial^{i-1-\eta}}{\partial x^{i-1-\eta}}\left[\frac{\partial^{j-l}}{\partial y^{j-l}}M(x,y)\right]\frac{\partial^{l+\eta}\phi(x-\tau_1,y-\tau_2)}{\partial x^{\eta}\partial y^{l}} \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-2}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-2-q}}{\partial x^{i-2-q}}\bigg|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}K(x,y,t,y)\right]\right)\right]\frac{\partial^{l+\eta}\phi(x-\tau_1,y-\tau_2)}{\partial x^{\eta}\partial y^{l}} \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_0^x \frac{\partial^{i-1}}{\partial x^{i-1}}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\right]\frac{\partial^{l}\phi(t-\tau_1,y-\tau_2)}{\partial y^{l}}dt \\
&+ \sum_{q=0}^{i-2}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_0^y \frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-2-q)}\partial_2^{(j)}K(x,y,x,s)\right]\frac{\partial^{\eta}\phi(x-\tau_1,s-\tau_2)}{\partial x^{\eta}}ds \\
&+ \int_0^x\int_0^y \partial_1^{(i-1)}\partial_2^{(j)}K(x,y,t,s)\phi(t-\tau_1,s-\tau_2)dsdt
\end{aligned}
$$

Moreover, when $(x,y)\in D^{0,\rho}_{n,m}$ from Equation (3.6) we get

$$
\begin{aligned}
\hat{v}^{0,\rho}_{n,m}(x,y) &= \phi(0,y) + \sum_{\eta=0}^{n-1}\int_{x^0_{\eta}}^{x^0_{\eta+1}} L(z,y)\, v^{0,\rho}_{\eta,m}(z,y)\,dz + \int_{x^0_n}^{x} L(z,y)\,\hat{v}^{0,\rho}_{n,m}(z,y)\,dz \\
&+ \int_0^x \left(g(z,y) + M(z,y)\,\phi(z-\tau_1,y-\tau_2) + \int_0^z\int_0^y K(z,y,t,s)\,\phi(t-\tau_1,s-\tau_2)\,ds\,dt\right)dz.
\end{aligned}
\tag{3.19}
$$

Then

$$
\begin{aligned}
\frac{\partial^j\hat{v}^{0,\rho}_{n,m}}{\partial y^j}(x,y) &= \frac{\partial^j\phi}{\partial y^j}(0,y) \\
&+ \sum_{\eta=0}^{n-1}\int_{x^0_{\eta}}^{x^0_{\eta+1}}\sum_{k=0}^{j}\binom{j}{k}\frac{\partial^k L}{\partial y^k}(z,y)\,\frac{\partial^{j-k}v^{0,\rho}_{\eta,m}}{\partial y^{j-k}}(z,y)\,dz \\
&+ \int_{x^0_n}^{x}\sum_{k=0}^{j}\binom{j}{k}\frac{\partial^k L}{\partial y^k}(z,y)\,\frac{\partial^{j-k}\hat{v}^{0,\rho}_{n,m}}{\partial y^{j-k}}(z,y)\,dz \\
&+ \int_0^x \frac{\partial^j}{\partial y^j}\Bigg(g(z,y) + M(z,y)\,\phi(z-\tau_1,y-\tau_2) \\
&+ \int_0^z\int_0^y K(z,y,t,s)\,\phi(t-\tau_1,s-\tau_2)\,ds\,dt\Bigg)dz.
\end{aligned}
\tag{3.20}
$$

Now in $\sigma_{n,0}^{0,\rho}$, such that $\hat{v}_{n,0}^{0,\rho}(x_n^0, y_0^\rho) = v_{n,M-1}^{0,\rho-1}(x_n^0, y_0^\rho)$,

Hence,

$$\frac{\partial^{i+j}\hat{v}_{n,0}^{0,\rho}(x_n^0, y_0^\rho)}{\partial x^i \partial y^j} = \partial_1^{(i-1)}\partial_2^{(j)} g(x_n^0, y_0^\rho) + \sum_{l=0}^{j}\sum_{\eta=0}^{i-1}\binom{j}{l}\binom{i-1}{\eta}\frac{\partial^{i-1-\eta}}{\partial x^{i-1-\eta}}\left[\frac{\partial^{j-l}}{\partial y^{j-l}}\left[L(x_n^0, y_0^\rho)\right]\right]\frac{\partial^{l+\eta}\hat{v}_{n,0}^{0,\rho}(x_n^0, y_0^\rho)}{\partial x^\eta \partial y^l}$$
$$+ \sum_{l=0}^{j}\sum_{\eta=0}^{i-1}\binom{j}{l}\binom{i-1}{\eta}\frac{\partial^{i-1-\eta}}{\partial x^{i-1-\eta}}\left[\frac{\partial^{j-l}}{\partial y^{j-l}}M(x_n^0, y_0^\rho)\right]\frac{\partial^{l+\eta}\phi(x_n^0-\tau_1, y_0^\rho-\tau_2)}{\partial x^\eta \partial y^l}$$
$$+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-2}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\left.\frac{\partial^{i-2-q}}{\partial x^{i-2-q}}\right|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}K(x,y,t,y)\right]\right)\right]_{x=x_n^0, y=y_0^\rho}\frac{\partial^{l+\eta}\phi(x_n^0-\tau_1, y_0^\rho-\tau_2)}{\partial x^\eta \partial y^l}$$
$$+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_0^{x_n^0}\frac{\partial^{i-1}}{\partial x^{i-1}}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x_n, y_0^\rho, t, y_0^\rho)]\right]\frac{\partial^l\phi(t-\tau_1, y_0^\rho-\tau_2)}{\partial y^l}dt$$
$$+ \sum_{q=0}^{i-2}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_0^{y_0^\rho}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-2-q)}\partial_2^{(j)}K(x_n^0, y_0^\rho, x_n^0, s)\right]\frac{\partial^\eta\phi(x_n^0-\tau_1, s-\tau_2)}{\partial x^\eta}ds$$
$$+ \int_0^{x_n^0}\int_0^{y_0^\rho}\partial_1^{(i-1)}\partial_2^{(j)}K(x_n^0, y_0^\rho, t, s)\phi(t-\tau_1, s-\tau_2)dsdt,$$

and

$$\begin{aligned}\frac{\partial^j\hat{v}_{n,0}^{0,\rho}}{\partial y^j}(x_n^0, y_0^\rho) &= \frac{\partial^j\phi}{\partial y^j}(0, y_0^\rho) \\ &+ \sum_{\eta=0}^{n-1}\int_{x_\eta^0}^{x_{\eta+1}^0}\sum_{k=0}^{j}\binom{j}{k}\frac{\partial^k L}{\partial y^k}(z, y_0^\rho)\,\frac{\partial^{j-k}v_{\eta,0}^{0,\rho}}{\partial y^{j-k}}(z, y_0^\rho)\,dz \\ &+ \int_0^{x_n^0}\frac{\partial^j}{\partial y^j}\bigg(g(z, y_0^\rho) + M(z, y_0^\rho)\,\phi(z-\tau_1, y_0^\rho-\tau_2) \\ &+ \int_0^z\int_0^{y_0^\rho}K(z, y_0^\rho, t, s)\,\phi(t-\tau_1, s-\tau_2)\,ds\,dt\bigg)dz.\end{aligned} \tag{3.21}$$

Now in $\sigma_{n,m}^{0,\rho}$, such that $\hat{v}_{n,m}^{0,\rho}(x_n^0, y_m^\rho) = v_{n,m-1}^{0,\rho}(x_n^0, y_m^\rho)$,

Hence,

$$\frac{\partial^{i+j}\hat{v}^{0,\rho}_{n,m}(x^0_n,y^\rho_m)}{\partial x^i\partial y^j}=\partial_1^{(i-1)}\partial_2^{(j)}g(x^0_n,y^\rho_m)+\sum_{l=0}^{j}\sum_{\eta=0}^{i-1}\binom{j}{l}\binom{i-1}{\eta}\frac{\partial^{i-1-\eta}}{\partial x^{i-1-\eta}}\left[\frac{\partial^{j-l}}{\partial y^{j-l}}\left[L(x^0_n,y^\rho_m)\right]\right]\frac{\partial^{l+\eta}\hat{v}^{0,\rho}_{n,m}(x^0_n,y^\rho_m)}{\partial x^\eta\partial y^l}$$
$$+\sum_{l=0}^{j}\sum_{\eta=0}^{i-1}\binom{j}{l}\binom{i-1}{\eta}\frac{\partial^{i-1-\eta}}{\partial x^{i-1-\eta}}\left[\frac{\partial^{j-l}}{\partial y^{j-l}}M(x^0_n,y^\rho_m)\right]\frac{\partial^{l+\eta}\phi(x^0_n-\tau_1,y^\rho_m-\tau_2)}{\partial x^\eta\partial y^l}$$
$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-2}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-2-q}}{\partial x^{i-2-q}}\bigg|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}K(x,y,t,y)\right]\right)\right]_{x=x^0_n,y=y^\rho_m}\frac{\partial^{l+\eta}\phi(x^0_n-\tau_1,y^\rho_m-\tau_2)}{\partial x^\eta\partial y^l}$$
$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_0^{x^0_n}\frac{\partial^{i-1}}{\partial x^{i-1}}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x^0_n,y^\rho_m,t,y^\rho_m)]\right]\frac{\partial^{l}\phi(t-\tau_1,y^\rho_m-\tau_2)}{\partial y^l}dt$$
$$+\sum_{q=0}^{i-2}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_0^{y^\rho_m}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-2-q)}\partial_2^{(j)}K(x^0_n,y^\rho_m,x^0_n,s)\right]\frac{\partial^{\eta}\phi(x^0_n-\tau_1,s-\tau_2)}{\partial x^\eta}ds$$
$$+\int_0^{x^0_n}\int_0^{y^\rho_m}\partial_1^{(i-1)}\partial_2^{(j)}K(x^0_n,y^\rho_m,t,s)\phi(t-\tau_1,s-\tau_2)dsdt,$$

and

$$\begin{aligned}\frac{\partial^j\hat{v}^{0,\rho}_{n,m}}{\partial y^j}(x^0_n,y^\rho_m)&=\frac{\partial^j\phi}{\partial y^j}(0,y^\rho_m)\\&+\sum_{\eta=0}^{n-1}\int_{x^0_\eta}^{x^0_{\eta+1}}\sum_{k=0}^{j}\binom{j}{k}\frac{\partial^k L}{\partial y^k}(z,y^\rho_m)\,\frac{\partial^{\,j-k}v^{0,\rho}_{\eta,m}}{\partial y^{\,j-k}}(z,y^\rho_m)\,dz\\&+\int_0^{x^0_n}\frac{\partial^j}{\partial y^j}\bigg(g(z,y^\rho_m)+M(z,y^\rho_m)\,\phi(z-\tau_1,y^\rho_m-\tau_2)\\&+\int_0^{z}\int_0^{y^\rho_m}K(z,y^\rho_m,t,s)\,\phi(t-\tau_1,s-\tau_2)\,ds\,dt\bigg)dz.\end{aligned}\tag{3.22}$$

3.2.4 Approximate solution in $D^{\epsilon,\rho}_{n,m}$

The polynomial $v^{\epsilon,\rho}_{n,m}(x,y)$ is employed to approximate $u(x,y)$ within the rectangle $D^{\epsilon,\rho}_{n,m}$, $n=0,...,N-1$, $m=1,...,M-1$, $\epsilon=1,...,r_1-1$, and $\rho=1,...,r_2-1$, such that

$$v^{\epsilon,\rho}_{n,m}(x,y)=\sum_{j+i=0}^{p}\frac{1}{i!j!}\frac{\partial^{i+j}\hat{v}^{\epsilon,\rho}_{n,m}(x^\epsilon_n,y^\rho_m)}{\partial x^i\partial y^j}(x-x^\epsilon_n)^i(y-y^\rho_m)^j;\quad (x,y)\in D^{\epsilon,\rho}_{n,m},\tag{3.23}$$

where $\dfrac{\partial \hat{v}^{\epsilon,\rho}_{n,m}(x,y)}{\partial x}$ is the exact solution of the integral equation:

$$
\begin{aligned}
\frac{\partial \hat{v}^{\epsilon,\rho}_{n,m}(x,y)}{\partial x} &= g(x,y) + L(x,y)\hat{v}^{\epsilon,\rho}_{n,m}(x,y) + M(x,y)v^{\epsilon-1,\rho-1}_{n,m}(x-\tau_1, y-\tau_2) \\
&+ \int_0^x \int_0^{\tau_2} K(x,y,t,s)\phi(t-\tau_1, s-\tau_2)dsdt \\
&+ \int_0^{\tau_1} \int_{\tau_2}^{y} K(x,y,t,s)\phi(t-\tau_1, s-\tau_2)dsdt \\
&+ \sum_{e=1}^{\epsilon-1}\sum_{p=1}^{\rho-1}\sum_{\eta=0}^{N-1}\sum_{\mu=0}^{M-1} \int_{x^e_\eta}^{x^e_{\eta+1}} \int_{y^p_\mu}^{y^p_{\mu+1}} K(x,y,t,s)v^{e-1,p-1}_{\eta,\mu}(t-\tau_1, s-\tau_2)dsdt. \\
&+ \sum_{e=1}^{\epsilon-1}\sum_{\eta=0}^{N-1}\sum_{\mu=0}^{m-1} \int_{x^e_\eta}^{x^e_{\eta+1}} \int_{y^\rho_\mu}^{y^\rho_{\mu+1}} K(x,y,t,s)v^{e-1,\rho-1}_{\eta,\mu}(t-\tau_1, s-\tau_2)dsdt. \\
&+ \sum_{p=1}^{\rho-1}\sum_{\eta=0}^{n-1}\sum_{\mu=0}^{M-1} \int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}} \int_{y^p_\mu}^{y^p_{\mu+1}} K(x,y,t,s)v^{\epsilon-1,p-1}_{\eta,\mu}(t-\tau_1, s-\tau_2)dsdt. \\
&+ \sum_{\eta=0}^{n-1}\sum_{\mu=0}^{m-1} \int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}} \int_{y^\rho_\mu}^{y^\rho_{\mu+1}} K(x,y,t,s)v^{\epsilon-1,\rho-1}_{\eta,\mu}(t-\tau_1, s-\tau_2)dsdt. \\
&+ \sum_{e=1}^{\epsilon-1}\sum_{\eta=0}^{N-1} \int_{x^e_\eta}^{x^e_{\eta+1}} \int_{y^\rho_m}^{y} K(x,y,t,s)v^{e-1,\rho-1}_{\eta,m}(t-\tau_1, s-\tau_2)dsdt. \qquad (3.24) \\
&+ \sum_{\eta=0}^{n-1} \int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}} \int_{y^\rho_m}^{y} K(x,y,t,s)v^{\epsilon-1,\rho-1}_{\eta,m}(t-\tau_1, s-\tau_2)dsdt. \\
&+ \sum_{p=1}^{\rho-1}\sum_{\mu=0}^{M-1} \int_{x^\epsilon_n}^{x} \int_{y^p_\mu}^{y^p_{\mu+1}} K(x,y,t,s)v^{\epsilon-1,p-1}_{n,\mu}(t-\tau_1, s-\tau_2)dsdt. \\
&+ \sum_{\mu=0}^{m-1} \int_{x^\epsilon_n}^{x} \int_{y^\rho_\mu}^{y^\rho_{\mu+1}} K(x,y,t,s)v^{\epsilon-1,\rho-1}_{n,\mu}(t-\tau_1, s-\tau_2)dsdt. \\
&+ \int_{x^\epsilon_n}^{x} \int_{y^\rho_m}^{y} K(x,y,t,s)v^{\epsilon-1,\rho-1}_{n,m}(t-\tau_1, s-\tau_2)dsdt.
\end{aligned}
$$

To compute $\dfrac{\partial^{i+j} \hat{v}^{\epsilon,\rho}_{n,m}(x,y)}{\partial x^i \partial y^j}$, we differentiate equation (3.24) j- times with respect to y and $(i-1)$- times with respect to x, obtaining

$$
\begin{aligned}
\frac{\partial^{i+j}\hat{v}_{n,m}^{\epsilon,\rho}(x,y)}{\partial x^i \partial y^j} &= \partial_1^{(i-1)}\partial_2^{(j)} g(x,y) + \partial_1^{(i-1)}\partial_2^{(j)}\left(\int_0^x \int_0^{\tau_2} K(x,y,t,s)\phi(t-\tau_1, s-\tau_2)dsdt\right) \\
&+ \partial_1^{(i-1)}\partial_2^{(j)}\left(\int_0^{\tau_1} \int_{\tau_2}^{y} K(x,y,t,s)\phi(t-\tau_1, s-\tau_2)dsdt\right) \\
&+ \sum_{l=0}^{j}\sum_{\sigma=0}^{i-1}\binom{j}{l}\binom{i-1}{\sigma}\frac{\partial^{i-1-\sigma}}{\partial x^{i-1-\sigma}}\left[\frac{\partial^{j-l}}{\partial y^{j-l}}L(x,y)\right]\frac{\partial^{l+\sigma}\hat{v}_{n,m}^{\epsilon,\rho}(x,y)}{\partial x^{\sigma}\partial y^{l}} \\
&+ \sum_{l=0}^{j}\sum_{\sigma=0}^{i-1}\binom{j}{l}\binom{i-1}{\sigma}\frac{\partial^{i-1-\sigma}}{\partial x^{i-1-\sigma}}\left[\frac{\partial^{j-l}}{\partial y^{j-l}}M(x,y)\right]\frac{\partial^{l+\sigma}v_{n,m}^{\epsilon-1,\rho-1}(x-\tau_1,y-\tau_2)}{\partial x^{\sigma}\partial y^{l}} \\
&+ \sum_{e=1}^{\epsilon-1}\sum_{p=1}^{\rho-1}\sum_{\eta=0}^{N-1}\sum_{\mu=0}^{M-1}\int_{x_\eta^e}^{x_{\eta+1}^e}\int_{y_\mu^p}^{y_{\mu+1}^p}\partial_1^{(i-1)}\partial_2^{(j)}K(x,y,t,s)v_{\eta,\mu}^{e-1,p-1}(t-\tau_1,s-\tau_2)dsdt \\
&+ \sum_{e=1}^{\epsilon-1}\sum_{\eta=0}^{N-1}\sum_{\mu=0}^{m-1}\int_{x_\eta^e}^{x_{\eta+1}^e}\int_{y_\mu^\rho}^{y_{\mu+1}^\rho}\partial_1^{(i-1)}\partial_2^{(j)}K(x,y,t,s)v_{\eta,\mu}^{e-1,\rho-1}(t-\tau_1,s-\tau_2)dsdt \\
&+ \sum_{p=1}^{\rho-1}\sum_{\eta=0}^{n-1}\sum_{\mu=0}^{M-1}\int_{x_\eta^\epsilon}^{x_{\eta+1}^\epsilon}\int_{y_\mu^p}^{y_{\mu+1}^p}\partial_1^{(i-1)}\partial_2^{(j)}K(x,y,t,s)v_{\eta,\mu}^{\epsilon-1,p-1}(t-\tau_1,s-\tau_2)dsdt \\
&+ \sum_{\eta=0}^{n-1}\sum_{\mu=0}^{m-1}\int_{x_\eta^\epsilon}^{x_{\eta+1}^\epsilon}\int_{y_\mu^\rho}^{y_{\mu+1}^\rho}\partial_1^{(i-1)}\partial_2^{(j)}K(x,y,t,s)v_{\eta,\mu}^{\epsilon-1,\rho-1}(t-\tau_1,s-\tau_2)dsdt \\
&+ \sum_{e=1}^{\epsilon-1}\sum_{\eta=0}^{N-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\eta^e}^{x_{\eta+1}^e}\frac{\partial^{i-1}}{\partial x^{i-1}}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\right]\frac{\partial^{l}v_{\eta,m}^{e-1,\rho-1}(t-\tau_1,y-\tau_2)}{\partial y^{l}}dt
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
&+ \sum_{\eta=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\eta^\epsilon}^{x_{\eta+1}^\epsilon}\frac{\partial^{i-1}}{\partial x^{i-1}}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x,y,t,y)]\right]\frac{\partial^{l}v_{\eta,m}^{\epsilon-1,\rho-1}(t-\tau_1,y-\tau_2)}{\partial y^{l}}dt \\
&+ \sum_{e=1}^{\epsilon-1}\sum_{\eta=0}^{N-1}\int_{x_\eta^e}^{x_{\eta+1}^e}\int_{y_m^\rho}^{y}\partial_1^{(i-1)}\partial_2^{(j)}K(x,y,t,s)v_{\eta,m}^{e-1,\rho-1}(t-\tau_1,s-\tau_2)dsdt \\
&+ \sum_{\eta=0}^{n-1}\int_{x_\eta^\epsilon}^{x_{\eta+1}^\epsilon}\int_{y_m^\rho}^{y}\partial_1^{(i-1)}\partial_2^{(j)}K(x,y,t,s)v_{\eta,m}^{\epsilon-1,\rho-1}(t-\tau_1,s-\tau_2)dsdt \\
&+ \sum_{p=1}^{\rho-1}\sum_{\mu=0}^{M-1}\sum_{q=0}^{i-2}\sum_{\sigma=0}^{q}\binom{q}{\sigma}\int_{y_\mu^p}^{y_{\mu+1}^p}\frac{\partial^{q-\sigma}}{\partial x^{q-\sigma}}\left[\partial_1^{(i-2-q)}\partial_2^{(j)}K(x,y,x,s)\right]\frac{\partial^{\sigma}v_{n,\mu}^{\epsilon-1,p-1}(x-\tau_1,s-\tau_2)}{\partial x^{\sigma}} \\
&+ \sum_{\mu=0}^{m-1}\sum_{q=0}^{i-2}\sum_{\sigma=0}^{q}\binom{q}{\sigma}\int_{y_\mu^\rho}^{y_{\mu+1}^\rho}\frac{\partial^{q-\sigma}}{\partial x^{q-\sigma}}\left[\partial_1^{(i-2-q)}\partial_2^{(j)}K(x,y,x,s)\right]\frac{\partial^{\sigma}v_{n,\mu}^{\epsilon-1,\rho-1}(x-\tau_1,s-\tau_2)}{\partial x^{\sigma}}ds
\end{aligned}
$$

$$+\sum_{p=1}^{\rho-1}\sum_{\mu=0}^{M-1}\int_{x_n^\epsilon}^{x}\int_{y_\mu^p}^{y_{\mu+1}^p}\partial_1^{(i-1)}\partial_2^{(j)}K(x,y,t,s)v_{n,\mu}^{\epsilon-1,p-1}(t-\tau_1,s-\tau_2)dsdt.$$

$$+\sum_{\mu=0}^{m-1}\int_{x_n^\epsilon}^{x}\int_{y_\mu^\rho}^{y_{\mu+1}^\rho}\partial_1^{(i-1)}\partial_2^{(j)}K(x,y,t,s)v_{n,\mu}^{\epsilon-1,\rho-1}(t-\tau_1,s-\tau_2)dsdt.$$

$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-2}\sum_{\sigma=0}^{q}\binom{r}{l}\binom{q}{\sigma}\frac{\partial^{q-\sigma}}{\partial x^{q-\sigma}}\left[\frac{\partial^{i-2-q}}{\partial x^{i-2-q}}\bigg|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}K(x,y,t,y)\right]\right)\right]\frac{\partial^{l+\sigma}\hat{v}_{n,m}^{\epsilon-1,\rho-1}(x-\tau_1,y-\tau_2)}{\partial x^\sigma\partial y^l}$$

$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_n^\epsilon}^{x}\frac{\partial^{i-1}}{\partial x^{i-1}}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}K(x,y,t,y)\right]\right]\frac{\partial^l v_{n,m}^{\epsilon-1,\rho-1}(t-\tau_1,y-\tau_2)}{\partial y^l}dt.$$

$$+\sum_{q=0}^{i-2}\sum_{\sigma=0}^{q}\binom{q}{\sigma}\int_{y_m^\rho}^{y}\frac{\partial^{q-\sigma}}{\partial x^{q-\sigma}}\left[\partial_1^{(i-2-q)}\partial_2^{(j)}K(x,y,x,s)\right]\frac{\partial^\sigma\hat{v}_{n,m}^{\epsilon-1,\rho-1}(x-\tau_1,s-\tau_2)}{\partial x^\sigma}ds$$

$$+\int_{x_n^\epsilon}^{x}\int_{y_m^\rho}^{y}\partial_1^{(i-1)}\partial_2^{(j)}K(x,y,t,s)v_{n,m}^{\epsilon-1,\rho-1}(t-\tau_1,s-\tau_2)dsdt.$$

by integrating Equation (3.24) from 0 to x

$$
\begin{aligned}
\hat{v}^{\epsilon,\rho}_{n,m}(x,y) &= \hat{v}^{\epsilon,\rho}_{n,m}(0,y) + \int_0^x g(z,y)\,dz + \sum_{e=0}^{\epsilon-1}\sum_{\eta=0}^{N-1}\int_{x^e_\eta}^{x^e_{\eta+1}} M(z,y)\, v^{e-1,\rho-1}_{\eta,m}(z-\tau_1, y-\tau_2)\,dz \\
&+ \sum_{\eta=0}^{n-1}\int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}} M(z,y)\, v^{\epsilon-1,\rho-1}_{\eta,m}(z-\tau_1, y-\tau_2)\,dz + \int_{x^\epsilon_n}^{x} M(z,y)\, v^{\epsilon-1,\rho-1}_{n,m}(z-\tau_1, y-\tau_2)\,dz \\
&+ \sum_{e=0}^{\epsilon-1}\sum_{\eta=0}^{N-1}\int_{x^e_\eta}^{x^e_{\eta+1}} L(z,y)\, \hat{v}^{e,\rho}_{\eta,m}(z,y)\,dz + \sum_{\eta=0}^{n-1}\int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}} L(z,y)\, \hat{v}^{\epsilon,\rho}_{\eta,m}(z,y)\,dz + \int_{x^\epsilon_n}^{x} L(z,y)\, \hat{v}^{\epsilon,\rho}_{n,m}(z,y)\,dz \\
&+ \int_0^x \left(\int_0^z \int_0^{\tau_2} K(z,y,t,s)\phi(t-\tau_1, s-\tau_2)\,ds\,dt \right) dz \qquad (3.26) \\
&+ \int_0^x \left(\int_0^{\tau_1} \int_{\tau_2}^{y} K(z,y,t,s)\phi(t-\tau_1, s-\tau_2)\,ds\,dt \right) dz \\
&+ \int_0^x \left(\sum_{e=1}^{\epsilon-1}\sum_{p=1}^{\rho-1}\sum_{\eta=0}^{N-1}\sum_{\mu=0}^{M-1} \int_{x^e_\eta}^{x^e_{\eta+1}} \int_{y^p_\mu}^{y^p_{\mu+1}} K(z,y,t,s) v^{e-1,p-1}_{\eta,\mu}(t-\tau_1, s-\tau_2)\,ds\,dt \right) dz \\
&+ \int_0^x \left(\sum_{e=1}^{\epsilon-1}\sum_{\eta=0}^{N-1}\sum_{\mu=0}^{m-1} \int_{x^e_\eta}^{x^e_{\eta+1}} \int_{y^\rho_\mu}^{y^\rho_{\mu+1}} K(z,y,t,s) v^{e-1,\rho-1}_{\eta,\mu}(t-\tau_1, s-\tau_2)\,ds\,dt \right) dz \\
&+ \int_0^x \left(\sum_{p=1}^{\rho-1}\sum_{\eta=0}^{n-1}\sum_{\mu=0}^{M-1} \int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}} \int_{y^p_\mu}^{y^p_{\mu+1}} K(z,y,t,s) v^{\epsilon-1,p-1}_{\eta,\mu}(t-\tau_1, s-\tau_2)\,ds\,dt \right) dz \\
&+ \int_0^x \left(\sum_{\eta=0}^{n-1}\sum_{\mu=0}^{m-1} \int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}} \int_{y^\rho_\mu}^{y^\rho_{\mu+1}} K(z,y,t,s) v^{\epsilon-1,\rho-1}_{\eta,\mu}(t-\tau_1, s-\tau_2)\,ds\,dt \right) dz \\
&+ \int_0^x \left(\sum_{e=1}^{\epsilon-1}\sum_{\eta=0}^{N-1} \int_{x^e_\eta}^{x^e_{\eta+1}} \int_{y^\rho_m}^{y} K(z,y,t,s) v^{e-1,\rho-1}_{\eta,m}(t-\tau_1, s-\tau_2)\,ds\,dt \right) dz \\
&+ \int_0^x \left(\sum_{\eta=0}^{n-1} \int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}} \int_{y^\rho_m}^{y} K(z,y,t,s) v^{\epsilon-1,\rho-1}_{\eta,m}(t-\tau_1, s-\tau_2)\,ds\,dt \right) dz \\
&+ \int_0^x \left(\sum_{p=1}^{\rho-1}\sum_{\mu=0}^{M-1} \int_{x^\epsilon_n}^{z} \int_{y^p_\mu}^{y^p_{\mu+1}} K(z,y,t,s) v^{\epsilon-1,p-1}_{n,\mu}(t-\tau_1, s-\tau_2)\,ds\,dt \right) dz \\
&+ \int_0^x \left(\sum_{\mu=0}^{m-1} \int_{x^\epsilon_n}^{z} \int_{y^\rho_\mu}^{y^\rho_{\mu+1}} K(z,y,t,s) v^{\epsilon-1,\rho-1}_{n,\mu}(t-\tau_1, s-\tau_2)\,ds\,dt \right) dz \\
&+ \int_0^x \left(\int_{x^\epsilon_n}^{z} \int_{y^\rho_m}^{y} K(z,y,t,s) v^{\epsilon-1,\rho-1}_{n,m}(t-\tau_1, s-\tau_2)\,ds\,dt \right) dz.
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{\partial^j \hat{v}^{\epsilon,\rho}_{n,m}}{\partial y^j}(x,y) &= \frac{\partial^j \phi}{\partial y^j}(0,y) + \int_0^x \frac{\partial^j}{\partial y^j} g(z,y)dz \\
&+ \sum_{\eta=0}^{N-1} \int_{x^0_\eta}^{x^0_{\eta+1}} \sum_{k=0}^{j} \binom{j}{k} \frac{\partial^k M}{\partial y^k}(z,y)\, \frac{\partial^{\,j-k}\phi}{\partial y^{\,j-k}}(z-\tau_1, y-\tau_2)\, dz \\
&+ \sum_{e=1}^{\epsilon-1} \sum_{\eta=0}^{N-1} \int_{x^0_\eta}^{x^0_{\eta+1}} \sum_{k=0}^{j} \binom{j}{k} \frac{\partial^k M}{\partial y^k}(z,y)\, \frac{\partial^{\,j-k} v^{e-1,\rho-1}_{\eta,m}}{\partial y^{\,j-k}}(z-\tau_1, y-\tau_2)\, dz \\
&+ \sum_{\eta=0}^{n-1} \int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}} \sum_{k=0}^{j} \binom{j}{k} \frac{\partial^k M}{\partial y^k}(z,y)\, \frac{\partial^{\,j-k} v^{\epsilon-1,\rho-1}_{\eta,m}}{\partial y^{\,j-k}}(z-\tau_1, y-\tau_2)\, dz \\
&+ \int_{x^\epsilon_n}^{x} \sum_{k=0}^{j} \binom{j}{k} \frac{\partial^k M}{\partial y^k}(z,y)\, \frac{\partial^{\,j-k} v^{\epsilon-1,\rho-1}_{n,m}}{\partial y^{\,j-k}}(z-\tau_1, y-\tau_2)\, dz \\
&+ \sum_{e=0}^{\epsilon-1} \sum_{\eta=0}^{N-1} \int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}} \sum_{k=0}^{j} \binom{j}{k} \frac{\partial^k L}{\partial y^k}(z,y)\, \frac{\partial^{\,j-k} v^{e,\rho-1}_{\eta,m}}{\partial y^{\,j-k}}(z,y)\, dz \\
&+ \sum_{\eta=0}^{n-1} \int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}} \sum_{k=0}^{j} \binom{j}{k} \frac{\partial^k L}{\partial y^k}(z,y)\, \frac{\partial^{\,j-k} v^{\epsilon,\rho-1}_{\eta,m}}{\partial y^{\,j-k}}(z,y)\, dz \\
&+ \int_{x^\epsilon_n}^{x} \sum_{k=0}^{j} \binom{j}{k} \frac{\partial^k L}{\partial y^k}(z,y)\, \frac{\partial^{\,j-k} \hat{v}^{\epsilon,\rho}_{n,m}}{\partial y^{\,j-k}}(z,y)\, dz
\end{aligned}
\tag{3.27}
$$

$$+\int_0^x \frac{\partial^j}{\partial y^j}\left(\int_0^z \int_0^{\tau_2} K(z,y,t,s)\phi(t-\tau_1,s-\tau_2)\,ds\,dt + \int_0^{\tau_1}\int_{\tau_2}^{y} K(z,y,t,s)\phi(t-\tau_1,s-\tau_2)\,ds\,dt\right)dz$$

$$+\int_0^x\left(\sum_{e=1}^{\epsilon-1}\sum_{p=1}^{\rho-1}\sum_{\eta=0}^{N-1}\sum_{\mu=0}^{M-1}\int_{x^e_\eta}^{x^e_{\eta+1}}\int_{y^p_\mu}^{y^p_{\mu+1}}\partial_2^{(j)}K(z,y,t,s)v_{\eta,\mu}^{e-1,p-1}(t-\tau_1,s-\tau_2)\,ds\,dt\right)dz$$

$$+\int_0^x\left(\sum_{e=1}^{\epsilon-1}\sum_{\eta=0}^{N-1}\sum_{\mu=0}^{m-1}\int_{x^e_\eta}^{x^e_{\eta+1}}\int_{y^\rho_\mu}^{y^\rho_{\mu+1}}\partial_2^{(j)}K(z,y,t,s)v_{\eta,\mu}^{e-1,\rho-1}(t-\tau_1,s-\tau_2)\,ds\,dt\right)dz$$

$$+\int_0^x\left(\sum_{p=1}^{\rho-1}\sum_{\eta=0}^{n-1}\sum_{\mu=0}^{M-1}\int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}}\int_{y^p_\mu}^{y^p_{\mu+1}}\partial_2^{(j)}K(z,y,t,s)v_{\eta,\mu}^{\epsilon-1,p-1}(t-\tau_1,s-\tau_2)\,ds\,dt\right)dz$$

$$+\int_0^x\left(\sum_{\eta=0}^{n-1}\sum_{\mu=0}^{m-1}\int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}}\int_{y^\rho_\mu}^{y^\rho_{\mu+1}}\partial_2^{(j)}K(z,y,t,s)v_{\eta,\mu}^{\epsilon-1,\rho-1}(t-\tau_1,s-\tau_2)\,ds\,dt\right)dz$$

$$+\int_0^x\left(\sum_{e=1}^{\epsilon-1}\sum_{\eta=0}^{N-1}\sum_{r=0}^{j-1}\sum_{k=0}^{r}\binom{r}{k}\int_{x^e_\eta}^{x^e_{\eta+1}}\frac{\partial^{r-k}}{\partial y^{r-k}}\left[\partial_2^{(j-1-r)}k(z,y,t,y)\right]\frac{\partial^k v_{\eta,m}^{e-1,\rho-1}(t-\tau_1,y-\tau_2)}{\partial y^k}\,dt\right)dz$$

$$+\int_0^x\left(\sum_{\eta=0}^{n-1}\sum_{r=0}^{j-1}\sum_{k=0}^{r}\binom{r}{k}\int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}}\frac{\partial^{r-k}}{\partial y^{r-k}}\left[\partial_2^{(j-1-r)}k(z,y,t,y)\right]\frac{\partial^k v_{\eta,m}^{\epsilon-1,\rho-1}(t-\tau_1,y-\tau_2)}{\partial y^k}\,dt\right)dz$$

$$+\int_0^x\left(\sum_{e=1}^{\epsilon-1}\sum_{\eta=0}^{N-1}\int_{x^e_\eta}^{x^e_{\eta+1}}\int_{y^\rho_m}^{y}\partial_2^{(j)}K(z,y,t,s)v_{\eta,m}^{e-1,\rho-1}(t-\tau_1,s-\tau_2)\,ds\,dt\right)dz$$

$$+\int_0^x\left(\sum_{\eta=0}^{n-1}\int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}}\int_{y^\rho_m}^{y}\partial_2^{(j)}K(z,y,t,s)v_{\eta,m}^{\epsilon-1,\rho-1}(t-\tau_1,s-\tau_2)\,ds\,dt\right)dz$$

$$+\int_0^x\left(\sum_{p=1}^{\rho-1}\sum_{\mu=0}^{M-1}\int_{x^\epsilon_n}^{z}\int_{y^p_\mu}^{y^p_{\mu+1}}\partial_2^{(j)}K(z,y,t,s)v_{n,\mu}^{\epsilon-1,p-1}(t-\tau_1,s-\tau_2)\,ds\,dt\right)dz$$

$$+\int_0^x\left(\sum_{\mu=0}^{m-1}\int_{x^\epsilon_n}^{z}\int_{y^\rho_\mu}^{y^\rho_{\mu+1}}\partial_2^{(j)}K(z,y,t,s)v_{n,\mu}^{\epsilon-1,\rho-1}(t-\tau_1,s-\tau_2)\,ds\,dt\right)dz$$

$$+\int_0^x\left(\sum_{r=0}^{j-1}\sum_{k=0}^{r}\binom{r}{k}\int_{x^\epsilon_n}^{z}\frac{\partial^{r-k}}{\partial y^{r-k}}\left[\partial_2^{(j-1-r)}K(z,y,t,y)\right]\frac{\partial^k v_{n,m}^{\epsilon-1,\rho-1}(t-\tau_1,y-\tau_2)}{\partial y^k}\,dt\right)dz.$$

$$+\int_0^x\left(\int_{x^\epsilon_n}^{z}\int_{y^\rho_m}^{y}\partial_2^{(j)}K(z,y,t,s)v_{n,m}^{\epsilon-1,\rho-1}(t-\tau_1,s-\tau_2)\,ds\,dt\right)dz.$$

Now in $\sigma^{\epsilon,\rho}_{n,0}$, such that $\hat{v}^{\epsilon,\rho}_{n,0}(x^\epsilon_n, y^\rho_0) = v^{\epsilon,\rho-1}_{n,M-1}(x^\epsilon_n, y^\rho_0)$,

Hence,

$$
\begin{aligned}
&\frac{\partial^{i+j}\hat{v}^{\epsilon,\rho}_{n,0}(x^\epsilon_n, y^\rho_0)}{\partial x^i \partial y^j} = \partial_1^{(i-1)}\partial_2^{(j)} g(x^\epsilon_n, y^\rho_0) \\
&+ \partial_1^{(i-1)}\partial_2^{(j)}\left(\int_0^x \int_0^{\tau_2} K(x,y,t,s)\phi(t-\tau_1, s-\tau_2)dsdt\right)(x^\epsilon_n, y^\rho_0) \\
&+ \partial_1^{(i-1)}\partial_2^{(j)}\left(\int_0^{\tau_1} \int_{\tau_2}^{y} K(x,y,t,s)\phi(t-\tau_1, s-\tau_2)dsdt\right)(x^\epsilon_n, y^\rho_0) \\
&+ \sum_{l=0}^{j}\sum_{\sigma=0}^{i-1}\binom{j}{l}\binom{i-1}{\sigma}\frac{\partial^{i-1-\sigma}}{\partial x^{i-1-\sigma}}\left[\frac{\partial^{j-l}}{\partial y^{j-l}}[L(x^\epsilon_n, y^\rho_0)\right]\frac{\partial^{l+\sigma}\hat{v}^{\epsilon,\rho}_{n,0}(x^\epsilon_n, y^\rho_0)}{\partial x^\sigma \partial y^l} \\
&+ \sum_{l=0}^{j}\sum_{\sigma=0}^{i-1}\binom{j}{l}\binom{i-1}{\sigma}\frac{\partial^{i-1-\sigma}}{\partial x^{i-1-\sigma}}\left[\frac{\partial^{j-l}}{\partial y^{j-l}}M(x^\epsilon_n, y^\rho_0)\right]\frac{\partial^{l+\sigma}v^{\epsilon-1,\rho-1}_{n,0}(x^\epsilon_n-\tau_1, y^\rho_0-\tau_2)}{\partial x^\sigma \partial y^l} \\
&+ \sum_{e=1}^{\epsilon-1}\sum_{\eta=0}^{N-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x^e_\eta}^{x^e_{\eta+1}}\frac{\partial^{i-1}}{\partial x^{i-1}}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x^\epsilon_n, y^\rho_0, t, y^\rho_0)]\right]\frac{\partial^{l}v^{e-1,\rho-1}_{\eta,0}(t-\tau_1, y^\rho_0-\tau_2)}{\partial y^l}dt
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{\eta=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}}\frac{\partial^{i-1}}{\partial x^{i-1}}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x^\epsilon_n, y^\rho_0, t, y^\rho_0)]\right]\frac{\partial^{l}v^{\epsilon-1,\rho-1}_{\eta,0}(t-\tau_1, y^\rho_0-\tau_2)}{\partial y^l}dt \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-2}\sum_{\sigma=0}^{q}\binom{r}{l}\binom{q}{\sigma}\frac{\partial^{q-\sigma}}{\partial x^{q-\sigma}}\left[\left.\frac{\partial^{i-2-q}}{\partial x^{i-2-q}}\right|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}K(x,y,t,y)\right]\right)\right]_{y=y^\rho_0}^{x=x^\epsilon_n}\frac{\partial^{l+\sigma}\hat{v}^{\epsilon-1,\rho-1}_{n,0}(x^\epsilon_n-\tau_1, y^\rho_0-\tau_2)}{\partial x^\sigma \partial y^l}
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial^j \hat{v}_{n,0}^{\epsilon,\rho}}{\partial y^j}(x_n^\epsilon, y_0^\rho) = \frac{\partial^j \phi}{\partial y^j}(0, y_0^\rho) + \int_0^{x_n^\epsilon} \frac{\partial^j}{\partial y^j} g(z, y_0^\rho) dz \\
&+ \sum_{e=0}^{\epsilon-1} \sum_{\eta=0}^{N-1} \int_{x_\eta^\epsilon}^{x_{\eta+1}^\epsilon} \sum_{k=0}^{j} \binom{j}{k} \frac{\partial^k M}{\partial y^k}(z, y_0^\rho)\, \frac{\partial^{\,j-k} v_{\eta,0}^{e-1,\rho-1}}{\partial y^{\,j-k}}(z-\tau_1, y_0^\rho - \tau_2)\, dz \\
&+ \sum_{\eta=0}^{n-1} \int_{x_\eta^\epsilon}^{x_{\eta+1}^\epsilon} \sum_{k=0}^{j} \binom{j}{k} \frac{\partial^k M}{\partial y^k}(z, y_0^\rho)\, \frac{\partial^{\,j-k} v_{\eta,0}^{\epsilon-1,\rho-1}}{\partial y^{\,j-k}}(z-\tau_1, y_0^\rho - \tau_2)\, dz \\
&+ \sum_{e=0}^{\epsilon-1} \sum_{\eta=0}^{N-1} \int_{x_\eta^\epsilon}^{x_{\eta+1}^\epsilon} \sum_{k=0}^{j} \binom{j}{k} \frac{\partial^k L}{\partial y^k}(z, y_0^\rho)\, \frac{\partial^{\,j-k} v_{\eta,0}^{e,\rho-1}}{\partial y^{\,j-k}}(z, y_0^\rho)\, dz \\
&+ \sum_{\eta=0}^{n-1} \int_{x_\eta^\epsilon}^{x_{\eta+1}^\epsilon} \sum_{k=0}^{j} \binom{j}{k} \frac{\partial^k L}{\partial y^k}(z, y_0^\rho)\, \frac{\partial^{\,j-k} v_{\eta,0}^{\epsilon,\rho-1}}{\partial y^{\,j-k}}(z, y_0^\rho)\, dz \\
&+ \int_0^{x_n^\epsilon} \frac{\partial^j}{\partial y^j} \left(\int_0^z \int_0^{\tau_2} K(z, y_0^\rho, t, s) \phi(t-\tau_1, s-\tau_2)\, ds\, dt + \int_0^{\tau_1} \int_{\tau_2}^{y_0^\rho} K(z, y_0^\rho, t, s) \phi(t-\tau_1, s-\tau_2)\, ds\, dt \right) dz \\
&+ \int_0^{x_n^\epsilon} \left(\sum_{e=1}^{\epsilon-1} \sum_{p=1}^{\rho-1} \sum_{\eta=0}^{N-1} \int_{x_\eta^e}^{x_{\eta+1}^e} \int_{y_0^p}^{y_1^p} \partial_2^{(j)} K(z, y_0^\rho, t, s) v_{\eta,0}^{e-1,p-1}(t-\tau_1, s-\tau_2)\, ds\, dt \right) dz \\
&+ \int_0^{x_n^\epsilon} \left(\sum_{p=1}^{\rho-1} \sum_{\eta=0}^{N-1} \int_{x_\eta^\epsilon}^{x_{\eta+1}^\epsilon} \int_{y_0^p}^{y_1^p} \partial_2^{(j)} K(z, y_0^\rho, t, s) v_{\eta,0}^{\epsilon-1,p-1}(t-\tau_1, s-\tau_2)\, ds\, dt \right) dz \\
&+ \int_0^{x_n^\epsilon} \left(\sum_{\eta=0}^{n-1} \int_{x_\eta^\epsilon}^{x_{\eta+1}^\epsilon} \int_{y_0^\rho}^{y_1^\rho} \partial_2^{(j)} K(z, y_0^\rho, t, s) v_{\eta,0}^{\epsilon-1,\rho-1}(t-\tau_1, s-\tau_2)\, ds\, dt \right) dz \\
&+ \int_0^{x_n^\epsilon} \left(\sum_{e=1}^{\epsilon-1} \sum_{\eta=0}^{N-1} \sum_{r=0}^{j-1} \sum_{k=0}^{r} \binom{r}{k} \int_{x_\eta^e}^{x_{\eta+1}^e} \frac{\partial^{r-k}}{\partial y^{r-k}} \left[\partial_2^{(j-1-r)} k(z, y_0^\rho, t, y_0^\rho) \right] \frac{\partial^k v_{\eta,0}^{e-1,\rho-1}(t-\tau_1, y_0^\rho - \tau_2)}{\partial y^k}\, dt \right) dz \\
&+ \int_0^{x_n^\epsilon} \left(\sum_{\eta=0}^{n-1} \sum_{r=0}^{j-1} \sum_{k=0}^{r} \binom{r}{k} \int_{x_\eta^\epsilon}^{x_{\eta+1}^\epsilon} \frac{\partial^{r-k}}{\partial y^{r-k}} \left[\partial_2^{(j-1-r)} k(z, y_0^\rho, t, y_0^\rho) \right] \frac{\partial^k v_{\eta,0}^{\epsilon-1,\rho-1}(t-\tau_1, y_0^\rho - \tau_2)}{\partial y^k}\, dt \right) dz \\
&+ \int_0^{x_n^\epsilon} \left(\sum_{p=1}^{\rho-1} \int_{x_n^\epsilon}^{z} \int_{y_0^p}^{y_1^p} \partial_2^{(j)} K(z, y_0^\rho, t, s) v_{n,0}^{\epsilon-1,p-1}(t-\tau_1, s-\tau_2)\, ds\, dt \right. \\
&+ \left. \int_{x_n^\epsilon}^{z} \int_{y_0^\rho}^{y_1^\rho} \partial_2^{(j)} K(z, y_0^\rho, t, s) v_{n,0}^{\epsilon-1,\rho-1}(t-\tau_1, s-\tau_2)\, ds\, dt \right) dz \\
&+ \int_0^{x_n^\epsilon} \left(\sum_{r=0}^{j-1} \sum_{k=0}^{r} \binom{r}{k} \int_{x_n^\epsilon}^{z} \frac{\partial^{r-k}}{\partial y^{r-k}} \left[\partial_2^{(j-1-r)} K(z, y_0^\rho, t, y_0^\rho) \right] \frac{\partial^k v_{n,0}^{\epsilon-1,\rho-1}(t-\tau_1, y_0^\rho - \tau_2)}{\partial y^k}\, dt \right) dz.
\end{aligned}
$$

Now in $\sigma_{n,m}^{\epsilon,\rho}$, such that $\hat{v}_{n,m}^{\epsilon,\rho}(x_n^\epsilon, y_m^\rho) = v_{n,m-1}^{\epsilon,\rho}(x_n^\epsilon, y_m^\rho)$,

Hence,

$$
\begin{aligned}
\frac{\partial^{i+j}\hat{v}_{n,m}^{\epsilon,\rho}(x_n^\epsilon, y_m^\rho)}{\partial x^i \partial y^j} &= \partial_1^{(i-1)}\partial_2^{(j)} g(x_n^\epsilon, y_m^\rho) \\
&+ \partial_1^{(i-1)}\partial_2^{(j)}\left(\int_0^x \int_0^{\tau_2} K(x,y,t,s)\phi(t-\tau_1, s-\tau_2)dsdt\right)(x_n^\epsilon, y_m^\rho) \\
&+ \partial_1^{(i-1)}\partial_2^{(j)}\left(\int_0^{\tau_1} \int_{\tau_2}^{y} K(x,y,t,s)\phi(t-\tau_1, s-\tau_2)dsdt\right)(x_n^\epsilon, y_m^\rho) \\
&+ \sum_{l=0}^{j}\sum_{\sigma=0}^{i-1}\binom{j}{l}\binom{i-1}{\sigma}\frac{\partial^{i-1-\sigma}}{\partial x^{i-1-\sigma}}\left[\frac{\partial^{j-l}}{\partial y^{j-l}}L(x_n^\epsilon, y_m^\rho)\right]\frac{\partial^{l+\sigma}\hat{v}_{n,m}^{\epsilon,\rho}(x_n^\epsilon, y_m^\rho)}{\partial x^\sigma \partial y^l} \\
&+ \sum_{l=0}^{j}\sum_{\sigma=0}^{i-1}\binom{j}{l}\binom{i-1}{\sigma}\frac{\partial^{i-1-\sigma}}{\partial x^{i-1-\sigma}}\left[\frac{\partial^{j-l}}{\partial y^{j-l}}M(x_n^\epsilon, y_m^\rho)\right]\frac{\partial^{l+\sigma}v_{n,m}^{\epsilon-1,\rho-1}(x_n^\epsilon-\tau_1, y_m^\rho-\tau_2)}{\partial x^\sigma \partial y^l} \\
&+ \sum_{e=1}^{\epsilon-1}\sum_{p=1}^{\rho-1}\sum_{\eta=0}^{N-1}\sum_{\mu=0}^{M-1}\int_{x_\eta^e}^{x_{\eta+1}^e}\int_{y_\mu^p}^{y_{\mu+1}^p}\partial_1^{(i-1)}\partial_2^{(j)}K(x_n^\epsilon, y_m^\rho, t, s)v_{\eta,\mu}^{e-1,p-1}(t-\tau_1, s-\tau_2)dsdt \\
&+ \sum_{e=1}^{\epsilon-1}\sum_{\eta=0}^{N-1}\sum_{\mu=0}^{m-1}\int_{x_\eta^e}^{x_{\eta+1}^e}\int_{y_\mu^\rho}^{y_{\mu+1}^\rho}\partial_1^{(i-1)}\partial_2^{(j)}K(x_n^\epsilon, y_m^\rho, t, s)v_{\eta,\mu}^{e-1,\rho-1}(t-\tau_1, s-\tau_2)dsdt \\
&+ \sum_{p=1}^{\rho-1}\sum_{\eta=0}^{n-1}\sum_{\mu=0}^{M-1}\int_{x_\eta^\epsilon}^{x_{\eta+1}^\epsilon}\int_{y_\mu^p}^{y_{\mu+1}^p}\partial_1^{(i-1)}\partial_2^{(j)}K(x_n^\epsilon, y_m^\rho, t, s)v_{\eta,\mu}^{\epsilon-1,p-1}(t-\tau_1, s-\tau_2)dsdt \\
&+ \sum_{\eta=0}^{n-1}\sum_{\mu=0}^{m-1}\int_{x_\eta^\epsilon}^{x_{\eta+1}^\epsilon}\int_{y_\mu^\rho}^{y_{\mu+1}^\rho}\partial_1^{(i-1)}\partial_2^{(j)}K(x_n^\epsilon, y_m^\rho, t, s)v_{\eta,\mu}^{\epsilon-1,\rho-1}(t-\tau_1, s-\tau_2)dsdt \\
&+ \sum_{e=1}^{\epsilon-1}\sum_{\eta=0}^{N-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\eta^e}^{x_{\eta+1}^e}\frac{\partial^{i-1}}{\partial x^{i-1}}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x_n^\epsilon, y_m^\rho, t, y_m^\rho)]\right]\frac{\partial^l v_{\eta,m}^{e-1,\rho-1}(t-\tau_1, y_m^\rho-\tau_2)}{\partial y^l}dt \\
&+ \sum_{\eta=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\eta^\epsilon}^{x_{\eta+1}^\epsilon}\frac{\partial^{i-1}}{\partial x^{i-1}}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K(x_n^\epsilon, y_m^\rho, t, y_m^\rho)]\right]\frac{\partial^l v_{\eta,m}^{\epsilon-1,\rho-1}(t-\tau_1, y_m^\rho-\tau_2)}{\partial y^l}dt \\
&+ \sum_{p=1}^{\rho-1}\sum_{\mu=0}^{M-1}\sum_{q=0}^{i-2}\sum_{\sigma=0}^{q}\binom{q}{\sigma}\int_{y_\mu^p}^{y_{\mu+1}^p}\frac{\partial^{q-\sigma}}{\partial x^{q-\sigma}}\left[\partial_1^{(i-2-q)}\partial_2^{(j)}K(x_n^\epsilon, y_m^\rho, x_n^\epsilon, s)\right]\frac{\partial^\sigma v_{n,\mu}^{\epsilon-1,p-1}(x_n^\epsilon-\tau_1, s-\tau_2)}{\partial x^\sigma}ds \\
&+ \sum_{\mu=0}^{m-1}\sum_{q=0}^{i-2}\sum_{\sigma=0}^{q}\binom{q}{\sigma}\int_{y_\mu^\rho}^{y_{\mu+1}^\rho}\frac{\partial^{q-\sigma}}{\partial x^{q-\sigma}}\left[\partial_1^{(i-2-q)}\partial_2^{(j)}K(x_n^\epsilon, y_m^\rho, x_n^\epsilon, s)\right]\frac{\partial^\sigma v_{n,\mu}^{\epsilon-1,\rho-1}(x_n^\epsilon-\tau_1, s-\tau_2)}{\partial x^\sigma}ds \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-2}\sum_{\sigma=0}^{q}\binom{r}{l}\binom{q}{\sigma}\frac{\partial^{q-\sigma}}{\partial x^{q-\sigma}}\left[\left.\frac{\partial^{i-2-q}}{\partial x^{i-2-q}}\right|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}K(x,y,t,y)\right]\right)\right]_{y=y_m^\rho}^{x=x_n^\epsilon}\frac{\partial^{l+\sigma}\hat{v}_{n,m}^{\epsilon-1,\rho-1}(x_n^\epsilon-\tau_1, y_m^\rho-\tau_2)}{\partial x^\sigma \partial y^l}
\end{aligned}
$$

and

$$\begin{aligned}
\frac{\partial^j \hat{v}^{\epsilon,\rho}_{n,m}}{\partial y^j}(x^\epsilon_n, y^\rho_m) &= \frac{\partial^j \phi}{\partial y^j}(0, y^\rho_m) + \int_0^x \frac{\partial^j}{\partial y^j} g(z, y^\rho_m) dz \\
&+ \sum_{\eta=0}^{n-1} \int_{x^0_\eta}^{x^0_{\eta+1}} \sum_{k=0}^{j} \binom{j}{k} \frac{\partial^k M}{\partial y^k}(z, y^\rho_m) \, \frac{\partial^{\, j-k} \phi}{\partial y^{\, j-k}}(z - \tau_1, y^\rho_m - \tau_2) \, dz \\
&+ \sum_{e=1}^{\epsilon-1} \sum_{\eta=0}^{N-1} \int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}} \sum_{k=0}^{j} \binom{j}{k} \frac{\partial^k M}{\partial y^k}(z, y^\rho_m) \, \frac{\partial^{\, j-k} v^{e-1,\rho-1}_{\eta,m}}{\partial y^{\, j-k}}(z - \tau_1, y^\rho_m - \tau_2) \, dz \\
&+ \sum_{\eta=0}^{n-1} \int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}} \sum_{k=0}^{j} \binom{j}{k} \frac{\partial^k M}{\partial y^k}(z, y^\rho_m) \, \frac{\partial^{\, j-k} v^{\epsilon-1,\rho-1}_{\eta,m}}{\partial y^{\, j-k}}(z - \tau_1, y^\rho_m - \tau_2) \, dz \\
&+ \sum_{e=0}^{\epsilon-1} \sum_{\eta=0}^{N-1} \int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}} \sum_{k=0}^{j} \binom{j}{k} \frac{\partial^k L}{\partial y^k}(z, y^\rho_m) \, \frac{\partial^{\, j-k} \hat{v}^{e,\rho}_{\eta,m}}{\partial y^{\, j-k}}(z, y^\rho_m) \, dz \\
&+ \sum_{\eta=0}^{n-1} \int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}} \sum_{k=0}^{j} \binom{j}{k} \frac{\partial^k L}{\partial y^k}(z, y^\rho_m) \, \frac{\partial^{\, j-k} \hat{v}^{\epsilon,\rho}_{\eta,m}}{\partial y^{\, j-k}}(z, y^\rho_m) \, dz \\
&+ \int_0^{x^\epsilon_n} \frac{\partial^j}{\partial y^j} \left(\int_0^z \int_0^{\tau_2} K(z, y^\rho_m, t, s) \phi(t - \tau_1, s - \tau_2) \, ds \, dt + \int_0^{\tau_1} \int_{\tau_2}^{y^\rho_m} K(z, y^\rho_m, t, s) \phi(t - \tau_1, s - \tau_2) \, ds \, dt \right) dz \\
&+ \int_0^{x^\epsilon_n} \left(\sum_{e=1}^{\epsilon-1} \sum_{p=1}^{\rho-1} \sum_{\eta=0}^{N-1} \sum_{\mu=0}^{M-1} \int_{x^e_\eta}^{x^e_{\eta+1}} \int_{y^p_\mu}^{y^p_{\mu+1}} \partial_2^{(j)} K(z, y^\rho_m, t, s) v^{e-1,p-1}_{\eta,\mu}(t - \tau_1, s - \tau_2) \, ds \, dt \right) dz \\
&+ \int_0^{x^\epsilon_n} \left(\sum_{e=1}^{\epsilon-1} \sum_{\eta=0}^{N-1} \sum_{\mu=0}^{m-1} \int_{x^e_\eta}^{x^e_{\eta+1}} \int_{y^\rho_\mu}^{y^\rho_{\mu+1}} \partial_2^{(j)} K(z, y^\rho_m, t, s) v^{e-1,\rho-1}_{\eta,\mu}(t - \tau_1, s - \tau_2) \, ds \, dt \right) dz \\
&+ \int_0^{x^\epsilon_n} \left(\sum_{p=1}^{\rho-1} \sum_{\eta=0}^{n-1} \sum_{\mu=0}^{M-1} \int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}} \int_{y^p_\mu}^{y^p_{\mu+1}} \partial_2^{(j)} K(z, y^\rho_m, t, s) v^{\epsilon-1,p-1}_{\eta,\mu}(t - \tau_1, s - \tau_2) \, ds \, dt \right) dz \\
&+ \int_0^{x^\epsilon_n} \left(\sum_{\eta=0}^{n-1} \sum_{\mu=0}^{m-1} \int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}} \int_{y^\rho_\mu}^{y^\rho_{\mu+1}} \partial_2^{(j)} K(z, y^\rho_m, t, s) v^{\epsilon-1,\rho-1}_{\eta,\mu}(t - \tau_1, s - \tau_2) \, ds \, dt \right) dz \\
&+ \int_0^{x^\epsilon_n} \left(\sum_{e=1}^{\epsilon-1} \sum_{\eta=0}^{N-1} \sum_{r=0}^{j-1} \sum_{k=0}^{r} \binom{r}{k} \int_{x^e_\eta}^{x^e_{\eta+1}} \frac{\partial^{r-k}}{\partial y^{r-k}} \left[\partial_2^{(j-1-r)} k(z, y^\rho_m, t, y^\rho_m) \right] \frac{\partial^k v^{e-1,\rho-1}_{\eta,m}(t - \tau_1, y^\rho_m - \tau_2)}{\partial y^k} \, dt \right) dz \\
&+ \int_0^{x^\epsilon_n} \left(\sum_{\eta=0}^{n-1} \sum_{r=0}^{j-1} \sum_{k=0}^{r} \binom{r}{k} \int_{x^\epsilon_\eta}^{x^\epsilon_{\eta+1}} \frac{\partial^{r-k}}{\partial y^{r-k}} \left[\partial_2^{(j-1-r)} k(z, y^\rho_m, t, y^\rho_m) \right] \frac{\partial^k v^{\epsilon-1,\rho-1}_{\eta,m}(t - \tau_1, y^\rho_m - \tau_2)}{\partial y^k} \, dt \right) dz \\
&+ \int_0^{x^\epsilon_n} \left(\sum_{p=1}^{\rho-1} \sum_{\mu=0}^{M-1} \int_{x^\epsilon_n}^{z} \int_{y^p_\mu}^{y^p_{\mu+1}} \partial_2^{(j)} K(z, y^\rho_m, t, s) v^{\epsilon-1,p-1}_{n,\mu}(t - \tau_1, s - \tau_2) \, ds \, dt \right) dz \\
&+ \int_0^{x^\epsilon_n} \left(\sum_{\mu=0}^{m-1} \int_{x^\epsilon_n}^{z} \int_{y^\rho_\mu}^{y^\rho_{\mu+1}} \partial_2^{(j)} K(z, y^\rho_m, t, s) v^{\epsilon-1,\rho-1}_{n,\mu}(t - \tau_1, s - \tau_2) \, ds \, dt \right) d \\
&+ \int_0^{x^\epsilon_n} \left(\sum_{r=0}^{j-1} \sum_{k=0}^{r} \binom{r}{k} \int_{x^\epsilon_n}^{z} \frac{\partial^{r-k}}{\partial y^{r-k}} \left[\partial_2^{(j-1-r)} K(z, y^\rho_m, t, y^\rho_m) \right] \frac{\partial^k v^{\epsilon-1,\rho-1}_{n,m}(t - \tau_1, y^\rho_m - \tau_2)}{\partial y^k} \, dt \right) dz.
\end{aligned}$$

3.3 Error Analysis

We need the following lemma gives the convergence of the presented method. We consider the space $L^\infty(D)$ with the norm

$$\left\|\varphi\right\|_{L^\infty(D)} = \inf\left\{C \in \mathbb{R} : \left|\varphi(x,y)\right| \leq C \text{ for a.e. } (x,y) \in D\right\} < \infty.$$

Lemma 3.3.1 *Let g, L, M, and K be $(p+1)$-times continuously differentiable functions on their respective domains. Then there exists a positive constant $\alpha(p)$ such that for all $n = 0,\dots,N-1$, $m = 0,\dots,M-1$, $\epsilon = 0,\dots,r_1-1$, $\rho = 0,\dots,r_2-1$, and for all integers i,j satisfying $i+j = 0,1,\dots,p+1$, we have*

$$\left\|\frac{\partial^{i+j}\hat{v}_{n,m}^{\epsilon,\rho}}{\partial x^i \partial y^j}\right\|_{L^\infty(D_{n,m}^{\epsilon,\rho})} \leq \alpha(p).$$

Moreover, for the initial rectangle we have

$$\hat{v}_{0,0}^{0,0}(x,y) = u(x,y), \qquad (x,y) \in D_{0,0}^{0,0}.$$

Proof. The proof follows the same steps and ideas as the proof of Lemma 3.6 in [12]. ■

Theorem 3.3.1 *Assume that the functions g, L, M, and K are $(p+1)$-times continuously differentiable on their respective domains, and that the history function ϕ is sufficiently smooth. Then the Taylor collocation scheme defined by equations (3.4), (3.8), (3.17), and (3.23) determines a unique approximate solution*

$$v \in S_{p,p}^{(0)}(\Pi_{N,M}^{\epsilon,\rho}).$$

Let $u(x,y)$ denote the exact solution of problem (3.1), and define the error function

$$e(x,y) := u(x,y) - v(x,y), \qquad (x,y) \in D.$$

Then the following estimate holds:

$$||e||_{L^\infty(D)} \leq C\,(h+k)^{p+1}\,,$$

where C is a positive constant independent of the stepsizes h and k.

Proof. Define the error $e(x,y)$ on $D_{n,m}^{\epsilon,\rho}$ by

$$e_{n,m}^{\epsilon,\rho}(x,y) = u(x,y) - v_{n,m}^{\epsilon,\rho}(x,y), \qquad (x,y) \in D_{n,m}^{\epsilon,\rho},$$

for all $n \in \{0,\ldots,N-1\}$, $m \in \{0,\ldots,M-1\}$, $\epsilon = 0,1,...,r_1-1;\ \ \rho = 0,1,...,r_2-1$.

Claim 1. There exists a constant C_1 independent of h and k such that

$$||e_{0,0}^{0,0}||_{L^\infty(D_{0,0}^{0,0})} \leq C_1(h+k)^{p+1}.$$

Let $(x,y) \in D_{0,0}^{0,0}$. Using Lemma 1.6.3, we obtain

$$|e_{0,0}^{0,0}(x,y)| \leq \sum_{i+j=p+1} \frac{1}{i!\,j!} \left\| \frac{\partial^{i+j}u}{\partial x^i \partial y^j} \right\|_{\infty,D} h^i k^j.$$

Hence, by Lemma 3.3.1, we have

$$|e_{0,0}^{0,0}(x,y)| \leq \alpha(p) \sum_{i+j=p+1} \frac{1}{i!\,j!} h^i k^j = \frac{\alpha(p)}{(p+1)!}(h+k)^{p+1}.$$

Therefore,

$$||e_{0,0}^{0,0}||_{L^\infty(D_{0,0}^{0,0})} \leq C_1(h+k)^{p+1}, \qquad C_1 = \frac{\alpha(p)}{(p+1)!}.$$

Claim 2. There exists a constant C_2 independent of h and k such that

$$||e_{n,m}^{\epsilon,0}||_{L^\infty(D_{n,m}^{\epsilon,0})} \leq C_2(h+k)^{p+1}, \qquad n = 0,\ldots,N-1, \qquad m = 0,\ldots,N-1, \qquad \epsilon = 0,\ldots,r_1-1.$$

and $(n, m, \epsilon) \neq (0,0,0)$. Let $(x, y) \in D_{n,m}^{\epsilon,0}$. From Equation (3.11), and the definition of the approximate solution $\hat{v}_{n,m}^{\epsilon,0}$, we obtain

$$\begin{aligned} u(x,y) - \hat{v}_{n,m}^{\epsilon,0}(x,y) &= \sum_{e=0}^{\epsilon-1}\sum_{\eta=0}^{N-1}\int_{x_\eta^e}^{x_{\eta+1}^e} L(z,y)\Big(u(z,y) - v_{\eta,m}^{e,0}(z,y)\Big)\,dz \\ &\quad + \sum_{\eta=0}^{n-1}\int_{x_\eta^\epsilon}^{x_{\eta+1}^\epsilon} L(z,y)\Big(u(z,y) - v_{\eta,m}^{\epsilon,0}(z,y)\Big)\,dz \\ &\quad + \int_{x_n^\epsilon}^{x} L(z,y)\Big(u(z,y) - \hat{v}_{n,m}^{\epsilon,0}(z,y)\Big)\,dz. \end{aligned}$$

Hence, taking absolute values and using the boundedness of L, we get

$$\begin{aligned} |u(x,y) - \hat{v}_{n,m}^{\epsilon,0}(x,y)| &\leq \|L\|_{\infty,D}\sum_{e=0}^{\epsilon-1}\sum_{\eta=0}^{N-1}\int_{x_\eta^e}^{x_{\eta+1}^e} |u(z,y) - v_{\eta,m}^{e,0}(z,y)|\,dz \\ &\quad + \|L\|_{\infty,D}\sum_{\eta=0}^{n-1}\int_{x_\eta^\epsilon}^{x_{\eta+1}^\epsilon} |u(z,y) - v_{\eta,m}^{\epsilon,0}(z,y)|\,dz \\ &\quad + \|L\|_{\infty,D}\int_{x_n^\epsilon}^{x} |u(z,y) - \hat{v}_{n,m}^{\epsilon,0}(z,y)|\,dz. \end{aligned}$$

Therefore,

$$\begin{aligned} |u(x,y) - \hat{v}_{n,m}^{\epsilon,0}(x,y)| &\leq h\|L\|_{\infty,D}\sum_{e=0}^{\epsilon-1}\sum_{\eta=0}^{N-1} \|e_{\eta,m}^{e,0}\|_{L^\infty(D_{\eta,m}^{e,0})} \\ &\quad + h\|L\|_{\infty,D}\sum_{\eta=0}^{n-1} \|e_{\eta,m}^{\epsilon,0}\|_{L^\infty(D_{\eta,m}^{\epsilon,0})} \\ &\quad + \|L\|_{\infty,D}\int_{x_n^\epsilon}^{x} |u(z,y) - \hat{v}_{n,m}^{\epsilon,0}(z,y)|\,dz. \end{aligned}$$

Set

$$\Psi(x) := \sup_{\substack{x_n^\epsilon \leq \xi \leq x \\ y \in \delta_m^0}} |u(\xi,y) - \hat{v}_{n,m}^{\epsilon,0}(\xi,y)|.$$

Then

$$\Psi(x) \le h\|L\|_{\infty,D} \sum_{e=0}^{\epsilon-1} \sum_{\eta=0}^{N-1} \|e^{e,0}_{\eta,m}\|_{L^\infty(D^{e,0}_{\eta,m})} + h\|L\|_{\infty,D} \sum_{\eta=0}^{n-1} \|e^{\epsilon,0}_{\eta,m}\|_{L^\infty(D^{\epsilon,0}_{\eta,m})} + \|L\|_{\infty,D} \int_{x_n^\epsilon}^{x} \Psi(z)\, dz.$$

Applying Gronwall's inequality, we obtain

$$\Psi(x) \le \Bigg(h\|L\|_{\infty,D} \sum_{e=0}^{\epsilon-1} \sum_{\eta=0}^{N-1} \|e^{e,0}_{\eta,m}\|_{L^\infty(D^{e,0}_{\eta,m})} + h\|L\|_{\infty,D} \sum_{\eta=0}^{n-1} \|e^{\epsilon,0}_{\eta,m}\|_{L^\infty(D^{\epsilon,0}_{\eta,m})}\Bigg) \exp\Big(\|L\|_{\infty,D} h\Big).$$

Hence,

$$\|u - \hat{v}^{\epsilon,0}_{n,m}\|_{L^\infty(D^{\epsilon,0}_{n,m})} \le C\Bigg(h \sum_{e=0}^{\epsilon-1} \sum_{\eta=0}^{N-1} \|e^{e,0}_{\eta,m}\|_{L^\infty(D^{e,0}_{\eta,m})} + h \sum_{\eta=0}^{n-1} \|e^{\epsilon,0}_{\eta,m}\|_{L^\infty(D^{\epsilon,0}_{\eta,m})}\Bigg),$$

where $C > 0$ is independent of h and k.

Now, by the decomposition

$$e^{\epsilon,0}_{n,m}(x,y) = u(x,y) - v^{\epsilon,0}_{n,m}(x,y) = \Big(u(x,y) - \hat{v}^{\epsilon,0}_{n,m}(x,y)\Big) + \Big(\hat{v}^{\epsilon,0}_{n,m}(x,y) - v^{\epsilon,0}_{n,m}(x,y)\Big),$$

we deduce that

$$\|e^{\epsilon,0}_{n,m}\|_{L^\infty(D^{\epsilon,0}_{n,m})} \le \|u - \hat{v}^{\epsilon,0}_{n,m}\|_{L^\infty(D^{\epsilon,0}_{n,m})} + \|\hat{v}^{\epsilon,0}_{n,m} - v^{\epsilon,0}_{n,m}\|_{L^\infty(D^{\epsilon,0}_{n,m})}.$$

Using Claim 1 and Lemma 3.3.1, we have

$$\|\hat{v}^{\epsilon,0}_{n,m} - v^{\epsilon,0}_{n,m}\|_{L^\infty(D^{\epsilon,0}_{n,m})} \le C(h+k)^{p+1}.$$

Set

$$T_\epsilon := \|e^{\epsilon,0}\|_{L^\infty(D^{\epsilon,0})}, \qquad \epsilon = 0, 1, \ldots, r_1 - 1,$$

where

$$\|e^{\epsilon,0}\|_{L^\infty(D^{\epsilon,0})} := \max_{\substack{0\le n\le N-1\\ 0\le m\le M-1}} \|e^{\epsilon,0}_{n,m}\|_{L^\infty(D^{\epsilon,0}_{n,m})}.$$

From the previous estimate, we have

$$\begin{aligned}\|e^{\epsilon,0}_{n,m}\|_{L^\infty(D^{\epsilon,0}_{n,m})} \le{}& C\Bigg(h\sum_{s=0}^{\epsilon-1}\sum_{\eta=0}^{N-1}\|e^{s,0}_{\eta,m}\|_{L^\infty(D^{s,0}_{\eta,m})} + h\sum_{\eta=0}^{n-1}\|e^{\epsilon,0}_{\eta,m}\|_{L^\infty(D^{\epsilon,0}_{\eta,m})}\Bigg)\\ &+ C(h+k)^{p+1}.\end{aligned}$$

Since

$$\|e^{s,0}_{\eta,m}\|_{L^\infty(D^{s,0}_{\eta,m})} \le \|e^{s,0}\|_{L^\infty(D^{s,0})} = T_s,$$

and since $Nh = \tau_1$, it follows that

$$\begin{aligned}\|e^{\epsilon,0}_{n,m}\|_{L^\infty(D^{\epsilon,0}_{n,m})} &\le C\Bigg(h\sum_{s=0}^{\epsilon-1}\sum_{\eta=0}^{N-1}T_s + h\sum_{\eta=0}^{n-1}\|e^{\epsilon,0}_{\eta,m}\|_{L^\infty(D^{\epsilon,0}_{\eta,m})}\Bigg) + C(h+k)^{p+1}\\ &= C\Bigg(\tau_1\sum_{s=0}^{\epsilon-1}T_s + h\sum_{\eta=0}^{n-1}\|e^{\epsilon,0}_{\eta,m}\|_{L^\infty(D^{\epsilon,0}_{\eta,m})}\Bigg) + C(h+k)^{p+1}.\end{aligned}$$

Taking the maximum over n and m, we obtain

$$T_\epsilon \le C\Bigg(\tau_1\sum_{s=0}^{\epsilon-1}T_s + h\sum_{\eta=0}^{n-1}\|e^{\epsilon,0}_{\eta,m}\|_{L^\infty(D^{\epsilon,0}_{\eta,m})}\Bigg) + C(h+k)^{p+1}. \tag{3.28}$$

Applying Lemma 1.6.1 to the sequence $(T_\epsilon)_{\epsilon=0}^{r_1-1}$, we get

$$T_\epsilon \le C\Bigg(h\sum_{\eta=0}^{n-1}\|e^{\epsilon,0}_{\eta,m}\|_{L^\infty(D^{\epsilon,0}_{\eta,m})} + (h+k)^{p+1}\Bigg)\exp(C\tau_1 r_1).$$

Hence, there exists a constant $C' > 0$, independent of h and k, such that

$$T_\epsilon \leq C'h \sum_{\eta=0}^{n-1} \|e_{\eta,m}^{\epsilon,0}\|_{L^\infty(D_{\eta,m}^{\epsilon,0})} + C'(h+k)^{p+1}. \tag{3.29}$$

Since

$$\|e_{n,m}^{\epsilon,0}\|_{L^\infty(D_{n,m}^{\epsilon,0})} \leq \|e^{\epsilon,0}\|_{L^\infty(D^{\epsilon,0})} = T_\epsilon,$$

it follows from (3.29) that

$$\|e_{n,m}^{\epsilon,0}\|_{L^\infty(D_{n,m}^{\epsilon,0})} \leq C'h \sum_{\eta=0}^{n-1} \|e_{\eta,m}^{\epsilon,0}\|_{L^\infty(D_{\eta,m}^{\epsilon,0})} + C'(h+k)^{p+1}.$$

Now, for each fixed ϵ and m, define

$$\Gamma_n := \|e_{n,m}^{\epsilon,0}\|_{L^\infty(D_{n,m}^{\epsilon,0})}, \qquad n = 0, 1, \ldots, N-1.$$

Then

$$\Gamma_n \leq C'h \sum_{\eta=0}^{n-1} \Gamma_\eta + C'(h+k)^{p+1}.$$

Applying Lemma 1.6.1 once again, this time to the sequence $(\Gamma_n)_{n=0}^{N-1}$, we obtain

$$\Gamma_n \leq C'(h+k)^{p+1} \exp(C'nh).$$

Since $nh \leq Nh = \tau_1$, we get

$$\Gamma_n \leq C'(h+k)^{p+1} \exp(C'\tau_1).$$

Therefore, there exists a constant $C_2 > 0$, independent of h and k, such that

$$\|e_{n,m}^{\epsilon,0}\|_{L^\infty(D_{n,m}^{\epsilon,0})} \leq C_2(h+k)^{p+1}.$$

where C_2 is a positive constant independent of h and k. This proves Claim 2.

Using the same arguments as those employed in Claim 2, we obtain

$$||e^{0,\rho}_{n,m}||_{L^\infty(D^{0,\rho}_{n,m})} \leq C_3(h+k)^{p+1}, \qquad n = 0,\dots,N-1, \qquad m = 0,\dots,M-1, \qquad \rho = 1,\dots,r_2-1.$$

Claim 3. There exists a constant C_4 independent of h and k such that

$$||e^{\epsilon,\rho}_{n,m}||_{L^\infty(D^{\epsilon,\rho}_{n,m})} \leq C_4(h+k)^{p+1},$$

for

$$n = 0,\dots,N-1, \qquad m = 0,\dots,M-1, \qquad \epsilon = 1,\dots,r_1-1, \qquad \rho = 1,\dots,r_2-1.$$

Let $(x, y) \in D^{\epsilon,\rho}_{n,m}$. From Equation (3.26) and the integral equation satisfied by the exact solution u, we obtain after subtraction

$$
\begin{aligned}
u(x,y)-\hat{v}^{\epsilon,\rho}_{n,m}(x,y) &= \sum_{e=0}^{\epsilon-1}\sum_{\eta=0}^{N-1}\int_{x^{e}_{\eta}}^{x^{e}_{\eta+1}} M(z,y)\Big(u(z-\tau_1,y-\tau_2)-v^{e-1,\rho-1}_{\eta,m}(z-\tau_1,y-\tau_2)\Big)\,dz \\
&+\sum_{\eta=0}^{n-1}\int_{x^{\epsilon}_{\eta}}^{x^{\epsilon}_{\eta+1}} M(z,y)\Big(u(z-\tau_1,y-\tau_2)-v^{\epsilon-1,\rho-1}_{\eta,m}(z-\tau_1,y-\tau_2)\Big)\,dz \\
&+\int_{x^{\epsilon}_{n}}^{x} M(z,y)\Big(u(z-\tau_1,y-\tau_2)-v^{\epsilon-1,\rho-1}_{n,m}(z-\tau_1,y-\tau_2)\Big)\,dz \\
&+\sum_{e=0}^{\epsilon-1}\sum_{\eta=0}^{N-1}\int_{x^{e}_{\eta}}^{x^{e}_{\eta+1}} L(z,y)\Big(u(z,y)-\hat{v}^{e,\rho}_{\eta,m}(z,y)\Big)\,dz \\
&+\sum_{\eta=0}^{n-1}\int_{x^{\epsilon}_{\eta}}^{x^{\epsilon}_{\eta+1}} L(z,y)\Big(u(z,y)-\hat{v}^{\epsilon,\rho}_{\eta,m}(z,y)\Big)\,dz \\
&+\int_{x^{\epsilon}_{n}}^{x} L(z,y)\Big(u(z,y)-\hat{v}^{\epsilon,\rho}_{n,m}(z,y)\Big)\,dz. \\
&+\int_0^x\left(\sum_{e=1}^{\epsilon-1}\sum_{p=1}^{\rho-1}\sum_{\eta=0}^{N-1}\sum_{\mu=0}^{M-1}\int_{x^{e}_{\eta}}^{x^{e}_{\eta+1}}\int_{y^{p}_{\mu}}^{y^{p}_{\mu+1}} K(z,y,t,s)\Big(u(t-\tau_1,s-\tau_2)-v^{e-1,p-1}_{\eta,\mu}(t-\tau_1,s-\tau_2)\Big)\,ds\,dt\right)dz \\
&+\int_0^x\left(\sum_{e=1}^{\epsilon-1}\sum_{\eta=0}^{N-1}\sum_{\mu=0}^{m-1}\int_{x^{e}_{\eta}}^{x^{e}_{\eta+1}}\int_{y^{\rho}_{\mu}}^{y^{\rho}_{\mu+1}} K(z,y,t,s)\Big(u(t-\tau_1,s-\tau_2)-v^{e-1,\rho-1}_{\eta,\mu}(t-\tau_1,s-\tau_2)\Big)\,ds\,dt\right)dz \\
&+\int_0^x\left(\sum_{p=1}^{\rho-1}\sum_{\eta=0}^{n-1}\sum_{\mu=0}^{M-1}\int_{x^{\epsilon}_{\eta}}^{x^{\epsilon}_{\eta+1}}\int_{y^{p}_{\mu}}^{y^{p}_{\mu+1}} K(z,y,t,s)\Big(u(t-\tau_1,s-\tau_2)-v^{\epsilon-1,p-1}_{\eta,\mu}(t-\tau_1,s-\tau_2)\Big)\,ds\,dt\right)dz \\
&+\int_0^x\left(\sum_{\eta=0}^{n-1}\sum_{\mu=0}^{m-1}\int_{x^{\epsilon}_{\eta}}^{x^{\epsilon}_{\eta+1}}\int_{y^{\rho}_{\mu}}^{y^{\rho}_{\mu+1}} K(z,y,t,s)\Big(u(t-\tau_1,s-\tau_2)-v^{\epsilon-1,\rho-1}_{\eta,\mu}(t-\tau_1,s-\tau_2)\Big)\,ds\,dt\right)dz \\
&+\int_0^x\left(\sum_{e=1}^{\epsilon-1}\sum_{\eta=0}^{N-1}\int_{x^{e}_{\eta}}^{x^{e}_{\eta+1}}\int_{y^{\rho}_{m}}^{y} K(z,y,t,s)\Big(u(t-\tau_1,s-\tau_2)-v^{e-1,\rho-1}_{\eta,m}(t-\tau_1,s-\tau_2)\Big)\,ds\,dt\right)dz \\
&+\int_0^x\left(\sum_{\eta=0}^{n-1}\int_{x^{\epsilon}_{\eta}}^{x^{\epsilon}_{\eta+1}}\int_{y^{\rho}_{m}}^{y} K(z,y,t,s)\Big(u(t-\tau_1,s-\tau_2)-v^{\epsilon-1,\rho-1}_{\eta,m}(t-\tau_1,s-\tau_2)\Big)\,ds\,dt\right)dz \\
&+\int_0^x\left(\sum_{p=1}^{\rho-1}\sum_{\mu=0}^{M-1}\int_{x^{\epsilon}_{n}}^{z}\int_{y^{p}_{\mu}}^{y^{p}_{\mu+1}} K(z,y,t,s)\Big(u(t-\tau_1,s-\tau_2)-v^{\epsilon-1,p-1}_{n,\mu}(t-\tau_1,s-\tau_2)\Big)\,ds\,dt\right)dz \\
&+\int_0^x\left(\sum_{\mu=0}^{m-1}\int_{x^{\epsilon}_{n}}^{z}\int_{y^{\rho}_{\mu}}^{y^{\rho}_{\mu+1}} K(z,y,t,s)\Big(u(t-\tau_1,s-\tau_2)-v^{\epsilon-1,\rho-1}_{n,\mu}(t-\tau_1,s-\tau_2)\Big)\,ds\,dt\right)dz \\
&+\int_0^x\left(\int_{x^{\epsilon}_{n}}^{z}\int_{y^{\rho}_{m}}^{y} K(z,y,t,s)\Big(u(t-\tau_1,s-\tau_2)-v^{\epsilon-1,\rho-1}_{n,m}(t-\tau_1,s-\tau_2)\Big)\,ds\,dt\right)dz.
\end{aligned}
$$

Taking absolute values and using the boundedness of L, M and K, we obtain

$$\begin{aligned}
\left|u(x,y)-\hat{v}^{\epsilon,\rho}_{n,m}(x,y)\right| \le{}& \sum_{s=0}^{\epsilon-1} \tau_1 L_0 \left|u(z,y)-\hat{v}^{s,\rho}(z,y)\right| + hL_0 \sum_{\eta=0}^{n-1} \|u(z,y)-\hat{v}^{\epsilon,\rho}_{\eta,m}(z,y)\| \\
&+ L_0 \int_{x_n^\epsilon}^{x} \left|u(z,y)-\hat{v}^{\epsilon,\rho}_{n,m}(z,y)\right| dz + r_1\tau_1 k_0\Big(hM_0+\tau_1 M_0 + r_1\tau_1 M_0 \\
&\tau_1\tau_2+\tau_1\tau_2+\tau_1\tau_2+\tau_1\tau_2+\tau_1 k+\tau_1 k+\tau_2 h+\tau_2 h+hk\Big) \sum_{s=1}^{\epsilon-1}\sum_{p=1}^{\rho-1} \|e^{s-1,p-1}\| \\
&\le k_0 r_1 \tau_1^2\Big(2M_0+r_1M_0+9\tau_1\tau_2\Big)\sum_{s=1}^{\epsilon-1}\sum_{p=1}^{\rho-1} \|e^{s-1,p-1}\| + \tau_1 L_0 \sum_{s=0}^{\epsilon-1}\left|u(z,y)-\hat{v}^{s,\rho}(z,y)\right| \\
&+ hL_0 \sum_{\eta=0}^{n-1}\left|u(z,y)-\hat{v}^{\epsilon,\rho}_{\eta,m}(z,y)\right| + L_0\int_{x_n^\epsilon}^{x}\left|u(z,y)-\hat{v}^{\epsilon,\rho}_{n,m}(z,y)\right| dz \\
&\le c_5 \sum_{s=1}^{\epsilon-1}\sum_{p=1}^{\rho-1} \|e^{s-1,p-1}\| + \tau_1 L_0 \sum_{s=0}^{\epsilon-1}\left|u(z,y)-\hat{v}^{s,\rho}(z,y)\right| \\
&+ hL_0 \sum_{\eta=0}^{n-1}\left|u(z,y)-\hat{v}^{\epsilon,\rho}_{\eta,m}(z,y)\right| + L_0\int_{x_n^\epsilon}^{x}\left|u(z,y)-\hat{v}^{\epsilon,\rho}_{n,m}(z,y)\right| dz.
\end{aligned}$$

Using Lemma 1.6.2, we get

$$\begin{aligned}
\left|u(x,y)-\hat{v}^{\epsilon,\rho}_{n,m}(x,y)\right| \le{}& \Bigg[c_5 \sum_{s=1}^{\epsilon-1}\sum_{p=1}^{\rho-1} \|e^{s-1,p-1}\| + \tau_1 L_0 \sum_{s=0}^{\epsilon-1}\left|u(z,y)-\hat{v}^{s,\rho}(z,y)\right| \\
&+ hL_0 \sum_{\eta=0}^{n-1}\left|u(z,y)-\hat{v}^{\epsilon,\rho}_{\eta,m}(z,y)\right|\Bigg]. \underbrace{exp(\tau_1.L_0)}_{c_6} \\
&\le c_5 c_6 \sum_{s=1}^{\epsilon-1}\sum_{p=1}^{\rho-1} \|e^{s-1,p-1}\| + c_6\tau_1 L_0 \sum_{s=0}^{\epsilon-1}\left|u(z,y)-\hat{v}^{s,\rho}(z,y)\right| \\
&+ c_6 hL_0 \sum_{\eta=0}^{n-1}\left|u(z,y)-\hat{v}^{\epsilon,\rho}_{\eta,m}(z,y)\right|.
\end{aligned}$$

Using Lemma 1.6.1, for fixed m, ϵ, ρ

$$\left|u(x,y) - \hat{v}^{\epsilon,\rho}_{n,m}(x,y)\right| \leq \left[c_5 c_6 \sum_{s=1}^{\epsilon-1}\sum_{p=1}^{\rho-1} ||e^{s-1,p-1}|| + c_6\tau_1 L_0 \sum_{s=0}^{\epsilon-1}\left|u(z,y) - \hat{v}^{s,\rho}(z,y)\right|\right].\underbrace{exp(c_6\tau_1 L_0)}_{c_7}.$$

Then

$$\left|u(x,y) - \hat{v}^{\epsilon,\rho}(x,y)\right| \leq c_5 c_6 c_7 \sum_{s=1}^{\epsilon-1}\sum_{p=1}^{\rho-1} ||e^{s-1,p-1}|| + c_6 c_7 \tau_1 L_0 \sum_{s=0}^{\epsilon-1}\left|u(z,y) - \hat{v}^{s,\rho}(z,y)\right|.$$

such That

$$v^{\epsilon,\rho} = \max_{\substack{0\leq n\leq N-1\\ 0\leq m\leq M-1}}\left\{v^{\epsilon,\rho}_{n,m}\right\}$$

Using Lemma 1.6.1, for fixed ρ

$$\left|u(x,y) - \hat{v}^{\epsilon,\rho}(x,y)\right| \leq \left(c_5 c_6 c_7 \sum_{s=1}^{\epsilon-1}\sum_{p=1}^{\rho-1} ||e^{s-1,p-1}||\right).\underbrace{exp(c_6 c_7 \tau_1 L_0 r_1)}_{c_8}.$$

$$\left|u(x,y) - \hat{v}^{\epsilon,\rho}(x,y)\right| \leq \underbrace{c_5 c_6 c_7 c_8}_{c_9} \sum_{s=1}^{\epsilon-1}\sum_{p=1}^{\rho-1} ||e^{s-1,p-1}||.$$

Using the decomposition

$$\begin{aligned} ||e^{\epsilon,\rho}|| &\leq ||u - \hat{v}^{\epsilon,\rho}|| + ||\hat{v}^{\epsilon,\rho} - v^{\epsilon,\rho}||\,, \\ &\leq c_9 \sum_{s=1}^{\epsilon-1}\sum_{p=1}^{\rho-1} ||e^{s-1,p-1}|| + \frac{\alpha(p)}{(p+1)!}(h+k)^{p+1}. \end{aligned}$$

Using Lemma 1.6.4, we get

$$\begin{aligned}
||e^{\epsilon,\rho}|| &\leq \frac{\alpha(p)}{(p+1)!}(h+k)^{p+1}.\prod_{s=0}^{\epsilon-1}\left[1+\underbrace{\sum_{p=0}^{\rho-1}c_9}_{r_2c_9}\right] \\
&\leq \frac{\alpha(p)}{(p+1)!}(h+k)^{p+1}.\,(1+r_2c_9)^{\epsilon} \\
&\leq c_{10}(h+k)^{p+1}.
\end{aligned}$$

By Claims 1 and 2 we already have

$$||e^{e,\rho}||_{L^\infty(D^{e,\rho})} \leq C(h+k)^{p+1}, \qquad ||e^{\epsilon,p}||_{L^\infty(D^{\epsilon,p})} \leq C(h+k)^{p+1}.$$

Hence

$$||e^{\epsilon,\rho}_{n,m}||_{L^\infty(D^{\epsilon,\rho}_{n,m})} \leq C_4(h+k)^{p+1},$$

where C_4 is independent of h and k. This proves Claim 3.

Now, from Claims 1–3, we know that there exist positive constants C_1, C_2, and C_4, independent of h and k, such that

$$||e^{0,0}_{0,0}||_{L^\infty(D^{0,0}_{n,m})} \leq C_1(h+k)^{p+1},$$

$$||e^{\epsilon,0}_{n,m}||_{L^\infty(D^{\epsilon,0}_{n,m})} \leq C_2(h+k)^{p+1},$$

$$||e^{0,\rho}_{n,m}||_{L^\infty(D^{0,\rho}_{n,m})} \leq C_3(h+k)^{p+1},$$

and

$$||e^{\epsilon,\rho}_{n,m}||_{L^\infty(D^{\epsilon,\rho}_{n,m})} \leq C_4(h+k)^{p+1},$$

for all admissible indices

$$n=0,\ldots,N-1, \qquad m=0,\ldots,M-1, \qquad \epsilon=0,\ldots,r_1-1, \qquad \rho=0,\ldots,r_2-1.$$

Set

$$C := \max\{C_1, C_2, C_3, C_4\}.$$

Since

$$D = \bigcup_{\epsilon=0}^{r_1-1} \bigcup_{\rho=0}^{r_2-1} \bigcup_{n=0}^{N-1} \bigcup_{m=0}^{M-1} D_{n,m}^{\epsilon,\rho},$$

we obtain

$$\begin{aligned} \|e\|_{L^\infty(D)} &= \max_{\substack{0\le\epsilon\le r_1-1 \\ 0\le\rho\le r_2-1 \\ 0\le n\le N-1 \\ 0\le m\le M-1}} \|e_{n,m}^{\epsilon,\rho}\|_{L^\infty(D_{n,m}^{\epsilon,\rho})} \\ &\le C(h+k)^{p+1}. \end{aligned}$$

Therefore,

$$\|u - v\|_{L^\infty(D)} \le C(h+k)^{p+1}.$$

which completes the proof of Theorem 3.3.1.

■

3.4 Experimental Results

In this section, we validate the theoretical results established in the previous sections by applying the proposed method to numerical examples. In each case, the exact solution is known, which allows a direct comparison with the approximate solution. The Taylor collocation method is then used to compute the numerical solution, and its accuracy is assessed by evaluating the absolute error between the exact solution u and the approximate solution v. This comparison illustrates the efficiency and accuracy of the proposed approach.

Example 3.4.1 *As a first example, we consider the following problem with time delays* $\tau_1 = \tau_2 = \dfrac{1}{5}$*:*

$$\begin{aligned}\frac{\partial u}{\partial x}(x,y) &= g(x,y) + xy\,u\left(x-\frac{1}{5}, y-\frac{1}{5}\right)\\ &\quad + \int_0^x \int_0^y \left(x+y+t+s^2\right) u\left(t-\frac{1}{5}, s-\frac{1}{5}\right) ds\,dt,\\ &\qquad\qquad\qquad\qquad\qquad x\in[0,1],\quad y\in[0,1].\end{aligned}$$

The exact solution of the problem is given by

$$u(x,y) = y^4 + \sin(x).$$

Accordingly, the function $g(x,y)$ is determined from this exact solution. Numerical results are obtained by applying the proposed Taylor collocation method for different values of the polynomial degree p. The absolute errors corresponding to $p = 3,4,5$ are reported in Table 3.1 and illustrated in Fig. 3.3 and Fig. 3.1. It can be clearly observed that the absolute error decreases as p increases, confirming the accuracy of the proposed method. Moreover, the exact and approximate solutions for $N = M = 5$ are shown in Fig. 3.2.

	$N = M = 5$		
(x,y)	$p=3$	$p=4$	$p=5$
$(0,0)$	0	0	0
$(0.1,0.1)$	9.99×10^{-5}	9.74×10^{-13}	9.74×10^{-13}
$(0.2,0.2)$	2.00×10^{-4}	2.68×10^{-11}	2.68×10^{-11}
$(0.3,0.3)$	2.99×10^{-4}	7.63×10^{-11}	7.63×10^{-11}
$(0.4,0.4)$	4.00×10^{-4}	7.49×10^{-12}	7.49×10^{-12}
$(0.5,0.5)$	5.01×10^{-4}	7.62×10^{-8}	5.23×10^{-10}
$(0.6,0.6)$	6.03×10^{-4}	7.73×10^{-7}	6.70×10^{-10}
$(0.7,0.7)$	7.01×10^{-4}	4.50×10^{-6}	4.38×10^{-8}
$(0.8,0.8)$	7.79×10^{-4}	1.89×10^{-5}	3.90×10^{-7}
$(0.9,0.9)$	7.84×10^{-4}	6.33×10^{-5}	2.14×10^{-6}
CPU time (sec)	9.02	9.98	10.67

Table 3.1 Numerical results for Example 3.4.1.

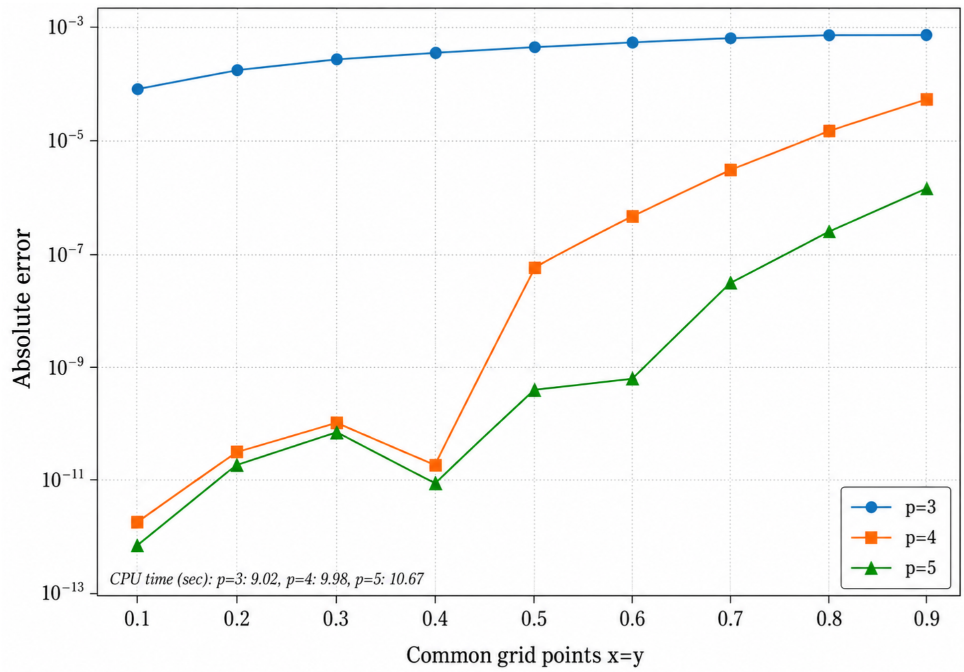
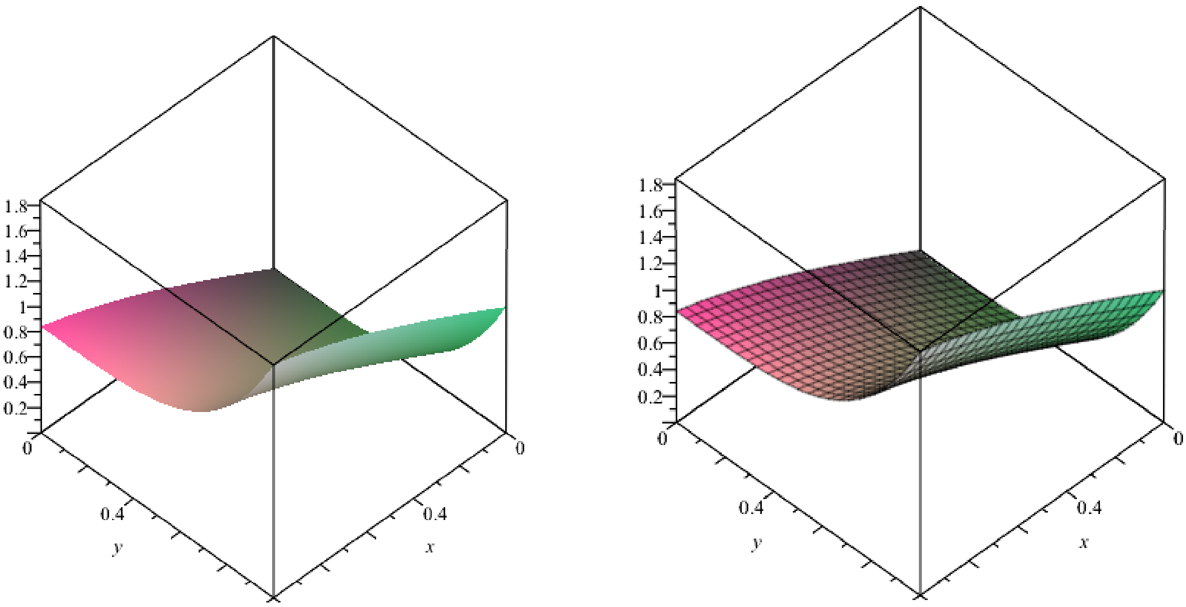


Figure 3.1: Numerical results for Example 3.4.1.



(a) *Exact solution* (b) *Approximate solution*

Figure 3.2: Exact and approximate solution of Example 3.4.1 for $N = M = 5$.

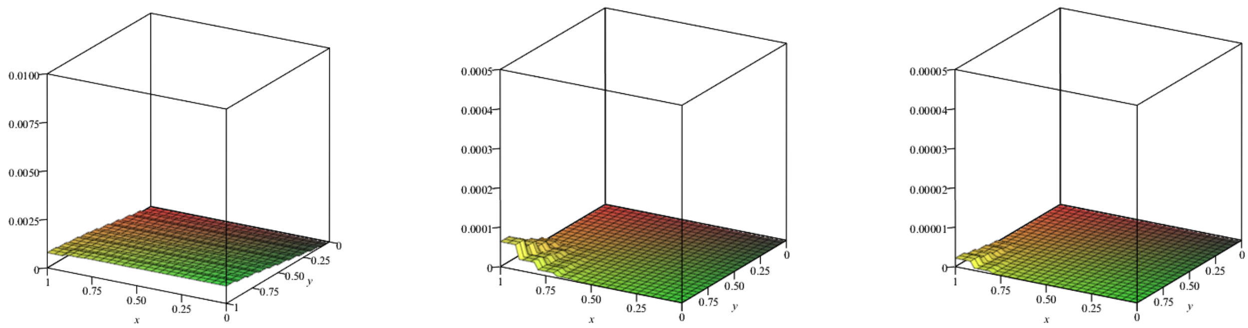


Figure 3.3: Absolute errors function of Example 3.4.1 for $p = 3, 4, 5$, respectively.

Example 3.4.2 *As a second example, we consider the following two-dimensional Volterra integro-differential problem with constant delays*

$$\tau_1 = \tau_2 = \frac{1}{2}.$$

The problem is defined on the domain

$$(x, y) \in [0, 1] \times [0, 1],$$

and is given by

$$\begin{aligned}\frac{\partial u}{\partial x}(x, y) = g(x, y) + L(x, y)u(x, y) + M(x, y)u\left(x - \frac{1}{2}, y - \frac{1}{2}\right)\\ + \int_0^x \int_0^y K(x, y, t, s)\, u\left(t - \frac{1}{2}, s - \frac{1}{2}\right) ds\, dt,\end{aligned}$$

where

$$L(x, y) = 0, \qquad M(x, y) = x \sin(y),$$

and

$$K(x, y, t, s) = s \cos(x) + 2 \sin(t + y).$$

The exact solution is chosen as

$$u(x, y) = y \cos(x).$$

Consequently, the function $g(x, y)$ is obtained by substituting the exact solution into the given *equation.*

The Taylor collocation method is applied for $N = M = 5$ and for different values of the polynomial degree p. The absolute error is computed by

$$E(x, y) = \left| u(x, y) - v(x, y) \right|,$$

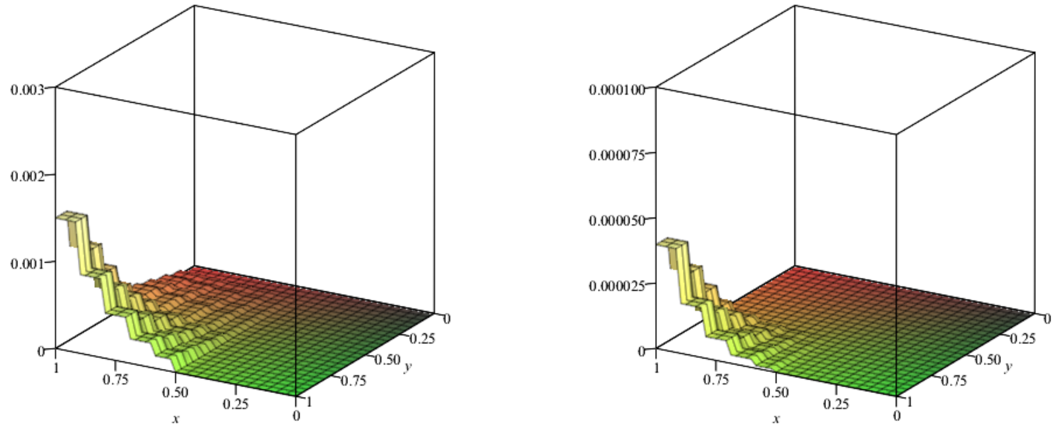
where u is the exact solution and v is the approximate solution. The numerical results corresponding to $p = 2, 3, 4$ are presented in Table 3.2. In addition, the absolute error profiles are illustrated in Fig. 3.4. The obtained results show that the proposed method provides accurate approximations for the considered delayed two-dimensional Volterra integro-differential problem.

	$N = M = 5$		
(x, y)	$p = 2$	$p = 3$	$p = 4$
$(0, 0)$	0	0	0
$(0.1, 0.1)$	3.53×10^{-12}	3.53×10^{-12}	3.53×10^{-12}
$(0.2, 0.2)$	3.18×10^{-11}	3.18×10^{-11}	3.18×10^{-11}
$(0.3, 0.3)$	4.79×10^{-13}	4.79×10^{-13}	4.79×10^{-13}
$(0.4, 0.4)$	3.06×10^{-11}	3.06×10^{-11}	3.06×10^{-11}
$(0.5, 0.5)$	2.74×10^{-5}	1.52×10^{-6}	1.33×10^{-8}
$(0.6, 0.6)$	1.03×10^{-4}	2.98×10^{-6}	6.56×10^{-7}
$(0.7, 0.7)$	2.76×10^{-4}	3.44×10^{-6}	3.58×10^{-6}
$(0.8, 0.8)$	6.64×10^{-4}	1.32×10^{-5}	3.47×10^{-6}
$(0.9, 0.9)$	1.49×10^{-3}	3.99×10^{-5}	3.97×10^{-5}

Table 3.2 Absolute errors for Example 3.4.2 with different values of p.

3.5 Conclusion

In this chapter, we proposed a new numerical method based on Taylor polynomials to construct a collocation solution for approximating the solution of delay Volterra integral equations (DVIEs) with a spatial variable. Unlike existing approaches, our method efficiently handles both time delay and spatial dependence in the integral equation, which is a key challenge in solving such problems



(a) $p = 2$ (b) $p = 4$

Figure 3.4: Absolute error profiles for Example 3.4.2 with $N = M = 5$.

A rigorous convergence analysis confirms that the proposed method is accurate and stable, with an $O(h + k)^{p+1}$ error bound. Numerical examples validate its effectiveness, demonstrating that it provides high-precision approximations while maintaining computational efficiency.

The novelty of our approach lies in the extension of the Taylor Collocation Method to DVIEs with spatial dependence,an area that has received limited attention in previous studies. Additionally, our method is computationally efficient as it computes the approximation coefficients iteratively without requiring the solution of any system of algebraic equations. Moreover, it can be adapted to various delay integral equations encountered in applied sciences and engineering.

These contributions make our method a powerful and efficient alternative for solving DVIEs with spatial variables,laying the groundwork for further extensions to higherdimensional problems.

CONCLUSION AND PERSPECTIVES

This thesis has been devoted to the numerical analysis and approximation of delayed and multi-dimensional Volterra integral and integro-differential equations. Such equations arise naturally in the modeling of dynamical systems with memory effects, where the current state depends not only on the present variables but also on past spatial-temporal interactions. Because of the presence of delay terms and integral operators, analytical solutions are generally unavailable, making the development of efficient numerical techniques an important research topic.

The main contribution of this thesis is the development of Taylor collocation methods for solving classes of delay Volterra integral and integro-differential equations with spatial variables. The proposed numerical framework combines Taylor polynomial approximations with collocation techniques to construct accurate approximations of the exact solution. A significant advantage of the developed algorithms is that the approximation coefficients are computed iteratively, avoiding the solution of large systems of algebraic equations and leading to a computationally efficient implementation.

Theoretical analyses have established the convergence and stability of the proposed methods under suitable assumptions, providing a rigorous mathematical justification for their application. These results guarantee that the numerical approximations converge to the exact solutions as the discretization parameters are refined.

The effectiveness of the proposed methods has been verified through several numerical examples. The obtained results demonstrate excellent agreement between the exact and approximate solutions, with very small absolute errors. Furthermore, comparisons with existing methods confirm that the proposed Taylor collocation approach achieves higher accuracy while maintaining a simple and efficient computational procedure.

Overall, the results of this thesis demonstrate that Taylor collocation methods provide reliable, accurate, and flexible numerical tools for the solution of delay Volterra integral and integro-differential equations. The proposed framework offers an effective approach for treating a broad class of problems arising in applied mathematics, physics, engineering, and related scientific disciplines.

Perspectives. Several directions for future research can be considered based on the results of this work. First, the proposed methods may be extended to nonlinear delay Volterra integro-differential equations and coupled systems. Second, they can be generalized to higher-dimensional problems involving more spatial variables. Another promising direction is the development of adaptive and higher-order collocation techniques to further improve accuracy and computational efficiency. In addition, combining the Taylor collocation framework with modern numerical approaches, such as spectral methods or machine learning techniques, may enhance its performance for large-scale problems. Finally, applying the proposed methods to realistic models arising in biology, physics, engineering, and other applied sciences represents an interesting topic for future investigation.

BIBLIOGRAPHY

[1] Ali I., Brunner H., Tang T., Spectral methods for pantograph-type differential and integral equations with multiple delays, *Frontiers of Mathematics in China*, **4**, 49–61, 2009. https://doi.org/10.1007/s11464-009-0010-z

[2] Bellman R., The stability of solutions of linear differential equations, *Duke Mathematical Journal*, **10**, 643–647, 1943. https://doi.org/10.1215/S0012-7094-43-01059-2

[3] Bellour A., Bousselsal M., A Taylor collocation method for solving delay integral equations, *Numerical Algorithms*, **65**(4), 843–857, 2014. https://doi.org/10.1007/s11075-013-9717-8

[4] Birem F., Boulmerka A., Laib H., Hennous C., An algorithm for solving first-kind two-dimensional Volterra integral equations using collocation method, *Nonlinear Dynamics and Systems Theory*, **23**(5), 475–486, 2023. https://doi.org/10.46300/9108.2023.23.5.8

[5] Bouzeraieb H., Laib H., Boulmerka A., Numerical solution of neutral double delay Volterra integral equations using Taylor collocation method, *Nonlinear Dynamics*

and Systems Theory, **24**(3), 236–245, 2024. https://doi.org/10.46300/9108.2024.24.3.3

[6] Christensen R.M., *Theory of Viscoelasticity: An Introduction*, 2nd ed., Academic Press, New York, 1982.

[7] Brunner H., *Collocation Methods for Volterra Integral and Related Functional Differential Equations*, Cambridge University Press, Cambridge, 2004. https://doi.org/10.1017/CBO9780511543234

[8] Brunner H., The numerical solution of two-dimensional Volterra integral equations by collocation and iterated collocation, *IMA Journal of Numerical Analysis*, **9**, 47–59, 1989. https://doi.org/10.1093/imanum/9.1.47

[9] Brunner H., Yatsenko Y., Spline collocation methods for nonlinear Volterra integral equations with unknown delay, *Journal of Computational and Applied Mathematics*, **71**, 67–81, 1996. https://doi.org/10.1016/0377-0427(95)00231-4

[10] Ghorpade S.R., Limaye B.V., *A Course in Multivariable Calculus and Analysis*, Springer, New York, 2010. https://doi.org/10.1007/978-1-4419-1620-1

[11] Hale J.K., Verduyn Lunel S.M., *Introduction to Functional Differential Equations*, Springer, New York, 1993. https://doi.org/10.1007/978-1-4612-4342-7

[12] Khennaoui C., Bellour A., Laib H., Taylor collocation method for solving two-dimensional partial Volterra integro-differential equations, *Mathematical Methods in the Applied Sciences*, **46**(12), 12735–12758, 2023. https://doi.org/10.1002/mma.9210

[13] Karamete A., Sezer M., Spectral Taylor matrix collocation method for linear integro-differential equations, *Applied Mathematics and Computation*, **131**(2), 273–286, 2002. https://doi.org/10.1016/S0096-3003(01)00179-8

[14] Laib H., Boulmerka A., Bellour A., Birem F., Numerical solution of two-dimensional linear and nonlinear Volterra integral equations using Taylor col-

location method, *Journal of Computational and Applied Mathematics*, **417**, 114537, 2023. https://doi.org/10.1016/j.cam.2022.114537

[15] Linz P., *Analytical and Numerical Methods for Volterra Equations*, Society for Industrial and Applied Mathematics (SIAM), 1985.

[16] McKee S., Tang T., Diogo T., An Euler-type method for two-dimensional Volterra integral equations of the first kind, *IMA Journal of Numerical Analysis*, **20**(3), 423–440, 2000. https://doi.org/10.1093/imanum/20.3.423

[17] Nemati S., Lima P.M., Ordokhani Y., Numerical solution of a class of two-dimensional nonlinear Volterra integral equations using Legendre polynomials, *Journal of Computational and Applied Mathematics*, **242**, 53–69, 2013. https://doi.org/10.1016/j.cam.2012.10.004

[18] Puu T., *Attractors, Bifurcations, and Chaos: Nonlinear Phenomena in Economics*, 2nd ed., Springer, Berlin, 2000. https://doi.org/10.1007/978-3-642-57113-8

[19] Renardy M., Rogers R.C., *An Introduction to Partial Differential Equations*, 2nd ed., Springer, New York, 2004. https://doi.org/10.1007/b97427

[20] Saidane S., Laib H., A Numerical Approach for Solving Delay Volterra Integral Equations with a Spatial Variable and Mixed Kernels, *Nonlinear Dynamics and Systems Theory*, **25**(6), 669–678, 2025. https://doi.org/10.52783/nds.v25i6.1904

[21] Smith H.L., *An Introduction to Delay Differential Equations with Applications to the Life Sciences*, Springer, New York, 2011. https://doi.org/10.1007/978-1-4419-7646-5

[22] Taylor P.M., *Multivariable Calculus Lecture Notes: Taylor's Theorem for Functions of Two Variables*, University of Lancaster, 2025.

[23] Wazwaz A.M., *Linear and Nonlinear Integral Equations: Methods and Applications*, Springer, 2011. https://doi.org/10.1007/978-3-642-21449-3

[24] Wazwaz A.M., *Partial Differential Equations and Boundary Value Problems with Volterra Integro-Differential Terms*, Springer, 2013. https://doi.org/10.1007/978-3-642-27308-7

[25] Zhang L., Huang J., Li H., Splitting extrapolation algorithms for solving linear delay Volterra integral equations with a spatial variable, *Applied Numerical Mathematics*, **178**, 372–385, 2022. https://doi.org/10.1016/j.apnum.2022.04.019

[26] Zheng W., Chen Y., Numerical analysis for Volterra integral equation with two kinds of delay, *Acta Mathematica Scientia*, **39**(2), 607–617, 2019. https://doi.org/10.1007/s10473-019-0211-3