الجمهوريـة الجزائـريـة الديمقراطيـة الشعبيـة
People's Democratic Republic of Algeria
وزارة التعليـم العالـي والبحـث العلمـي
Ministry of Higher Education and Scientific Research

University of Mila
جـــامعة ميلـة

Faculty of Mathematics and Computer Sciences
Order N°: …………………
Registration number:

Department: Mathematics
Field : Mathematics
Specialty: Applied mathematics

Thesis

Submitted for the degree of
Doctorate LMD

Numerical Study of Volterra Integral Equations with Delay

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University year: 2025/2026

ACKNOWLEDGMENTS

First and foremost, I would like to express my sincere gratitude to my supervisor, Dr. Hafida Laib, for her invaluable guidance, continuous support, and constructive advice throughout the preparation of this thesis. Her scientific insight, patience, and encouragement have greatly contributed to the successful completion of this work.

I would also like to extend my sincere thanks to the members of the jury for kindly accepting to evaluate this thesis and for the time and effort they devoted to reviewing this work. Their valuable comments and suggestions are highly appreciated.

My gratitude also goes to all the professors and staff members of the Department of Mathematics at the University of Mila for the quality of the education and the support provided during my academic journey.

I would also like to thank my colleagues and friends for their encouragement, discussions, and moral support during the course of my doctoral studies.

Finally, my deepest gratitude goes to my family for their unconditional love, patience, and constant encouragement. Their support has been a fundamental source of motivation throughout this work.

DEDICATION

To my parents.

To my brothers.

To my children, my husband.

To everyone who encouraged me.

ABSTRACT

This thesis is devoted to the development of Taylor collocation method for a class of Volterra-type problems, including linear delay Volterra integral equations, and Volterra integro-differential equations with two constant delays. Such models naturally arise in many applied fields, including population dynamics, epidemiology, viscoelasticity, control theory, and systems with memory effects, where hereditary phenomena and time delays play a crucial role.

The solution is approximated by piecewise Taylor polynomials on each subinterval, leading to an explicit and direct numerical scheme. A key advantage of the method is that it avoids the solution of large algebraic systems, which significantly reduces computational cost and enhances numerical stability. The coefficients of the local Taylor expansions are computed recursively using the governing equations, ensuring a consistent and globally defined approximation.

A rigorous convergence analysis is carried out under appropriate smoothness assumptions on the data. Using discrete Grönwall-type inequalities, it is shown that the proposed method converges. Several numerical experiments are presented to validate the theoretical results and to illustrate the accuracy, and efficiency of the method for different delay configurations.

Key Words: Volterra delay integral equation, Volterra delay integro-differential equation, Collocation method, Taylor polynomials.

RÉSUMÉ

Cette thèse est consacrée au développement de la méthode de collocation de Taylor pour une classe de problèmes de type Volterra, incluant les équations intégrales de Volterra linéaires à retard ainsi que les équations intégro-différentielles de Volterra avec deux retards constants. De tels modèles apparaissent naturellement dans de nombreux domaines appliqués, notamment la dynamique des populations, lépidémiologie, la viscoélasticité, la théorie du contrôle et les systèmes à mémoire, où les phénomènes héréditaires et les retards temporels jouent un rôle essentiel.

La solution est approchée par des polynômes de Taylor définis par morceaux sur chaque sous-intervalle, ce qui conduit à un schéma numérique explicite et direct. Un avantage majeur de la méthode réside dans le fait quelle ne nécessite pas la résolution de grands systèmes algébriques, ce qui permet de réduire significativement le coût de calcul et daméliorer la stabilité numérique. Les coefficients des développements de Taylor locaux sont calculés de manière récursive à partir des équations gouvernantes, garantissant ainsi une approximation cohérente et globalement définie.

Une analyse rigoureuse de la convergence est menée sous des hypothèses appropriées de régularité des données. En utilisant des inégalités de type Grönwall discrètes, il est démontré que la méthode proposée est convergente. Plusieurs expériences numériques sont présentées afin de valider les résultats théoriques et dillustrer la précision et lefficacité de la méthode pour différentes configurations de retard.

Mots-clés : équation intégrale de Volterra à retard, équation intégro-différentielle de Volterra à retard, méthode de collocation, polynômes de Taylor.

ملخص

في هذه الأطروحة تم تطوير طريقة التقريب بالتجميع (Collocation) المعتمدة على كثير حدود تايلور لفئة من مسائل فولتيرا، بما في ذلك معادلات فولتيرا التكاملية الخطية ذات التأخير ومعادلات فولتيرا التكاملية-التفاضلية ذات تأخيرين ثابتين. وتظهر هذه النماذج بطبيعتها في العديد من المجالات التطبيقية، مثل ديناميكيات السكان، وعلم الأوبئة، والمرونة اللزجة، ونظرية التحكم، والأنظمة ذات الذاكرة، حيث تلعب الظواهر الوراثية والتأخيرات الزمنية دورًا محوريًا.

يتم تقريب الحل باستخدام كثيرات حدود تايلور المعرَّفة على أجزاء فوق كل مجال فرعي، مما يؤدي إلى مخطط عددي صريح ومباشر. وتكمن إحدى المزايا الأساسية لهذه الطريقة في أنها لا تتطلب حل أنظمة جبرية كبيرة، الأمر الذي يُسهم في تقليل الكلفة الحسابية بشكل ملحوظ وتحسين الاستقرار العددي. ويتم حساب معاملات التوسعات المحلية لمتسلسلات تايلور بصورة تراجعية اعتمادًا على المعادلات الحاكمة، بما يضمن تقريبًا متسقًا ومُعرَّفًا على المجال كاملاً.

تم إجراء تحليل تقاربي تحت فرضيات مناسبة تتعلق بنعومة المعطيات. وباستخدام متراجحات من نوع غرونوول المتقطعة، يُثبت أن الطريقة المقترحة متقاربة. كما تم عرض عدة امثلة عددية للتحقق من صحة النتائج النظرية وإبراز دقة وكفاءة الطريقة لمختلف إعدادات التأخير.

الكلمات المفتاحية: معادلة فولتيرا التكاملية ذات تأخير، معادلة فولتيرا التكاملية-التفاضلية ذات تأخير، طريقة التجميع، كثير حدود تايلور.

Contents

General Introduction

Volterra integral equations constitute one of the most important classes of functional equations in applied mathematics and have attracted considerable attention due to their ability to model systems with memory and hereditary effects [23, 28, 29, 30]. These equations are named after the Italian mathematician Vito Volterra, whose pioneering work in population dynamics laid the foundations for mathematical modeling of time-dependent phenomena [28, 29]. Since then, Volterra-type equations have undergone significant theoretical and numerical development and have found wide-ranging applications in science and engineering [12, 13, 23].

In their classical form, Volterra integral equations involve functions of a single independent variable, typically time or space. Their distinguishing feature is the dependence of the current state of the system on its past history, which makes them particularly suitable for describing cumulative processes and systems with memory effects [13, 30, 31]. This characteristic allows Volterra equations to provide realistic and flexible models for complex dynamical behaviors that cannot be adequately represented by ordinary differential equations alone [4, 17, 20].

The study of Volterra integral equations encompasses both theoretical and numerical aspects. From a theoretical perspective, attention is devoted to issues such as existence, uniqueness, and stability of solutions [13, 17, 23]. From a numerical point of view, the focus lies on the construction of efficient and accurate approximation methods, especially in situations where closed-form analytical solutions are unavailable [12, 16, 19]. Due to the increasing complexity of models arising in practical applications, numerical methods have become indispensable, motivating the development of a wide variety of specialized algorithms [23, 12, 30].

Volterra integral and integro-differential equations arise in numerous scientific fields, including population dynamics, epidemic modeling, viscoelasticity, control theory, semiconductor physics, and material science [15, 8, 13, 30]. In particular, models incorporating time delays play a crucial role when the influence of past states on the current behavior of a system is not instantaneous [17, 20, 4]. The inclusion of two constant delays enables the simultaneous representation of multiple delayed effects, thereby enhancing the realism of the mathematical model [5, 25].

A typical example of a Volterra integral equation with two constant delays is given by

$$x(t) = \begin{cases} g(t) + \displaystyle\int_{t-\tau_2}^{t-\tau_1} k(t,s)\, f\Big(s, x(s)\Big)\, ds, & t \in [\tau_2, +\infty), \\ \Phi(t), & t \in [0, \tau_2), \end{cases} \tag{1}$$

where g is a given continuous function, $k(t,s)$ is the kernel of the integral operator, f is a nonlinear function, and Φ denotes the prescribed initial function [5, 25].

Equations of the form (1) naturally arise in the mathematical modeling of age-structured population dynamics. In such models, the population density at time t depends on the contribution of individuals whose ages lie within a specific interval. In particular, the delays τ_1 and τ_2 may represent the maturation age and the maximum life expectancy, respectively. This formulation allows the simultaneous consideration of multiple age classes within the same population and provides a realistic description of biological growth processes. Similar models have been studied extensively in the literature; see, for example, [10, 11, 18].

In epidemiology, the spread of infectious diseases with incubation and infection delays can be modeled using a Volterra integral equation of the form [14, 8, 32, 21]:

$$I(t) = \int_0^t \beta(t - s - \tau_1)\, G(I(s - \tau_2))\, ds, \tag{2}$$

where $I(t)$ denotes the number of infected individuals, τ_1 corresponds to the incubation period, τ_2 represents delayed infectivity, and β is the transmission kernel.

Predatorprey interactions with delayed biological responses can be described by a system of neutral Volterra integro-differential equations with two constant delays [12, 15, 20]

$$\begin{cases} \dfrac{dN_1(t)}{dt} = N_1(t)\Big(a_1 - b_1 N_2(t-\tau_1) - \displaystyle\int_0^{t-\tau_2} F_1(t,s)N_1(s)\,ds\Big), \\ \dfrac{dN_2(t)}{dt} = N_2(t)\Big(a_2 - b_2 N_1(t-\tau_1) - \displaystyle\int_0^{t-\tau_2} F_2(t,s)N_2(s)\,ds\Big), \end{cases} \tag{3}$$

where $N_1(t)$ and $N_2(t)$ represent prey and predator populations, respectively, and the delays account for gestation and adaptation effects.

In viscoelasticity, materials with memory effects are commonly modeled by Volterra integral equations involving delayed strain [28, 13, 30]:

$$\sigma(t) = \int_0^t G(t-s-\tau_1)\,\varepsilon(s-\tau_2)\,ds, \tag{4}$$

where $\sigma(t)$ denotes stress, $\varepsilon(t)$ is the strain, and G is the relaxation kernel characterizing the material properties.

These representative models demonstrate how Volterra integral and integro-differential equations with two constant delays naturally arise in real-world applications [15, 20, 4, 5]. They also highlight the necessity of developing efficient numerical methods capable of accurately approximating solutions to such equations, which constitutes the main objective of the present thesis [12, 6, 7].

Analytical solutions for such equations are often difficult or impossible to obtain, which has led to the development of modern numerical techniques [23, 12, 30]. Among the most widely used approaches are collocation methods, spectral methods, Runge–Kutta schemes, and Bernstein or tau methods [16, 12, 19]. In particular, collocation methods based on Taylor polynomials offer several advantages, including simplicity of implementation, explicit formulation of the approximate solution, and guaranteed convergence under suitable conditions [19, 6, 7, 22].

Motivated by these considerations, the present thesis is devoted to the numerical study of Volterra integral and integro-differential equations of neutral type involving two constant delays. The main objective is to develop efficient numerical algorithms based on the Taylor collocation method and to analyze their convergence and accuracy through rigorous theoretical results and illustrative numerical examples [9].

The thesis is organized as follows.

Chapter 1 presents the preliminary concepts and auxiliary results required for the subsequent analysis. Fundamental definitions, notations, and theoretical tools are introduced to establish a solid mathematical framework.

Chapter 2 is devoted to the numerical approximation of linear neutral Volterra integral equations with two constant delays. A direct collocation method based on Taylor polynomials is proposed, and a computational algorithm is developed. Several numerical examples are provided to confirm the theoretical results and demonstrate the convergence of the method.

Chapter 3 extends the proposed approach to linear neutral Volterra integro-differential equations with two constant delays. A Taylor collocation-based numerical scheme is constructed, and its effectiveness is validated through numerical experiments illustrating the accuracy and convergence of the method.

This work aims to contribute to the development of reliable and efficient numerical tools for the solution of delay Volterra equations, which continue to play a central role in modeling real-world phenomena with memory effects.

Chapter 1

Preliminary and Auxiliary Results

This chapter is devoted to the presentation of the basic concepts, notations, and preliminary theoretical results that form the foundation of the subsequent mathematical analysis. The aim is to establish a clear and rigorous framework by introducing essential definitions and fundamental theorems required for the development of the proposed numerical methods. These preliminary results ensure logical consistency and mathematical coherence throughout the thesis and facilitate a smooth transition toward more advanced topics.

1.1 Taylor Expansion

The Taylor expansion provides a powerful tool for approximating a sufficiently smooth function in the neighborhood of a given point by a polynomial whose coefficients depend on the derivatives of the function at that point. It is widely used in mathematical analysis, numerical methods, and engineering applications, as it allows complex functions to be locally approximated with high accuracy and facilitates both theoretical investigations and computational implementations.

Definition 1.1.1 *(Taylor's Theorem) Let f be a function of class $\mathcal{C}^n$ on an interval I containing the point x_0. Then, for all $x \in I$, the function f can be expressed as*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + (x - x_0)^n \, \varepsilon(x - x_0),$$

where the remainder term $\varepsilon(x - x_0)$ satisfies

$$\lim_{x \to x_0} \varepsilon(x - x_0) = 0.$$

This representation shows that f can be approximated by a polynomial of degree n around the point x_0, with an error term that vanishes as x approaches x_0.

1.2 Integral Equations

An integral equation is an equation in which the unknown function appears under the integral sign. Integral equations constitute fundamental and powerful tools in both theoretical and applied mathematics, as they naturally arise in the modeling and analysis of a wide range of physical, biological, and engineering phenomena.

In many situations, problems governed by ordinary or partial differential equations, together with initial or boundary conditions, can be reformulated as equivalent integral equations. This transformation often simplifies the analysis and provides alternative approaches for both theoretical investigations and numerical approximations.

A general form of a linear integral equation can be written as

$$G(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} K(x,s)\, g(s)\, ds, \tag{1.1}$$

where $\alpha(x)$ and $\beta(x)$ denote the limits of integration, λ is a constant parameter, $K(x,s)$ is a known function called the kernel of the integral equation, and $f(x)$ is a given function. The function $g(x)$ is the unknown to be determined and may appear either inside the integral only or both inside and outside the integral.

The limits of integration $\alpha(x)$ and $\beta(x)$ may be fixed constants, variable functions, or a combination of both, leading to different classes of integral equations.

1.2.1 Fredholm Linear Integral Equations

Fredholm integral equations are characterized by fixed limits of integration and play an important role in many physical and engineering applications. They are classified into two types depending on the position of the unknown function.

A Fredholm integral equation of the first kind has the form

$$f(x) = \lambda \int_{a}^{b} K(x,s)\, g(s)\, ds, \tag{1.2}$$

where the unknown function $g(s)$ appears only under the integral sign.

A Fredholm integral equation of the second kind is given by

$$g(x) = f(x) + \lambda \int_a^b K(x,s)\, g(s)\, ds, \tag{1.3}$$

where the unknown function appears both inside and outside the integral. This class of equations is particularly important due to its favorable analytical and numerical properties.

1.2.2 Volterra Linear Integral Equations

Volterra integral equations are distinguished by the presence of at least one variable limit of integration.

A Volterra integral equation of the first kind is expressed as

$$f(x) = \int_0^x K(x,s)\, g(s)\, ds, \tag{1.4}$$

where the unknown function appears only inside the integral.

A Volterra integral equation of the second kind takes the form

$$g(x) = f(x) + \lambda \int_0^x K(x,s)\, g(s)\, ds. \tag{1.5}$$

1.2.3 Volterra–Fredholm Integral Equations

Volterra–Fredholm integral equations arise in a variety of applications, including boundary value problems for parabolic partial differential equations, spatio-temporal epidemic models, and other physical and biological systems. They typically appear in the following forms:

$$g(x) = f(x) + \lambda_1 \int_a^x K_1(x,s)\, g(s)\, ds + \lambda_2 \int_a^b K_2(x,s)\, g(s)\, ds, \tag{1.6}$$

or

$$g(x) = f(x) + \lambda \int_a^x \int_a^b K(r,s)\, g(s)\, dr\, ds, \tag{1.7}$$

where $f(x)$ and the kernels K_1, K_2, and K are given functions.

The first form involves separate Volterra and Fredholm components, while the second form represents a mixed-type integral equation combining characteristics of both classes.

1.2.4 Linear Integro-Differential Equations

Integro-differential equations naturally arise when differential equations are reformulated in integral form or when systems exhibit both local and nonlocal interactions. These equations involve an unknown function that appears both through its derivatives and under an integral sign.

A Fredholm integro-differential equation of order k is typically written as

$$g^{(k)}(x) = f(x) + \lambda \int_a^b K(x,s)\, g(s)\, ds, \qquad g^{(k)} = \frac{d^k g}{dx^k}. \tag{1.8}$$

Similarly, a Volterra integro-differential equation of order k has the form

$$g^{(k)}(x) = f(x) + \lambda \int_a^x K(x,s)\, g(s)\, ds, \qquad g^{(k)} = \frac{d^k g}{dx^k}. \tag{1.9}$$

1.2.5 Linearity and Homogeneity

Integral and integro-differential equations can also be classified according to their linearity and homogeneity.

a) Linearity

An integral or integro-differential equation is said to be linear if the unknown function $g(x)$ appears linearly, that is, only to the first power and not inside nonlinear functions.

If the equation involves nonlinear expressions of $g(x)$, such as $g^2(x)$, $\exp(g(x))$, $\sin(g(x))$, or $\ln(1+g(x))$, it is classified as nonlinear.

b) Homogeneity

Integral and integro-differential equations of the second kind are called homogeneous if the function $f(x)$ is identically zero. If $f(x)$ is not identically zero, the equation is said to be inhomogeneous. This classification applies exclusively to equations of the second kind.

1.3 Leibniz Rule for Differentiation of Integrals

The Leibniz rule provides a fundamental formula for differentiating integrals whose integrands and limits of integration depend on a variable parameter. This rule plays a crucial role in the analysis of Volterra integral and integro-differential equations, especially in the presence of variable limits and delay terms.

Let $\psi(x,s)$ be a continuous function, and assume that its partial derivative with respect to x, denoted by $\partial\psi/\partial x$, is also continuous in a region of the (x,s)-plane containing the domain of integration. Let $u(x)$ and $v(x)$ be differentiable functions such that $u(x) \leq v(x)$ for all x in the interval of interest.

Define the function

$$\Psi(x) = \int_{u(x)}^{v(x)} \psi(x,s)\, ds. \tag{1.10}$$

Then $\Psi(x)$ is differentiable with respect to x, and its derivative is given by

$$\Psi'(x) = \psi(x, v(x))\, v'(x) - \psi(x, u(x))\, u'(x) + \int_{u(x)}^{v(x)} \frac{\partial \psi(x,s)}{\partial x}\, ds. \tag{1.11}$$

In the special case where the limits of integration are constants, that is, $u(x) = a$ and

$v(x) = b$, the Leibniz rule reduces to

$$\Psi'(x) = \int_a^b \frac{\partial \psi(x,s)}{\partial x}\, ds. \tag{1.12}$$

This result corresponds to a particular case of differentiation under the integral sign.

Example. Consider the delayed Volterra integral equation

$$x(t) = g(t) + \int_{t-\tau_2}^{t-\tau_1} K(t,s)\, x(s)\, ds, \qquad 0 < \tau_1 < \tau_2, \tag{1.13}$$

where $g(t)$ and $K(t,s)$ are sufficiently smooth functions.

Define

$$\Phi(t) = \int_{t-\tau_2}^{t-\tau_1} K(t,s)\, x(s)\, ds.$$

Applying the Leibniz rule with $u(t) = t - \tau_2$ and $v(t) = t - \tau_1$, we obtain

$$\begin{aligned}\Phi'(t) = K(t, t-\tau_1)\, x(t-\tau_1) - K(t, t-\tau_2)\, x(t-\tau_2) \\ + \int_{t-\tau_2}^{t-\tau_1} \frac{\partial K(t,s)}{\partial t}\, x(s)\, ds.\end{aligned}$$

Consequently, differentiating equation (1.13) with respect to t yields

$$x'(t) = g'(t) + K(t, t-\tau_1)\, x(t-\tau_1) - K(t, t-\tau_2)\, x(t-\tau_2) + \int_{t-\tau_2}^{t-\tau_1} \frac{\partial K(t,s)}{\partial t}\, x(s)\, ds.$$

1.4 Piecewise Polynomial Spaces

Piecewise polynomial spaces provide a flexible and efficient framework for approximating functions over a given interval by partitioning the domain into smaller subintervals and associating a polynomial with each subinterval. These spaces play a central role in numerical analysis, finite element methods, computer-aided design (CAD), and data fitting, as they allow for local approximation while maintaining global control over regularity.

Let

$$G_k = \{X_m : 0 = X_0 < X_1 < \cdots < X_M = a\}$$

be a grid (or mesh) defined on the interval $G = [0, a]$, with uniform step size

$$k = \frac{a}{M}.$$

The corresponding subintervals are given by

$$\sigma_m = [X_m, X_{m+1}], \qquad m = 0, 1, \ldots, M-1.$$

Definition 1.4.1 ([12]) *Let G_k be a grid on $G = [0, a]$. For integers $\rho \geq 0$ and $-1 \leq d \leq \rho$, the space of piecewise polynomial functions $\mathcal{S}_\rho^{(d)}(G_k)$ is defined by*

$$\mathcal{S}_\rho^{(d)}(G_k) := \left\{ v \in C^d(G) \ : \ v|_{\sigma_m} \in \Pi_\rho, \ m = 0, 1, \ldots, M-1 \right\},$$

where Π_ρ denotes the set of all real polynomials of degree less than or equal to ρ.

It can be shown that $\mathcal{S}_\rho^{(d)}(G_k)$ is a real vector space of finite dimension, given by

$$\dim \mathcal{S}_\rho^{(d)}(G_k) = M(\rho - d) + d + 1.$$

Remark 1.4.1 *The particular piecewise polynomial space $\mathcal{S}_{n+d}^{(d)}(G_k)$, with $n \geq 1$ and $d \geq -1$, has dimension $Mn + d + 1$ and is commonly employed as a collocation space for approximating solutions of initial-value problems associated with ordinary differential equations (ODEs) and Volterra-type equations.*

The choice of the smoothness parameter d depends on the number of prescribed initial conditions. The quantity Mn indicates that each of the M subintervals σ_m contains n distinct collocation points. Accordingly, suitable choices of d are:

- *For Volterra integral equations without initial conditions, $d = -1$.*
- *For first-order ordinary differential equations or Volterra integro-differential equations with a single initial condition, $d = 0$.*

- *For ordinary differential equations or Volterra integro-differential equations of order $k \geq 2$, where k initial conditions are prescribed, $d = k - 1$.*

In higher dimensions, the space $\mathcal{S}^{(-1)}_{\mu-1,\nu-1}(\Omega_{K,L})$ consists of bivariate piecewise polynomial spline functions of degree $\mu - 1$ in the variable x and $\nu - 1$ in the variable y. This space is constructed as a tensor product of the univariate spline spaces $\mathcal{S}^{(-1)}_{\mu-1}(\Omega_K)$ and $\mathcal{S}^{(-1)}_{\nu-1}(\Omega_L)$. Functions in this space may exhibit jump discontinuities across the internal grid lines $x = x_k$, $k = 1, \ldots, K - 1$, and $y = y_l$, $l = 1, \ldots, L - 1$.

1.5 Collocation Methods

Collocation methods constitute an important class of numerical techniques for the approximate solution of differential, integral, and integro-differential equations, including ordinary differential equations (ODEs), partial differential equations (PDEs), and integral equations. The central idea of these methods is to approximate the exact solution by a function belonging to a finite-dimensional approximation space and to enforce the governing equation at a finite set of discrete points, called collocation points, within the computational domain.

In the following, we briefly review several collocation approaches commonly used for ODEs, PDEs, and integral equations.

1.5.1 Collocation Methods for Ordinary Differential Equations

Various collocation strategies have been developed for the numerical solution of ordinary differential equations, among which the most notable are:

- **Chebyshev Collocation Method:** The approximate solution is expressed as a truncated expansion in Chebyshev polynomials, leading to highly accurate spectral approximations.

- **Legendre–Gauss Collocation Method:** This approach employs Legendre–Gauss interpolation nodes and is particularly effective for initial value problems associated with second-order ordinary differential equations.
- **Bernstein Collocation Method:** Bernstein polynomial bases are used to approximate the solution, providing numerical stability and efficiency, especially for nonlinear ordinary differential equations.
- **Block Hybrid Collocation Method:** This technique combines block and hybrid collocation strategies and is designed for the direct numerical solution of higher-order ordinary differential equations.

1.5.2 Collocation Methods for Partial Differential Equations

Collocation techniques have also been successfully applied to partial differential equations. Representative approaches include:

- **Sparse Grid Stochastic Collocation Method:** This method is based on Smolyak sparse grids and is used to efficiently approximate statistical quantities associated with PDEs involving random inputs.
- **Wavelet-Based Collocation Method:** A numerical approach that exploits the localization and multiresolution properties of wavelets, particularly Daubechies wavelets with compact support.
- **Spline-Based Collocation Method:** This method employs B-spline basis functions to construct smooth and flexible approximations within the collocation framework.

1.5.3 Collocation Methods for Integral Equations

Several collocation schemes have been proposed for the numerical solution of integral equations, depending on the type and structure of the problem. Among them are:

- **Galerkin-Type Collocation Methods:** Applied primarily to integral equations of the first kind.
- **Iterative Collocation Methods:** Designed to handle nonlinear Volterra integral equations of the second kind.
- **Taylor expansion Collocation Method:** A widely used approach for solving Volterra integral equations of both the first and second kinds by employing truncated Taylor expansions.

1.6 Taylor Collocation Method

The Taylor collocation method is an efficient numerical technique for solving differential, integral, and integro-differential equations. The method is based on approximating the unknown solution by a truncated Taylor expansion expansion and determining the coefficients of this expansion by enforcing the governing equation at a prescribed set of collocation points.

Originally developed for the numerical solution of Volterra integral equations, the Taylor collocation method has been progressively extended and refined by several authors. Notable contributions include the work of Karamete and Sezer [19], who introduced a Taylor matrix collocation approach for linear integro-differential equations, and Bellour and Bousselsal [6, 7], who proposed Taylor-based numerical schemes for integral and delay integro-differential equations. More recently, Laib and collaborators [22] applied the method to systems of nonlinear Volterra integro-differential equations with time delays, significantly broadening its range of applications.

Owing to its simplicity, explicit formulation, and guaranteed convergence under suitable conditions, the Taylor collocation method has become a powerful tool for solving a wide class of integral and integro-differential equations. In this thesis, both linear and nonlinear problems are considered, with the aim of extending the applicability of the Taylor collocation method and developing efficient numerical algorithms for a broader class of Volterra-type equations.

1.7 Discrete Variants of Inequalities

The following lemmas will be used in following chapters.

Lemma 1.7.1 *Let $u \in \mathcal{C}^p([a,b])$ and let $\xi \in [a,b]$. For every $x \in [a,b]$, there exists a point*

$$\zeta = (1-\lambda)\xi + \lambda x \in (a,b), \qquad \lambda \in (0,1),$$

such that

$$u(x) = \sum_{m=0}^{p-1} \frac{u^{(m)}(\xi)}{m!}(x-\xi)^m + \frac{u^{(p)}(\zeta)}{p!}(x-\xi)^p.$$

Lemma 1.7.2 *(Discrete Grönwall-type inequality [12]) Let $\{a_m\}_{m=0}^N$ be a nonnegative sequence and assume that the sequence $\{u_n\}$ satisfies $u_0 \le c_0$ and*

$$u_n \le c_0 + \sum_{i=0}^{n-1} a_i\, u_i, \qquad n \ge 1,$$

where $c_0 \ge 0$. Then u_n satisfies the bound

$$u_n \le c_0 \exp\left(\sum_{m=0}^{n-1} a_m\right), \qquad n \ge 1.$$

Lemma 1.7.3 *(Discrete Grönwall-type inequality [1]) Let $\{u_k\}_{k\ge0}$, $\{a_k\}_{k\ge0}$ and $\{b_k\}_{k\ge0}$ be nonnegative sequences satisfying*

$$u_k \le a_k + \sum_{j=0}^{k-1} b_j\, u_j, \qquad k \ge 0.$$

Then the following estimate holds:

$$u_k \le a_k + \sum_{j=0}^{k-1} a_j\, b_j \exp\left(\sum_{m=0}^{k-1} b_m\right), \qquad k \ge 0.$$

Lemma 1.7.4 *[16] Let $\{\delta_k\}_{k\geq 0}$ be a sequence of nonnegative real numbers satisfying*

$$\delta_k \leq \alpha\delta_{k-1} + \beta\sum_{j=0}^{k-1}\delta_j + C, \qquad k \geq 0,$$

where α, β, and C are nonnegative constants. Then the following estimate holds:

$$\delta_k \leq \frac{\delta_0}{\lambda_2 - \lambda_1}\left[(\lambda_2 - 1)\lambda_2^k + (1 - \lambda_1)\lambda_1^k\right] + \frac{C}{\lambda_2 - \lambda_1}\left(\lambda_2^k - \lambda_1^k\right),$$

where

$$\lambda_1 = \frac{1 + \alpha + \beta - \sqrt{(1-\alpha)^2 + \beta^2 + 2\alpha\beta + 2\beta}}{2},$$
$$\lambda_2 = \frac{1 + \alpha + \beta + \sqrt{(1-\alpha)^2 + \beta^2 + 2\alpha\beta + 2\beta}}{2}.$$

Moreover, the following relation holds:

$$0 \leq \lambda_1 \leq 1 \leq \lambda_2.$$

Chapter 2

Numerical solution of linear neutral integral equations with double constant delays

2.1 Introduction

We consider the linear Volterra integral equation with two constant delays τ_1 and τ_2 of the form

$$x(t) = g(t) + x(t-\tau_1) + x(t-\tau_2) + \int_{t-\tau_2}^{t-\tau_1} k(t,s)\, x(s)\, ds, \tag{2.1}$$

for $t \in [\tau_2, T]$, together with the initial condition

$$x(t) = \Phi(t), \qquad t \in [0, \tau_2].$$

Throughout this work, we assume that the given functions g, k, and Φ are sufficiently smooth. Moreover, the following compatibility condition is supposed to hold:

$$\Phi(\tau_2) = g(\tau_2) + \Phi(\tau_2 - \tau_1) + \Phi(0) + \int_0^{\tau_2-\tau_1} k(\tau_2, s)\, \Phi(s)\, ds.$$

Delay Volterra integral equations arise naturally in a wide range of scientific and engineering applications, including biology [27], epidemiology [21, 32], chemical engineering [24], control theory [2], physics [26], and social sciences [18]. In particular, double-delay Volterra integral equations (DDVIEs), which incorporate both memory and delayed effects, have been used to model age-structured population dynamics, where two distinct age classes coexist within the same population [10, 11]. In this context, the delays appearing in equation (2.1) typically represent the maturation time and the maximum life expectancy, respectively.

The presence of two constant delays significantly increases the mathematical complexity of the problem, often rendering analytical solutions infeasible. Consequently, the development of efficient and accurate numerical methods for solving DDVIEs has become an important topic of contemporary research. Despite their relevance, the numerical analysis of equation (2.1) has only recently attracted attention, and the available specialized literature remains relatively limited. A numerical approach for this class of equations was proposed in [25].

Motivated by these considerations, the present study investigates the application of the Taylor collocation method [22] to the numerical solution of linear double-delay Volterra integral equations. By combining the flexibility of Taylor polynomial approximations with the accuracy of collocation techniques, this method provides an effective and reliable framework for addressing the challenges associated with DDVIEs.

2.2 Numerical Discretization and Approximation Framework

We assume that the final time satisfies $T = (r+1)\tau_2$, where $r \in \{1, 2, 3, \ldots\}$. Let Π_N denote a uniform partition of the interval $I = [\tau_2, T]$, defined by the grid points

$$t_n^i = (i+1)\tau_2 + nh, \qquad n = 0, 1, \ldots, N, \quad i = 0, 1, \ldots, r-1,$$

where the step size is given by

$$h = t_{n+1}^i - t_n^i.$$

We further assume that the step size satisfies

$$h = \frac{\tau_1}{N_1} = \frac{\tau_2}{N},$$

with N and N_1 being positive integers.

The corresponding subintervals are defined as

$$\sigma_n^i = [t_n^i,\, t_{n+1}^i), \qquad n = 0, 1, \ldots, N-1, \quad i = 0, 1, \ldots, r-2,$$

and

$$\sigma_{N-1}^{r-1} = [t_{N-1}^{r-1},\, t_N^{r-1}].$$

Let Π_m denote the set of all real polynomials of degree less than or equal to m. We introduce the real polynomial spline space of degree $m-1$, defined by

$$S_{m-1}^{(-1)}(\Pi_N) = \left\{ u \in C^{-1}(I, \mathbb{R}) : u|_{\sigma_n^i} \in \Pi_{m-1},\ n = 0, \ldots, N-1,\ i = 0, \ldots, r-1 \right\}.$$

This space consists of piecewise polynomial functions of degree at most $m-1$ defined on the partition Π_N. Its dimension is equal to rNm, which coincides with the total number of coefficients of the local polynomial approximations u_n^i on each subinterval.

To determine these coefficients, the Taylor collocation method is employed. On each subinterval σ_n^i, the approximate solution is represented by a truncated Taylor polynomial, whose coefficients are computed by enforcing the governing integral equation at appropriately chosen collocation points.

2.3 Construction of the Taylor Collocation Approximation

First, to approximate x by u_0^0 so that

$$u_0^0(t) = \sum_{j=0}^{m-1} \frac{x^{(j)}(t_0^0)}{j!}(t - t_0^0)^j; \quad t \in \sigma_0^0, \tag{2.2}$$

Now, for all $j = 1, ..., m-1$, the formula for computing the values of the coefficients $x^{(j)}(t_0^0)$ can be obtained by differentiating (2.1), we get the following formulas:

$$x^{(j)}(t_0^0) = g^{(j)}(t_0^0) + \phi^{(j)}(t_0^0 - \tau_1) + \phi^{(j)}(t_0^0 - \tau_2) + \left(\int_{t-\tau_2}^{t-\tau_1} K(t,s)\phi(s)ds \right)^{(j)} (t_0^0). \tag{2.3}$$

Second, for each $n \in \{1, \ldots, N-1\}$, the exact solution x is approximated on the subinterval σ_n^0 by a local Taylor polynomial u_n^0. This construction assumes that x has already been approximated by u_k^0 on all preceding subintervals σ_k^0, for $0 \le k < n$. The approximation on σ_n^0 is defined as

$$u_n^0(t) = \sum_{j=0}^{m-1} \frac{\hat{u}_{n,0}^{(j)}(t_n^0)}{j!} (t - t_n^0)^j, \qquad t \in \sigma_n^0, \tag{2.4}$$

where $\hat{u}_{n,0}$ denotes the exact solution of an auxiliary integral equation.

For $n \in \{0, \ldots, N_1 - 1\}$, the function $\hat{u}_{n,0}$ is given by

$$\hat{u}_{n,0}(t) = g(t) + \phi(t - \tau_1) + \phi(t - \tau_2) + \int_{t-\tau_2}^{t-\tau_1} K(t,s)\,\phi(s)\,ds. \tag{2.5}$$

For $n \in \{N_1, \ldots, N - 1\}$, $\hat{u}_{n,0}$ satisfies

$$\begin{aligned} \hat{u}_{n,0}(t) = g(t) + u_{n-N_1}^0(t - \tau_1) + \phi(t - \tau_2) + \int_{t-\tau_2}^{\tau_2} K(t,s)\,\phi(s)\,ds \\ + \sum_{i=0}^{n-N_1-1} \int_{t_i^0}^{t_{i+1}^0} K(t,s)\,u_i^0(s)\,ds + \int_{t_n^0-\tau_1}^{t-\tau_1} K(t,s)\,u_{n-N_1}^0(s)\,ds. \end{aligned} \tag{2.6}$$

For $j = 1, \ldots, m - 1$, the coefficients $\hat{u}_{n,0}^{(j)}(t_n^0)$ are obtained by differentiating equations (2.5) and (2.6). For $n \in \{0, \ldots, N_1 - 1\}$, this yields

$$\begin{aligned} \hat{u}_{n,0}^{(j)}(t) = g^{(j)}(t) + \phi^{(j)}(t - \tau_1) + \phi^{(j)}(t - \tau_2) \\ + \left(\int_{t-\tau_2}^{t-\tau_1} K(t,s)\phi(s)ds\right)^{(j)}(t), \;\; n \in \{0, ..., N_1 - 1\}, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \hat{u}_{n,0}^{(j)}(t) = g^{(j)}(t) + \hat{u}_{n-N_1,0}^{(j)}(t - \tau_1) + \phi^{(j)}(t - \tau_2) + \left(\int_{t-\tau_2}^{\tau_2} K(t,s)\phi(s)ds\right)^{(j)}(t) \\ + \sum_{i=0}^{n-N_1-1} \sum_{l=0}^{m-1} \frac{\hat{u}_{i,0}^{(l)}(t_i^0)}{l!} \int_{t_i^0}^{t_{i+1}^0} \partial_1^{(j)} K(t,s)(s - t_i^0)^l ds \\ + \sum_{i=0}^{j-1} \sum_{l=0}^{i} \binom{i}{l} \hat{u}_{n-N_1,0}^{(l)}(t_{n-N_1}^0)[\partial_1^{(j-1-i)} K(t, t - \tau_1)]^{(i-l)}(t) \\ + \int_{t-\tau_1}^{t-\tau_1} \partial_1^{(j)} K(t,s) u_{n-N_1}^0(s)ds, \;\; n \in \{N_1, ..., N - 1\}, \end{aligned} \tag{2.8}$$

hence

$$\hat{u}_{n,0}^{(j)}(t_n^0) = g^{(j)}(t_n^0) + \phi^{(j)}(t_n^0 - \tau_1) + \phi^{(j)}(t_n^0 - \tau_2) + \left(\int_{t-\tau_2}^{t-\tau_1} K(t,s)\phi(s)ds\right)^{(j)}(t_n^0), \;\; n \in \{0, ..., N_1 - 1\},$$

and

$$\begin{aligned}\hat{u}_{n,0}^{(j)}(t_n^0) &= g^{(j)}(t_n^0) + \hat{u}_{n-N_1,0}^{(j)}(t_{n-N_1}^0 - \tau_1) + \phi^{(j)}(t_n^0 - \tau_2) \\ &+ \left(\int_{t-\tau_2}^{\tau_2} K(t,s)\phi(s)ds\right)^{(j)}(t_n^0) \\ &+ \sum_{i=0}^{n-N_1-1}\sum_{l=0}^{m-1} \frac{\hat{u}_{i,0}^{(l)}(t_i^0)}{l!}\int_{t_i^0}^{t_{i+1}^0} \partial_1^{(j)} K(t_n^0, s)(s - t_i^0)^l ds \\ &+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\hat{u}_{n-N_1,0}^{(l)}(t_{n-N_1}^0)[\partial_1^{(j-1-i)}K(t, t-\tau_1)]^{(i-l)}(t_n^0), \;\; n \in \{N_1, ..., N-1\}.\end{aligned}$$

Third, for x to be approximated by u_0^p ($n \in \{0, 1, ..., N-1\}$ and $p \in \{1, 2, ..., r-1\}$) on the interval σ_0^p, x must be approximated by u_k^j ($0 \le k < n$ and $0 \le j \le p$) on each interval σ_k^j so that

$$u_0^p(t) = \sum_{j=0}^{m-1} \frac{\hat{u}_{0,p}^{(j)}(t_0^p)}{j!}(t - t_0^p)^j; \quad t \in \sigma_0^p, \tag{2.9}$$

where $\hat{u}_{0,p}$ is the exact solution of the integral equations

$$\begin{aligned}\hat{u}_{0,p}(t) &= g(t) + u_0^{p-1}(t - \tau_1) + u_0^{p-1}(t - \tau_2) \\ &+ \int_{t-\tau_2}^{t_1^{p-1}} K(t,s)u_0^{p-1}(s)ds + \sum_{d=1}^{N-N_1-1}\int_{t_d^{p-1}}^{t_{d+1}^{p-1}} K(t,s)u_d^{p-1}(s)ds \\ &+ \int_{t_0^p - \tau_1}^{t-\tau_1} K(t,s)u_{N-N_1}^{p-1}(s)ds, \;\; n \in \{1, ..., N-1\},\end{aligned}$$

The coefficients $\hat{u}_{0,p}^{(j)}(t_0^p)$ are given by the following formulas:

$$\begin{aligned}\hat{u}_{0,p}^{(j)}(t) &= g^{(j)}(t) + \hat{u}_{0,p-1}^{(j)}(t - \tau_1) + \hat{u}_{0,p-1}^{(j)}(t - \tau_2) \\ &+ \int_{t-\tau_2}^{t_1^{p-1}} \partial_1^{(j)} K(t,s)u_0^{p-1}(s)ds + \sum_{d=1}^{N-N_1-1}\int_{t_d^{p-1}}^{t_{d+1}^{p-1}} \partial_1^{(j)} K(t,s)u_d^{p-1}(s)ds \\ &+ \sum_{i=0}^{j-1}\left[\partial_1^{(j-1-i)}K(t,s)u_{N-N_1}^{p-1}(t)\right]^i + \int_{t_0^p-\tau_1}^{t-\tau_1} \partial_1^{(j)} K(t,s)u_{N-N_1}^{p-1}(s)ds, \;\; n \in \;\{0, ..., N-1\},\end{aligned}$$

and

$$\begin{aligned}
\hat{u}^{(j)}_{0,p}(t^p_0) &= g^{(j)}(t^p_0) + \hat{u}^{(j)}_{N-N_1,p-1}(t^{p-1}_{N-N_1}) + \hat{u}^{(j)}_{n-N,p-1}(t^{p-1}_{n-N}) \\
&\quad - \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\hat{u}^{(l)}_{0,p-1}(t^{p-1}_0)[\partial_1^{(j-1-i)}K(t,t-\tau_2)]^{(i-l)}(t^p_0) \\
&\quad + \sum_{l=0}^{m-1}\frac{\hat{u}^{(l)}_{0,p-1}(t^{p-1}_0)}{l!}\int_{t^p_0-\tau_2}^{t^{p-1}_1}\partial_1^{(j)}K(t^p_0,s)(s-t^{p-1}_0)^l ds \\
&\quad + \sum_{d=1}^{N-N_1-1}\sum_{l=0}^{m-1}\frac{\hat{u}^{(l)}_{d,p-1}(t^{p-1}_d)}{l!}\int_{t^{p-1}_d}^{t^{p-1}_{d+1}}\partial_1^{(j)}K(t^p_0,s)(s-t^{p-1}_d)^l ds \\
&\quad + \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\hat{u}^{(l)}_{N-N_1,p-1}(t^{p-1}_{N-N_1})[\partial_1^{(j-1-i)}K(t,t-\tau_1)]^{(i-l)}(t^p_0),
\end{aligned}$$

Finally, for x to be approximated by u^p_n ($n \in \{0,1,...,N-1\}$ and $p \in \{1,2,...,r-1\}$) on the interval σ^p_n, x must be approximated by u^j_k ($0 \le k < n$ and $0 \le j \le p$) on each interval σ^j_k so that

$$u^p_n(t) = \sum_{j=0}^{m-1}\frac{\hat{u}^{(j)}_{n,p}(t^p_n)}{j!}(t-t^p_n)^j; \quad t \in \sigma^p_n, \tag{2.10}$$

where $\hat{u}_{n,p}$ is the exact solution of the integral equations

$$\begin{aligned}
\hat{u}_{n,p}(t) &= g(t) + u^{p-1}_{N-N_1+n}(t-\tau_1) + u^{p-1}_n(t-\tau_2) \\
&\quad + \int_{t-\tau_2}^{t^{p-1}_{n+1}}K(t,s)u^{p-1}_n(s)ds + \sum_{d=n+1}^{N-N_1+n-1}\int_{t^{p-1}_d}^{t^{p-1}_{d+1}}K(t,s)u^{p-1}_d(s)ds \\
&\quad + \int_{t^p_n-\tau_1}^{t-\tau_1}K(t,s)u^{p-1}_{N-N_1+n}(s)ds, \;\; n \in \{0,...,N_1-1\},
\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
\hat{u}_{n,p}(t) &= g(t) + u^p_{n-N_1}(t-\tau_1) + u^{p-1}_n(t-\tau_2) + \int_{t-\tau_2}^{t^{p-1}_{n+1}}K(t,s)u^{p-1}_n(s)ds \\
&\quad + \sum_{d=n+1}^{N-1}\int_{t^{p-1}_d}^{t^{p-1}_{d+1}}K(t,s)u^{p-1}_d(s)ds + \sum_{i=0}^{n-N_1-1}\int_{t^p_i}^{t^p_{i+1}}K(t,s)u^p_i(s)ds \\
&\quad + \int_{t^p_n-\tau_1}^{t-\tau_1}K(t,s)u^p_{n-N_1}(s)ds, \;\; n \in \{N_1,...,N-1\}.
\end{aligned} \tag{2.12}$$

The coefficients $\hat{u}^{(j)}_{n,p}(t^p_n)$ are given by the following formulas:

For $n \in \{0, ..., N_1 - 1\}$,

$$\begin{aligned}
\hat{u}^{(j)}_{n,p}(t) &= g^{(j)}(t) + \hat{u}^{(j)}_{N-N_1+n,p-1}(t-\tau_1) + \hat{u}^{(j)}_{n,p-1}(t-\tau_2) \\
&+ \int_{t-\tau_2}^{t^{p-1}_{n+1}} \partial_1^{(j)} K(t,s) u^{p-1}_n(s) ds \\
&+ \sum_{d=n+1}^{N-N_1+n-1} \int_{t^{p-1}_d}^{t^{p-1}_{d+1}} \partial_1^{(j)} K(t,s) u^{p-1}_d(s) ds + \sum_{i=0}^{j-1} \left[\partial_1^{(j-1-i)} K(t,s) u^{p-1}_{N-N_1+n}(t)\right]^i \\
&+ \int_{t^p_n-\tau_1}^{t-\tau_1} \partial_1^{(j)} K(t,s) u^{p-1}_{N-N_1+n}(s) ds, \quad n \in \ \{0, ..., N_1 - 1\},
\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}
\hat{u}^{(j)}_{n,p}(t^p_n) &= g^{(j)}(t^p_n) + \hat{u}^{(j)}_{N-N_1+n,p-1}(t^{p-1}_{N-N_1+n}) + \hat{u}^{(j)}_{n,p-1}(t^{p-1}_n) \\
&- \sum_{i=0}^{j-1} \sum_{l=0}^{i} \binom{i}{l} \hat{u}^{(l)}_{n,p-1}(t^{p-1}_n)[\partial_1^{(j-1-i)} K(t, t-\tau_2)]^{(i-l)}(t^p_n) \\
&+ \sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{n,p-1}(t^{p-1}_n)}{l!} \int_{t^p_n-\tau_2}^{t^{p-1}_{n+1}} \partial_1^{(j)} K(t^p_n, s)(s-t^{p-1}_n)^l ds \\
&+ \sum_{d=n+1}^{N-N_1+n-1} \sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{d,p-1}(t^{p-1}_d)}{l!} \int_{t^{p-1}_d}^{t^{p-1}_{d+1}} \partial_1^{(j)} K(t^p_n, s)(s-t^{p-1}_d)^l ds \\
&+ \sum_{i=0}^{j-1} \sum_{l=0}^{i} \binom{i}{l} \hat{u}^{(l)}_{N-N_1+n,p-1}(t^{p-1}_{N-N_1+n})[\partial_1^{(j-1-i)} K(t, t-\tau_1)]^{(i-l)}(t^p_n),
\end{aligned}$$

and for $n \in \{N_1, ..., N - 1\}$

$$\begin{aligned}
\hat{u}^{(j)}_{n,p}(t) &= g^{(j)}(t) + \hat{u}^{(j)}_{n-N_1,p}(t-\tau_1) + \hat{u}^{(j)}_{n,p-1}(t-\tau_2) \\
&+ \int_{t-\tau_2}^{t^{p-1}_{n+1}} \partial_1^{(j)} K(t,s) u^{p-1}_n(s) ds + \sum_{d=n+1}^{N-1} \int_{t^{p-1}_d}^{t^{p-1}_{d+1}} \partial_1^{(j)} K(t,s) u^{p-1}_d(s) ds \\
&+ \sum_{i=0}^{n-N_1-1} \int_{t^p_i}^{t^p_{i+1}} \partial_1^{(j)} K(t,s) u^p_i(s) ds + \sum_{i=0}^{n-N_1-1} \left[\partial_1^{(j-1-i)} K(t,s) u^p_{n-N_1}(t)\right]^i \\
&+ \int_{t^p_n-\tau_1}^{t-\tau_1} \partial_1^{(j)} K(t,s) u^p_{n-N_1}(s) ds
\end{aligned} \tag{2.14}$$

and

$$
\begin{aligned}
\hat{u}^{(j)}_{n,p}(t^p_n) &= g^{(j)}(t^p_n) + \hat{u}^{(j)}_{n-N_1,p}(t^p_{n-N_1}) + \hat{u}^{(j)}_{n,p-1}(t^{p-1}_n) \\
&\quad - \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\hat{u}^{(l)}_{n,p-1}(t^{p-1}_n)[\partial_1^{(j-1-i)}K(t,t-\tau_2)]^{(i-l)}(t^p_n) \\
&\quad + \sum_{l=0}^{m-1}\frac{\hat{u}^{(l)}_{n,p-1}(t^{p-1}_n)}{l!}\int_{t^p_n-\tau_2}^{t^{p-1}_{n+1}}\partial_1^{(j)}K(t^p_n,s)(s-t^{p-1}_n)^l ds \\
&\quad + \sum_{d=n+1}^{N-1}\sum_{l=0}^{m-1}\frac{\hat{u}^{(l)}_{d,p-1}(t^{p-1}_d)}{l!}\int_{t^{p-1}_d}^{t^{p-1}_{d+1}}\partial_1^{(j)}K(t^p_n,s)(s-t^{p-1}_d)^l ds \\
&\quad + \sum_{i=0}^{n-N_1-1}\sum_{l=0}^{m-1}\frac{\hat{u}^{(l)}_{i,p}(t^p_i)}{l!}\int_{t^p_i}^{t^p_{i+1}}\partial_1^{(j)}K(t^p_n,s)(s-t^p_i)^l ds \\
&\quad + \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\hat{u}^{(l)}_{n-N_1,p}(t^p_{n-N_1})[\partial_1^{(j-1-i)}K(t,t-\tau_1)]^{(i-l)}(t^p_n).
\end{aligned}
$$

2.4 Convergence and Error Estimates

The following lemma shows that the coefficients of the polynomial approximation are bounded.

Lemma 2.4.1 *Let the functions g and K be m times continuously differentiable on their respective domains. Then there exists a positive constant $\alpha(m)$ such that, for all $n = 0, 1, \ldots, N-1$, $p = 0, 1, \ldots, r-1$, and $j = 0, 1, \ldots, m$, the following estimate holds:*

$$
\left\|\hat{u}^{(j)}_{n,p}\right\|_{L^\infty(\sigma^p_n)} \leq \alpha(m),
$$

where $\hat{u}_{0,0}(t) = x(t)$ for $t \in \sigma^0_0$.

Proof. The proof is carried out in two steps.

Step 1. There exists a positive constant $\alpha_1(m)$ such that

$$
\left\|\hat{u}^{(j)}_{n,0}\right\|_{L^\infty(\sigma^0_n)} \leq \alpha_1(m), \qquad n = 0, 1, \ldots, N-1, \quad j = 0, 1, \ldots, m.
$$

Let

$$a_n^j = \left\| \hat{u}_{n,0}^{(j)} \right\|_{L^\infty(\sigma_n^0)}.$$

Then, for $j = 0, 1, \ldots, m$, we have

$$a_0^j \leq \max\left\{ \left\| x^{(j)} \right\|_{L^\infty(\sigma_0^0)} : j = 0, 1, \ldots, m \right\} =: \alpha_1^{(1)}(m). \tag{2.15}$$

Let Let $n = 1, 2, \ldots, N-1$ and $j = 1, \ldots, m$. Then, from equations (2.7) and (2.8), we obtain

$$\begin{aligned} a_n^j &\leq c + d_1^2 h \sum_{l=j}^{m-1} a_{n-N_1}^l + h d_1^1 \sum_{i=0}^{n-N_1} \sum_{l=0}^{m-1} a_i^l + m^2 h d_1^2 \sum_{l=0}^{m-1} a_{n-1-N_1}^l + h d_1^1 \sum_{l=0}^{m-1} a_{n-N_1}^l \\ &\leq c + \underbrace{\left(d_1^2 (1 + m^2) + d_1^1 \right)}_{=:d_1} h \sum_{l=0}^{m-1} a_{n-N_1}^l + h d_1^1 \sum_{i=0}^{n-1} \sum_{l=0}^{m-1} a_i^l. \end{aligned} \tag{2.16}$$

Here, for $j = 0, 1, \ldots, m-1$, the constants are defined by

$$c_1 = \max\left\{ \left\| g^{(j)} + \phi^{(j)}(t - \tau_1) + \phi^{(j)}(t - \tau_2) + \left(\int_{t-\tau_2}^{t-\tau_1} K(t,s)\, \phi(s)\, ds \right)^{(j)} \right\|_{L^\infty(\sigma_n^0)} \right\},$$

$$c = \max\left\{ \left\| g^{(j)} + \hat{u}_{n-N_1,0}^{(j)}(t - \tau_1) + \phi^{(j)}(t - \tau_2) + \left(\int_{t-\tau_2}^{\tau_2} K(t,s)\, \phi(s)\, ds \right)^{(j)} \right\|_{L^\infty(\sigma_n^0)} \right\},$$

$$d_1^1 = \max\left\{ \frac{1}{l!} \left\| \partial_1^{(j)} K(t,s)\, (s - t_i^0)^l \right\|_{L^\infty(\sigma_n^0)} : j = 0, \ldots, m-1 \right\},$$

$$d_1^2 = \max\left\{ \binom{i}{l} \left\| \left[\partial_1^{(j-1-i)} K(t, t - \tau_1) \right]^{(i-l)} \right\|_{L^\infty(\sigma_n^0)} : \begin{array}{l} j = 1, \ldots, m-1, \\ i = 0, \ldots, j-1, \\ l = 0, \ldots, i \end{array} \right\}.$$

The constants c, d_1^1, and d_1^2 are positive and independent of the discretization parameter N.

On the other hand, for the case $j = 0$, equations (2.5) and (2.6) yield

$$a_n^0 \leq c + (d_1^1 + d_1^2)\, h \sum_{l=0}^{m-1} a_{n-N_1}^l + hd_1^1 \sum_{i=0}^{n-1} \sum_{l=0}^{m-1} a_i^l. \tag{2.17}$$

Combining estimates (2.16) and (3.22), we obtain, for each fixed $n \geq 1$ and for all $j = 0, \ldots, m$, the uniform bound

$$a_n^j \leq c + 2hd_1 \sum_{i=0}^{n-1} \sum_{l=0}^{m-1} a_i^l. \tag{2.18}$$

Define the sequence

$$z_n = \sum_{j=0}^{m} a_n^j, \qquad n \geq 0.$$

Then, summing inequality (2.18) over $j = 0, \ldots, m$, we obtain

$$z_n \leq \underbrace{(m+1)c}_{=:c_1} + h \underbrace{2(m+1)d_1}_{=:d_2} \sum_{i=0}^{n-1} z_i. \tag{2.19}$$

Moreover, from (2.3) it follows that

$$z_0 \leq (m+1)\alpha_1^{(1)}(m) =: c_2. \tag{2.20}$$

Combining (3.23) and (3.24), we deduce that for all $n = 0, 1, \ldots, N-1$,

$$z_n \leq c_3 + hd_2 \sum_{i=0}^{n-1} z_i, \qquad c_3 = \max\{c_1, c_2\}.$$

Applying Lemma 1.7.2 (discrete Grönwall inequality), we conclude that

$$z_n \leq c_3 \exp(\tau_2 d_2) =: \alpha_1(m), \qquad n = 0, 1, \ldots, N-1.$$

This completes the proof of the first step.

Step 2. There exists a positive constant $\alpha_2(m)$ such that

$$\left\|\hat{u}_{n,p}^{(j)}\right\|_{L^\infty(\sigma_n^p)} \leq \alpha_2(m),$$

for all $n = 0, 1, \ldots, N-1$, $j = 0, 1, \ldots, m$, and $p = 1, 2, \ldots, r-1$.

Let

$$a_{n,p}^{(j)} = \left\|\hat{u}_{n,p}^{(j)}\right\|_{L^\infty(\sigma_n^p)},$$

and define

$$\varepsilon_p = \max\left\{a_{i,p}^{(j)} : j = 0, \ldots, m,\ i = 0, \ldots, N-1\right\}, \qquad p = 1, \ldots, r-1.$$

Proceeding analogously to Step 1., and from equations (2.13) and (2.14), we obtain for all $n = 1, \ldots, N-1$ the estimate

$$a_{n,p}^{(j)} \leq c_4 + b_1\, \varepsilon_{p-1} + d_3 h \sum_{i=0}^{n-1} \sum_{l=0}^{m-1} a_{i,p}^{(l)}, \tag{2.21}$$

where c_4, b_1, and d_3 are positive constants independent of N.

Define the auxiliary sequence

$$y_n = \sum_{j=0}^{m} a_{n,p}^{(j)}, \qquad n = 0, 1, \ldots, N-1.$$

Summing inequality (2.21) over $j = 0, \ldots, m$ and applying Lemma 1.7.2 (discrete Grönwall inequality), we obtain

$$\begin{aligned} y_n &\leq ((m+1)c_4 + (m+1)b_1\varepsilon_{p-1}) \exp\left(\sum_{i=0}^{n-1}(m+1)d_3 h\right) \\ &\leq \underbrace{(m+1)c_4 \exp(d_4)}_{=:c_5} + \underbrace{(m+1)b_1 \exp(d_4)}_{=:b_2}\, \varepsilon_{p-1}, \end{aligned}$$

where $d_4 = (m+1)d_3\tau_2$.

Using the bound $\varepsilon_{p-1} \leq \alpha_1(m)$ obtained in Step 1., we finally deduce

$$y_n \leq \underbrace{c_6 + b_3\alpha_1(m)}_{=:\alpha_2(m)}, \qquad n = 0, 1, \ldots, N-1,$$

where c_6 and b_3 are positive constants.

Hence, Lemma 2.4.1 follows by setting

$$\alpha(m) = \max\{\alpha_1(m),\, \alpha_2(m)\}.$$

□

The following theorem establishes the convergence of the proposed method.

Theorem 2.4.1 *Assume that the functions g and K are m times continuously differentiable on their respective domains. Then equations* (3.3) *and* (3.8) *define a unique approximate solution*

$$u \in S_{m-1}^{(-1)}(\Pi_N).$$

Moreover, the associated error function $e = x - u$ satisfies the estimate

$$\|e\|_{L^\infty(I)} \leq C\, h^m,$$

provided that the stepsize h is sufficiently small, where C is a positive constant independent of h.

Proof. The proof is divided into two steps.

Claim 1. There exists a constant C_1, independent of the stepsize h, such that

$$\|e^0\|_{L^\infty(\sigma^0)} \leq C_1 h^m,$$

where the local error $e^0 = e|_{\sigma^0}$ is defined on each subinterval σ_n^0 by

$$e^0(t) = e_n^0(t) = |x(t) - u_n^0(t)|, \qquad n = 0, 1, \ldots, N-1.$$

Let $t \in \sigma_0^0$. For sufficiently small h, the Taylor remainder theorem yields

$$|e_0^0(t)| = |x(t) - u_0^0(t)| \leq \frac{\|x^{(m)}\|_{L^\infty(\sigma_0^0)}}{m!}\, h^m \leq \frac{\alpha(m)}{m!}\, h^m.$$

For $n = 1, \ldots, N-1$, it follows from equations (2.5) and (2.6) that

$$\|x - \hat{u}_{n,0}\|_{L^\infty(\sigma_n^0)} \leq h\, \|K\|_{L^\infty(I)} \sum_{i=0}^{n-1} \|e_i^0\|_{L^\infty(\sigma_i^0)} + \|e_{n-N_1}^0\|_{L^\infty(\sigma_{n-N_1}^0)}.$$

Denoting $k = \|K\|_{L^\infty(I)}$, and invoking Lemma 1.7.1 and Lemma 2.4.1, we obtain

$$\begin{aligned}
\|e_n^0\|_{L^\infty(\sigma_n^0)} &\leq \|x - \hat{u}_{n,0}\|_{L^\infty(\sigma_n^0)} + \|\hat{u}_{n,0} - u_n^0\|_{L^\infty(\sigma_n^0)} \\
&\leq hk \sum_{i=0}^{n-1} \|e_i^0\|_{L^\infty(\sigma_i^0)} + \|e_{n-N_1}^0\|_{L^\infty(\sigma_{n-N_1}^0)} + \frac{\alpha(m)}{m!} h^m \\
&\leq hk \sum_{i=0}^{n-1} \|e_i^0\|_{L^\infty(\sigma_i^0)} + 2\, \frac{\alpha(m)}{m!} h^m.
\end{aligned}$$

Hence, by Lemma 1.7.2, it follows that for all $n = 1, \ldots, N_1 - 1$,

$$\|e_n^0\|_{L^\infty(\sigma_n^0)} \leq \underbrace{\left(\frac{2\,\alpha(m)}{m!}\, \exp\!\big(\tau_2 k\big)\right)}_{=:C_1^{(1)}} h^m.$$

By setting

$$C_1 = \max\left\{\frac{\alpha(m)}{m!},\ C_1^{(1)}\right\},$$

we conclude that Claim 1 holds.

Claim 2. There exists a constant C_2, independent of the stepsize h, such that

$$\|e\|_{L^\infty(I)} \leq C_2 h^m.$$

Define the local error $e^p(t)$ on σ^p by

$$e^p(t) = x(t) - u^p(t),$$

and, more specifically, on each subinterval σ_n^p by

$$e^p(t) = e_n^p(t) = x(t) - u_n^p(t), \qquad n = 0, \dots, N-1, \;\; p = 1, \dots, r-1.$$

Let $t \in \sigma_n^p$ for $n = 0, 1, \dots, N-1$. From equations (2.11) and (2.12), we obtain

$$\begin{aligned}\|x - \hat{u}_{n,p}\|_{L^\infty(\sigma_n^p)} \leq \|e_{N-N_1+n}^{p-1}\|_{L^\infty(\sigma_{N-N_1+n}^{p-1})} + \|e_n^{p-1}\|_{L^\infty(\sigma_n^{p-1})} + \|e_{n-N_1}^p\|_{L^\infty(\sigma_{n-N_1}^p)} \\ + 2hk \sum_{i=0}^{N-1} \|e_i^{p-1}\|_{L^\infty(\sigma_i^{p-1})} + hk \sum_{i=0}^{n-1} \|e_i^p\|_{L^\infty(\sigma_i^p)},\end{aligned}$$

where $k = \|K\|_{L^\infty(I)}$.

Consequently, the above inequality can be simplified as

$$\begin{aligned}\|x - \hat{u}_{n,p}\|_{L^\infty(\sigma_n^p)} \leq 2(1 + \tau_2 k) \|e^{p-1}\|_{L^\infty(\sigma^{p-1})} + \|e_{n-N_1}^p\|_{L^\infty(\sigma_{n-N_1}^p)} \\ + hk \sum_{i=0}^{n-1} \|e_i^p\|_{L^\infty(\sigma_i^p)}.\end{aligned}$$

Therefore, by Lemma 1.7.1 and Lemma 2.4.1, it follows that for $n = 1, 2, \dots, N-1$,

$$\begin{aligned}\|e_n^p\|_{L^\infty(\sigma_n^p)} &\leq \|x - \hat{u}_{n,p}\|_{L^\infty(\sigma_n^p)} + \|\hat{u}_{n,p} - u_n^p\|_{L^\infty(\sigma_n^p)} \\ &\leq \|x - \hat{u}_{n,p}\|_{L^\infty(\sigma_n^p)} + \frac{\alpha(m)}{m!} h^m.\end{aligned}$$

For a fixed $p \geq 1$, inequality (2.22) is iteratively used for $n = 1, 2, \dots, aN_1 - 1$, where

$a \in \{1, 2, \ldots, \tau_2/\tau_1\}$, we obtain

$$\begin{aligned} \|e_n^p\|_{L^\infty(\sigma_n^p)} \leq 2(1+\tau_2 k)\, \|e^{p-1}\|_{L^\infty(\sigma^{p-1})} + hk \sum_{i=0}^{n-1} \|e_i^p\|_{L^\infty(\sigma_i^p)} \\ + \|e_{n-N_1}^p\|_{L^\infty(\sigma_{n-N_1}^p)} + \frac{\alpha(m)}{m!}\, h^m. \end{aligned} \tag{2.22}$$

The inequality expresses the local error as a combination of accumulated, delayed, and inherited errors, together with a Taylor truncation term, evaluated on successive subintervals of length τ_1 covering $[0, \tau_2[$

Applying Lemma 1.7.2 (discrete Grönwall inequality), we deduce that for all $n = 0, 1, \ldots, N-1$,

$$\|e_n^p\|_{L^\infty(\sigma_n^p)} \leq \frac{\alpha(m) C_1 \exp(\tau_2 k)}{m!}\, h^m + 2(1+\tau_2 k)\exp(\tau_2 k)\, \|e^{p-1}\|_{L^\infty(\sigma^{p-1})}.$$

Consequently, for all $p = 1, \ldots, r-1$, there exists a constant C_2 such that

$$\|e_n^p\|_{L^\infty(\sigma_n^p)} \leq C_2 h^m.$$

Thus, the proof is completed by setting

$$C = \max\{C_1,\, C_2\}.$$

□

2.5 Numerical examples

In this section, we present numerical experiments that evaluate the effectiveness of the Taylor collocation method (TCM) in solving problems described by (2.1). We demonstrate convergence of order m by providing numerical examples that exhibit convergence rates of up to m when applying this method. Additionally, we record the computational time (in CPU time/s) for each example. Our numerical experiments were conducted using Maple version 17 on a personal computer equipped with an Intel Core i7-1165G7 CPU @ 2.80GHz and 16 GB of RAM, running the MS Windows 7 operating system. As the values of both n and m increase, the absolute error function decreases, thereby confirming the theoretical estimates outlined in section 4.

Example 2.5.1 *Consider the neutral double delay Volterra integral equation*

$$x(t) = g(t) + x(t - \frac{1}{2}) + x(t-1) + \int_{t-1}^{t-\frac{1}{2}} (ts + \cos(t+s))x(s)ds$$

for $t \in [1, 5]$*, and* g *is chosen so that the exact solution* $x(t) = \cos(t)+1$*,* $x(t) = \Phi(t)$ *for* $t \in [0, 1]$*. The absolute errors and computational time for* $(N, m) = \{(2, 2), (4, 4), (6, 6), (8, 8)\}$ *are presented in Table 2.1.The exact and approximate solutions for* $N = m = 8$ *are shown in Figure 2.2.*

Figure 2.1 illustrates the decay of the absolute error as the parameters N *and* m *increase, confirming the theoretical convergence order of the proposed method.*

t	$N=m=2$	$N=m=4$	$N=m=6$	$N=m=8$
1.00	0.0	0.0	0.0	0.0
2.00	$1.58e-02$	$1.02e-5$	$1.29e-9$	$7.08e-11$
3.00	$1.27e-02$	$2.89e-6$	$2.52e-9$	$9.70e-10$
4.00	$4.45e-02$	$2.53e-4$	$1.17e-7$	$2.53e-08$
5.00	$1.44e-01$	$7.31e-3$	$3.81e-6$	$8.50e-07$
CPU	1.79 s	1.95 s	3.18 s	5.29 s

Table 2.1: Absolute errors for Example 2.5.1

Example 2.5.2 *Consider the neutral double delay Volterra integral equation*

$$x(t) = g(t) + x(t-0.4) + x(t-0.8) + \int_{t-0.8}^{t-0.4} (t-s)e^{t-s}x(s)ds$$

for $t \in [0.8, 8]$*, and* g *is chosen so that the exact solution* $x(t) = e^{-t}\sin(t)$*,* $\Phi(t) = x(t)$

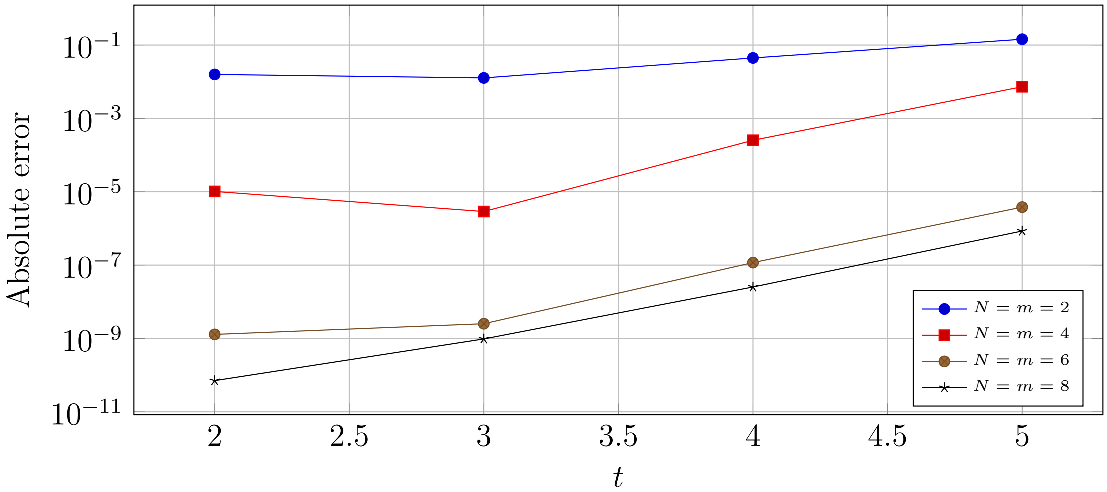

Figure 2.1: Logarithmic plot of absolute errors for Example 2.5.1

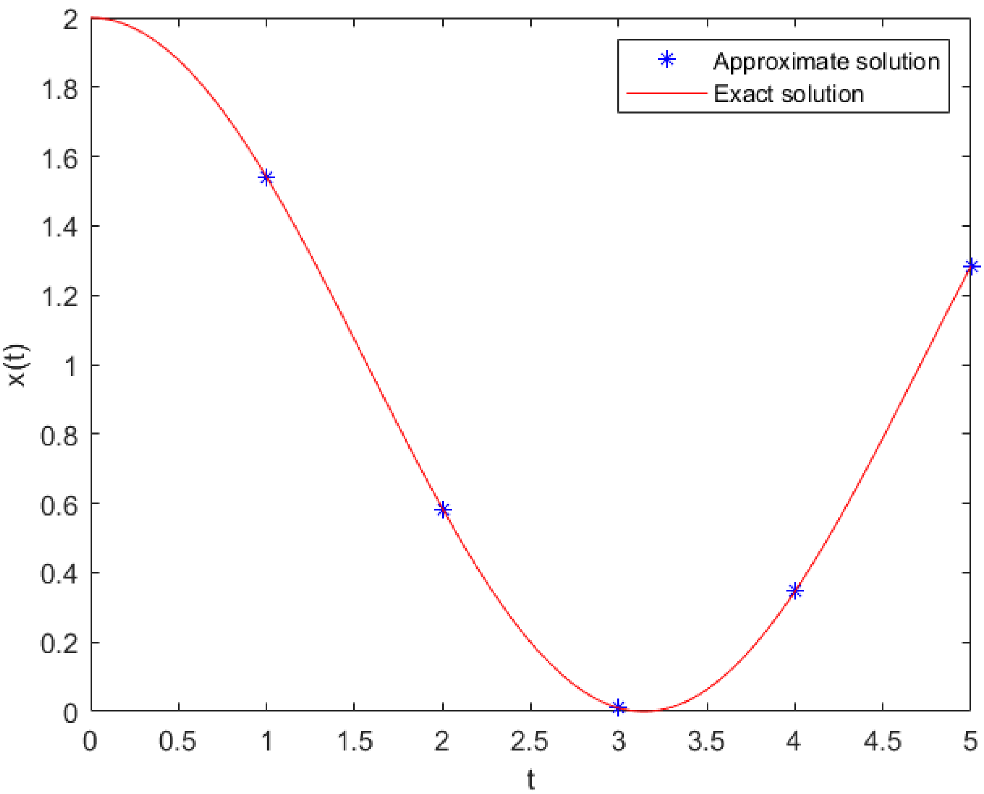

Figure 2.2: The exact and approximate solutions for $N = m = 8$.

for $t \in [0, 0.8]$. *The absolute errors and computational time for*

$$(N, m) = \{(2, 6), (2, 8), (2, 10), (4, 4), (4, 6), (4, 8)\}$$

are presented in Table 2.2. Figure 2.4 displays the exact and approximate solutions.

	$N=2$			$N=4$		
t	$m=6$	$m=8$	$m=10$	$m=4$	$m=6$	$m=8$
0.8	0	0	0	0	0	0
1.6	$5.22e-07$	$2.26e-09$	$2.82e-10$	$6.28e-06$	$9.24e-09$	$2.81e-10$
2.4	$1.57e-06$	$9.66e-09$	$1.05e-09$	$2.59e-05$	$2.64e-08$	$1.08e-09$
3.2	$5.31e-06$	$3.57e-08$	$3.53e-09$	$9.19e-05$	$8.57e-08$	$3.75e-09$
4.0	$1.88e-05$	$1.30e-07$	$1.18e-08$	$3.21e-04$	$2.95e-07$	$1.29e-08$
4.8	$6.69e-05$	$4.67e-07$	$3.97e-08$	$1.12e-03$	$1.02e-06$	$4.46e-08$
5.6	$3.65e-04$	$1.66e-06$	$1.34e-07$	$3.91e-03$	$3.55e-06$	$1.54e-07$
6.4	$8.30e-04$	$5.86e-06$	$4.56e-07$	$1.35e-02$	$1.23e-05$	$5.32e-07$
7.2	$2.90e-03$	$2.06e-05$	$1.55e-06$	$4.71e-02$	$4.25e-05$	$1.83e-06$
8.0	$1.07e-02$	$7.81e-05$	$4.98e-06$	$1.65e-01$	$1.49e-04$	$6.28e-06$
CPU	1.57 s	1.90 s	3.85 s	1.75 s	2.56 s	3.82 s

Table 2.2: Absolute errors of Example 2.5.2.

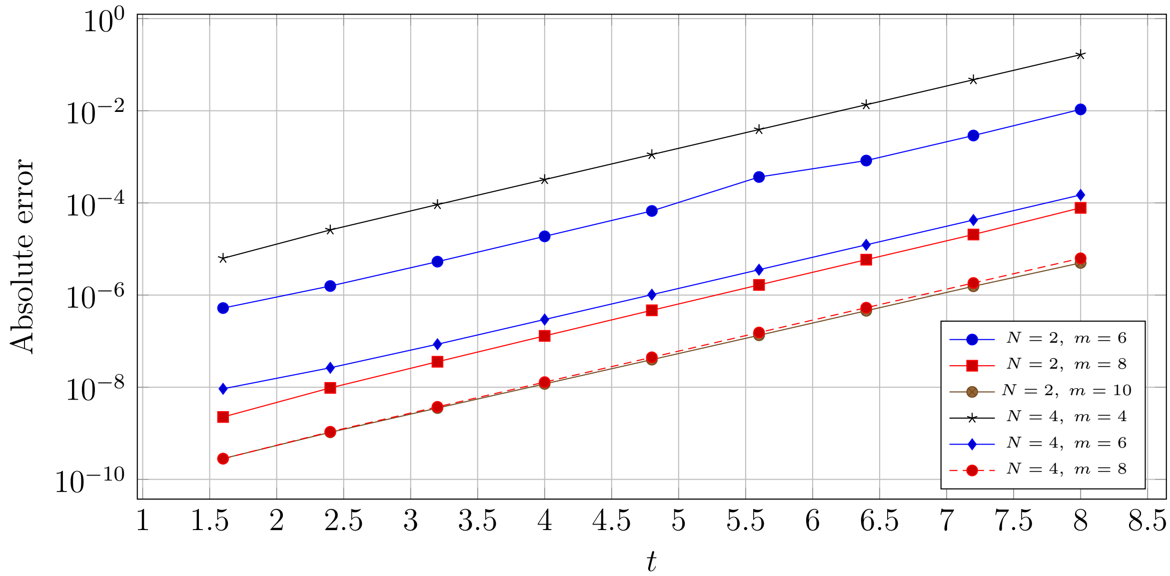

Figure 2.3: Logarithmic plot of absolute errors for Example 2.5.2 (Table 2.2).

Example 2.5.3 *Consider the neutral double delay Volterra integral equation*

$$x(t) = g(t) + x(t-\frac{1}{4}) + x(t-\frac{1}{2}) + \int_{t-\frac{1}{2}}^{t-\frac{1}{4}} (e^t + s)x(s)ds$$

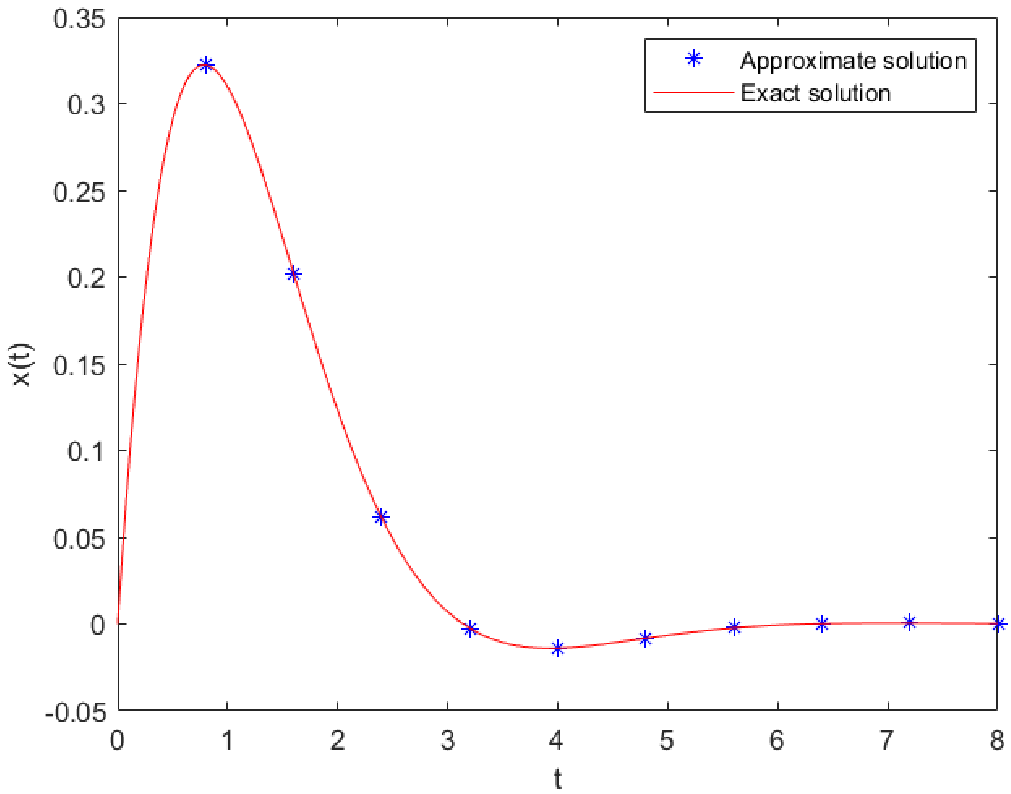

Figure 2.4: The exact and approximate solutions for $N = 4, m = 8$.

for $t \in [0.5, 2.5]$*, and* g *is chosen so that the exact solution* $x(t) = t^2 + 4t^5$*,* $\Phi(t) = x(t)$ *for* $t \in [0, 0.5]$*. The absolute errors and computational time for* $(N, m) = \{(2, 4), (4, 4), (6, 6), (8, 8)\}$ *are presented in Table 2.3. Figure 2.5 displays the exact and approximate solutions for N=m=8.*

t	$N = 2, m = 4$	$N = 4, m = 4$	$N = 6, m = 6$	$N = 8, m = 8$
0.50	0.0	0.0	0.0	0.0
0.75	$8.66e-10$	$8.66e-10$	$8.66e-10$	$8.66e-10$
1.00	$7.25e-03$	$4.83e-04$	$1.75e-09$	$1.75e-09$
1.25	$2.10e-02$	$1.49e-03$	$7.04e-09$	$6.65e-09$
1.50	$6.26e-02$	$4.50e-03$	$1.61e-08$	$1.87e-08$
1.75	$1.70e-01$	$1.26e-02$	$5.43e-08$	$5.70e-08$
2.00	$4.90e-01$	$3.70e-02$	$1.58e-07$	$1.74e-07$
2.25	$1.51e-00$	$1.16e-01$	$5.27e-07$	$5.66e-07$

Table 2.3: Absolute errors of Example 2.5.3.

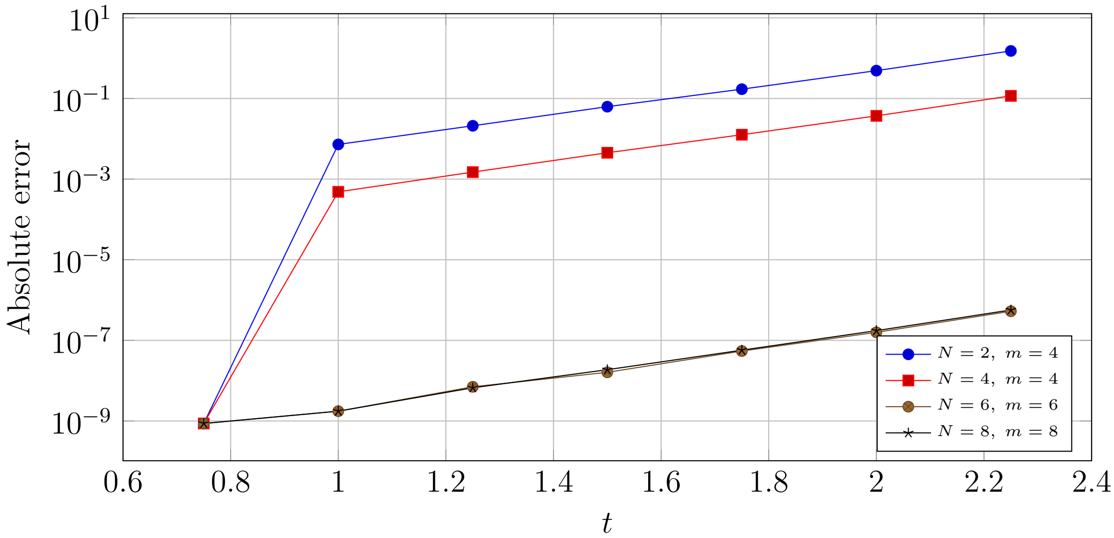

Figure 2.5: Logarithmic plot of absolute errors for Example 2.5.3 (Table 2.3).

Example 2.5.4 *Consider the neutral double delay Volterra integral equation*

$$x(t) = g(t) + x(t - 0.45) + x(t - 0.9) + \int_{t-0.9}^{t-0.45} (\sin(3t) + s)x(s)ds$$

for $t \in [0.9, 9.9]$*, and* g *is chosen so that the exact solution* $x(t) = \cos(t)$*,* $\Phi(t) = x(t)$ *for* $t \in [0, 0.9]$*. The absolute errors and computational time for* $(N, m) = \{(10, 5), (10, 10), (20, 5), (20, 10)\}$ *are presented in Table 2.4. Figure 2.6 displays the exact and approximate solutions.*

	$N = 10$		$N = 20$	
t	$m = 5$	$m = 10$	$m = 5$	$m = 10$
0.9	$3.00e-20$	$3.00e-20$	$3.00e-20$	$3.00e-20$
1.8	$1.27e-09$	$2.07e-12$	$4.17e-11$	$3.17e-12$
2.7	$2.65e-08$	$2.92e-10$	$1.15e-09$	$3.42e-10$
3.6	$1.25e-07$	$2.01e-09$	$6.18e-09$	$2.36e-09$
4.5	$8.31e-07$	$1.78e-08$	$4.64e-08$	$2.11e-08$
5.4	$5.90e-06$	$1.39e-07$	$3.46e-07$	$1.66e-07$
6.3	$4.60e-05$	$1.10e-06$	$2.70e-06$	$1.36e-06$
7.2	$4.70e-04$	$1.10e-05$	$2.80e-05$	$1.40e-05$

Table 2.4: Absolute errors of Example 2.5.4.

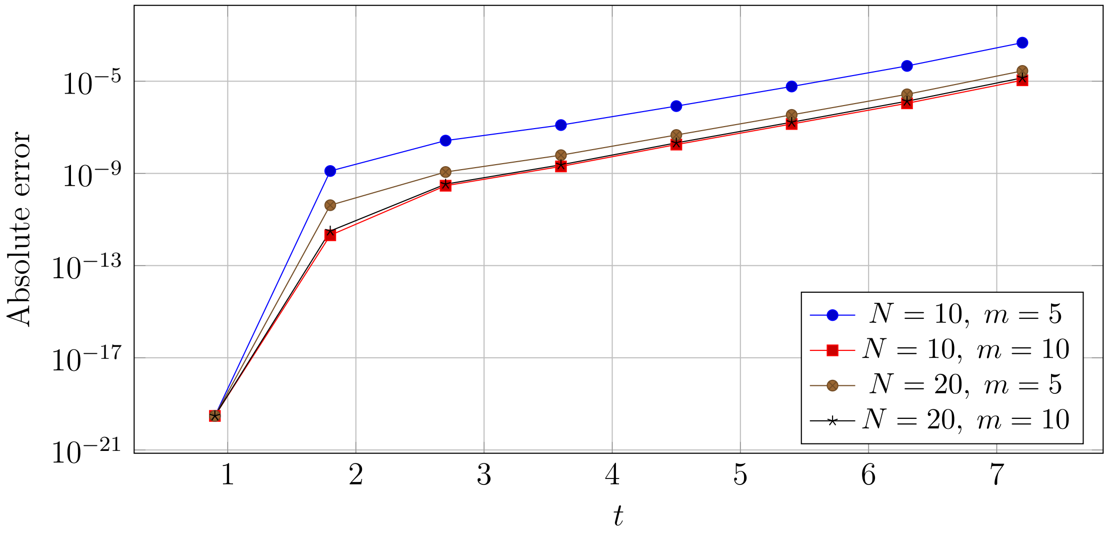

Figure 2.6: Logarithmic plot of absolute errors for Example 2.5.4 (Table 2.4).

2.6 Conclusion

This chapter was devoted to the numerical resolution of linear Volterra integral equations with two constant delays using a Taylor collocation framework. The main objective was to construct an efficient approximation scheme and to provide a rigorous theoretical analysis of its convergence properties.

First, the problem was formulated within a suitable piecewise polynomial spline space, allowing the solution to be approximated locally on successive subintervals. By employing truncated Taylor expansions at carefully chosen collocation points, an explicit and easily implementable numerical algorithm was derived. The presence of two constant delays was handled through a stepwise construction of the approximate solution, ensuring that delayed terms depend only on previously computed approximations.

A detailed convergence analysis was then carried out. By establishing the boundedness of the coefficients of the polynomial approximation and applying discrete Grönwall-type inequalities, it was shown that the proposed method converges with order $O(h^m)$, provided that the exact solution and the kernel are sufficiently smooth. The analysis also clarified how the delayed error terms are absorbed into the accumulated error estimates, guaranteeing the stability of the method despite the presence of delays.

Finally, several numerical examples were presented to illustrate the effectiveness and accuracy of the proposed approach. The reported results confirm the theoretical convergence rate and demonstrate that increasing the polynomial degree significantly improves the accuracy, while mesh refinement further enhances the numerical performance.

In conclusion, the Taylor collocation method developed in this chapter provides a reliable and accurate tool for solving linear Volterra integral equations with double constant delays. The results obtained here form a solid foundation for the extensions considered in the following chapters, particularly for more general integro-differential equations and higher-order or neutral delay problems.

Chapter 3

Taylor collocation method for neutral double delay Volterra integro-differential equations

3.1 Introduction

In this chapter, we investigate the numerical solution of a linear Volterra integro-differential equation with two constant delays τ_1 and τ_2, defined as

$$\begin{aligned} x'(t) = g(t) + A(t)x(t-\tau_1) + B(t)x(t-\tau_2) + \int_0^{t-\tau_2} k_3(t,s)x(s)\,ds \\ + \int_{t-\tau_2}^{t-\tau_1} k_2(t,s)x(s)\,ds + \int_{t-\tau_1}^{t} k_1(t,s)x(s)\,ds, \end{aligned} \tag{3.1}$$

for $t \in [\tau_2, T]$, together with the initial function

$$x(t) = \Phi(t), \quad t \in [0, \tau_2].$$

Throughout this work, we assume that the given functions g, A, B, k_1, k_2, k_3, and Φ are sufficiently smooth. Moreover, the following compatibility condition at $t = \tau_2$ is imposed:

$$\begin{aligned} \Phi'(\tau_2) = g(\tau_2) + A(\tau_2)\Phi(\tau_2 - \tau_1) + B(\tau_2)\Phi(0) + \int_{\tau_2-\tau_1}^{\tau_2} k_1(\tau_2, s)\Phi(s)\,ds \\ + \int_0^{\tau_2-\tau_1} k_2(\tau_2, s)\Phi(s)\,ds. \end{aligned}$$

Existence and uniqueness of solutions to problem (3.1) can be established by adapting classical results from the theory of Volterra integro-differential equations (see, for instance, [1]).

It is worth emphasizing that equation (3.1) provides a general modeling framework that encompasses several important classes of functional equations as special cases.

In particular, equation (3.1) also includes, as a special case, linear Volterra integro-differential equations with two discrete delays and a single Volterra integral term. Indeed, by setting the kernels $k_2(t,s)$ and $k_3(t,s)$ equal to zero, equation (3.1) reduces to

$$x'(t) = g(t) + A(t)x(t-\tau_1) + B(t)x(t-\tau_2) + \int_{t-\tau_1}^{t} k_1(t,s)x(s)\,ds,$$

which represents a linear delay integro-differential equation with two constant delays and a memory effect acting over the most recent time interval $[t - \tau_1,\, t]$.

Such equations frequently arise in applications where the system dynamics depend both on discrete delayed states and on accumulated effects from the recent past. Therefore, the proposed Taylor collocation method is directly applicable to this reduced model, further illustrating the flexibility and generality of the numerical approach developed in this chapter.

Moreover, by setting all integral kernels equal to zero, namely $k_1(t, s) = k_2(t, s) = k_3(t, s) = 0$, equation (3.1) degenerates into the linear delay differential equation

$$x'(t) = g(t) + A(t)x(t - \tau_1) + B(t)x(t - \tau_2),$$

which is a classical model in the theory of functional differential equations.

This hierarchical structure demonstrates that equation (3.1) unifies delay differential equations and Volterra integro-differential equations with multiple delays. Consequently, the numerical method developed in this chapter applies not only to the full model (3.1), but also to its various simplified forms, highlighting the generality and robustness of the proposed Taylor collocation approach.

The main objective of this chapter is to develop a numerical collocation method based on truncated Taylor polynomials for the approximate solution of linear Volterra integro-differential equations with two constant delays of the form (3.1). The proposed method is explicit and direct, achieves a provable order of convergence, and avoids the solution of algebraic systems. These features make the method computationally efficient and easy to implement.

The remainder of this chapter is organized as follows. In Section 2, the Taylor collocation scheme and the construction of the approximate solution are presented. Section 3 is devoted to the convergence analysis and the derivation of the order of convergence. In Section 4, several numerical examples are provided to demonstrate the accuracy and effectiveness of the proposed method. Finally, concluding remarks are given in Section 5.

3.2 Numerical Scheme and Discretization

We assume that the final time T satisfies $T = (r+1)\tau_2$, where $r \in \{1, 2, 3, \ldots\}$. Let Π_N denote a uniform partition of the interval $I = [\tau_2, T]$, defined by the grid points

$$t_n^i = (i+1)\tau_2 + nh, \qquad n = 0, 1, \ldots, N,\ i = 0, 1, \ldots, r-1,$$

where the step size is given by

$$h = t_{n+1}^i - t_n^i.$$

We further assume that

$$h = \frac{\tau_1}{N_1} = \frac{\tau_2}{N},$$

with N and N_1 being positive integers.

The corresponding subintervals are defined as

$$\sigma_n^i = [t_n^i,\, t_{n+1}^i), \quad n = 0, 1, \ldots, N-1,\ i = 0, 1, \ldots, r-2,$$

and

$$\sigma_{N-1}^{r-1} = [t_{N-1}^{r-1},\, t_N^{r-1}].$$

Let π_m denote the set of all real polynomials of degree at most m. We introduce the real polynomial spline space of degree m by

$$S_m^{(0)}(\Pi_N) = \left\{ u \in C^0(I, \mathbb{R}) : u|_{\sigma_n^i} \in \pi_m,\ n = 0, \ldots, N-1,\ i = 0, 1, \ldots, r-1 \right\}.$$

This space consists of piecewise polynomial functions of degree at most m that are continuous on I. Its dimension is equal to $rNm+1$, which corresponds to the total number of unknown coefficients of the local polynomial representations u_n^i, $n = 0, \ldots, N-1$, $i = 0, 1, \ldots, r-1$.

To determine these coefficients, we approximate the solution on each subinterval by means of a truncated Taylor polynomial expansion.

First, we approximate the solution x on the initial subinterval σ_0^0 by a truncated Taylor polynomial centered at $t = \tau_2$. Specifically, we define

$$u_0^0(t) = \sum_{j=0}^{m} \frac{x^{(j)}(\tau_2)}{j!}\,(t-\tau_2)^j, \qquad t \in \sigma_0^0, \tag{3.2}$$

where $x^{(j)}(\tau_2)$, $j = 0, \ldots, m$, denotes the exact value of the j-th derivative of x at $t = \tau_2$. This construction requires the solution x to be sufficiently smooth in a neighborhood of τ_2.

To compute the coefficients $x^{(j)}(\tau_2)$, we differentiate equation (3.1) j times with respect to t. For $j = 0, 1, \ldots, m$, this yields

$$\begin{aligned}
x^{(j+1)}(t) &= g^{(j)}(t) + \Big(A(t)\Phi(t-\tau_1)\Big)^{(j)} + \Big(B(t)\Phi(t-\tau_2)\Big)^{(j)} \\
&\quad + \left(\int_{t-\tau_1}^{\tau_2} k_1(t,s)\Phi(s)\,ds + \int_{t-\tau_2}^{t-\tau_1} k_2(t,s)\Phi(s)\,ds + \int_0^{t-\tau_2} k_3(t,s)\Phi(s)\,ds\right)^{(j)} \\
&\quad + \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\Big[\partial_1^{(j-1-i)}k_1(t,t)\Big]^{(i-l)} x^{(l)}(t) + \int_{\tau_2}^{t} \partial_1^{(j)}k_1(t,s)\,x(s)\,ds.
\end{aligned}$$

Evaluating the above expression at $t = \tau_2$, we obtain

$$\begin{aligned}
x^{(j+1)}(\tau_2) &= g^{(j)}(\tau_2) + \Big(A(t)\Phi(t-\tau_1)\Big)^{(j)}\Big|_{t=\tau_2} + \Big(B(t)\Phi(t-\tau_2)\Big)^{(j)}\Big|_{t=\tau_2} \\
&\quad + \left(\int_{t-\tau_1}^{\tau_2} k_1(t,s)\Phi(s)\,ds + \int_{t-\tau_2}^{t-\tau_1} k_2(t,s)\Phi(s)\,ds + \int_0^{t-\tau_2} k_3(t,s)\Phi(s)\,ds\right)^{(j)}\Bigg|_{t=\tau_2} \\
&\quad + \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\Big[\partial_1^{(j-1-i)}k_1(t,t)\Big]^{(i-l)}\Big|_{t=\tau_2} x^{(l)}(\tau_2),
\end{aligned}$$

where $x(\tau_2) = \Phi(\tau_2)$.

This condition is introduced to ensure the continuity of the piecewise polynomial approximation. Since the function $x$is employed to construct the Taylor expansion on the initial subintervalσ_0^0, its value at the left endpoint must coincide with the value provided by the approximation on the preceding subinterval. Enforcing this compatibility condition guarantees the continuity of the global numerical solution and eliminates the possibility of spurious discontinuities arising at the mesh nodes.

Second, for $n \in \{1, 2, \ldots, N-1\}$, the approximation of the solution x on the subinterval σ_n^0 is constructed recursively. More precisely, in order to approximate x by u_n^0 on σ_n^0, the approximations u_k^0 must already be available on all previous subintervals σ_k^0, $k = 0, 1, \ldots, n-1$.

On each subinterval σ_n^0, we define the approximation u_n^0 by a Taylor polynomial of degree m centered at t_n^0, namely

$$u_n^0(t) = \sum_{j=0}^{m} \frac{\hat{u}_{n,0}^{(j)}(t_n^0)}{j!} (t - t_n^0)^j, \qquad t \in \sigma_n^0, \tag{3.3}$$

where $\hat{u}_{n,0}$ denotes the exact solution of an auxiliary Volterra integral equation defined on σ_n^0.

For $t \in \sigma_n^0$ and $n \in \{1, 2, \ldots, N_1 - 1\}$, the function $\hat{u}_{n,0}$ satisfies

$$\begin{aligned} \hat{u}'_{n,0}(t) &= g(t) + A(t)\Phi(t-\tau_1) + B(t)\Phi(t-\tau_2) \\ &\quad + \int_{t-\tau_1}^{\tau_2} k_1(t,s)\Phi(s)\,ds + \int_{t-\tau_2}^{t-\tau_1} k_2(t,s)\Phi(s)\,ds + \int_{0}^{t-\tau_2} k_3(t,s)\Phi(s)\,ds \\ &\quad + \sum_{i=0}^{n-1} \int_{t_i^0}^{t_{i+1}^0} k_1(t,s)u_i^0(s)\,ds + \int_{t_n^0}^{t} k_1(t,s)\hat{u}_{n,0}(s)\,ds. \end{aligned} \tag{3.4}$$

For $t \in \sigma_n^0$ and $n \in \{N_1, N_1+1, \ldots, N-1\}$, the delayed term $x(t-\tau_1)$ involves previously computed approximations, and $\hat{u}_{n,0}$ satisfies

$$\begin{aligned}
\hat{u}'_{n,0}(t) = g(t) &+ A(t)u^0_{n-N_1}(t-\tau_1) + B(t)\Phi(t-\tau_2) \\
&+ \int_0^{t-\tau_2} k_3(t,s)\Phi(s)\,ds + \int_{t-\tau_2}^{\tau_2} k_2(t,s)\Phi(s)\,ds \\
&+ \sum_{i=0}^{n-N_1-1}\int_{t^0_i}^{t^0_{i+1}} k_2(t,s)u^0_i(s)\,ds + \int_{t^0_n-\tau_1}^{t-\tau_1} k_2(t,s)u^0_{n-N_1}(s)\,ds \\
&+ \int_{t-\tau_1}^{t^0_{n+1}-\tau_1} k_1(t,s)u^0_{n-N_1}(s)\,ds + \sum_{i=n+1-N_1}^{n-1}\int_{t^0_i}^{t^0_{i+1}} k_1(t,s)u^0_i(s)\,ds \\
&+ \int_{t^0_n}^{t} k_1(t,s)\hat{u}_{n,0}(s)\,ds.
\end{aligned} \tag{3.5}$$

Now, for all $j = 0,1,\ldots,m$, the coefficients $\hat{u}^{(j)}_{n,0}(t^0_n)$ appearing in the Taylor expansion (3.3) can be computed by arguments analogous to those used for deriving the expressions of $x^{(j)}(\tau_2)$. Differentiating the auxiliary equation satisfied by $\hat{u}_{n,0}$ j times, we obtain the following recurrence relations.

For $n \in \{1,2,\ldots,N_1-1\}$, we have

$$\begin{aligned}
\hat{u}^{(j+1)}_{n,0}(t) = g^{(j)}(t) &+ \Big(A(t)\Phi(t-\tau_1)\Big)^{(j)} + \Big(B(t)\Phi(t-\tau_2)\Big)^{(j)} \\
&+ \left(\int_{t-\tau_1}^{\tau_2} k_1(t,s)\Phi(s)\,ds + \int_{t-\tau_2}^{t-\tau_1} k_2(t,s)\Phi(s)\,ds + \int_0^{t-\tau_2} k_3(t,s)\Phi(s)\,ds\right)^{(j)} \\
&+ \sum_{i=0}^{n-1}\sum_{l=0}^{m}\frac{\hat{u}^{(l)}_{i,0}(t^0_i)}{l!}\int_{t^0_i}^{t^0_{i+1}} \partial_1^{(j)}k_1(t,s)\,(s-t^0_i)^l\,ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\Big[\partial_1^{(j-1-i)}k_1(t,t)\Big]^{(i-l)}\hat{u}^{(l)}_{n,0}(t) + \int_{t^0_n}^{t}\partial_1^{(j)}k_1(t,s)\hat{u}_{n,0}(s)\,ds.
\end{aligned} \tag{3.6}$$

Evaluating the above expression at $t = t_n^0$ yields

$$\begin{aligned}
\hat{u}_{n,0}^{(j+1)}(t_n^0) &= g^{(j)}(t_n^0) + \Big(A(t)\Phi(t-\tau_1)\Big)^{(j)}\Big|_{t=t_n^0} + \Big(B(t)\Phi(t-\tau_2)\Big)^{(j)}\Big|_{t=t_n^0} \\
&+ \left(\int_{t-\tau_1}^{\tau_2} k_1(t,s)\Phi(s)\,ds + \int_{t-\tau_2}^{t-\tau_1} k_2(t,s)\Phi(s)\,ds + \int_0^{t-\tau_2} k_3(t,s)\Phi(s)\,ds\right)^{(j)}\Bigg|_{t=t_n^0} \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\Big[\partial_1^{(j-1-i)}k_1(t,t)\Big]^{(i-l)}(t_n^0)\,\hat{u}_{n,0}^{(l)}(t_n^0) \\
&+ \sum_{i=0}^{n-1}\sum_{l=0}^{m}\frac{\hat{u}_{i,0}^{(l)}(t_i^0)}{l!}\int_{t_i^0}^{t_{i+1}^0}\partial_1^{(j)}k_1(t_n^0,s)\,(s-t_i^0)^l\,ds.
\end{aligned}$$

For $n \in \{N_1, N_1+1, \ldots, N-1\}$, differentiating the auxiliary equation satisfied by $\hat{u}_{n,0}$ j times yields

$$\begin{aligned}
\hat{u}_{n,0}^{(j+1)}(t) &= g^{(j)}(t) + \Big(A(t)u_{n-N_1}^0(t-\tau_1)\Big)^{(j)} + \Big(B(t)\Phi(t-\tau_2)\Big)^{(j)} \\
&+ \left(\int_0^{t-\tau_2} k_3(t,s)\Phi(s)\,ds + \int_{t-\tau_2}^{\tau_2} k_2(t,s)\Phi(s)\,ds\right)^{(j)} \\
&+ \sum_{i=0}^{n-N_1-1}\sum_{l=0}^{m}\frac{\hat{u}_{i,0}^{(l)}(t_i^0)}{l!}\int_{t_i^0}^{t_{i+1}^0}\partial_1^{(j)}k_2(t,s)\,(s-t_i^0)^l\,ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m}\binom{i}{l}\frac{\hat{u}_{n-N_1,0}^{(r)}(t_{n-N_1}^0)}{(r-l)!}\Big[\partial_1^{(j-1-i)}k_2(t,t-\tau_1)\Big]^{(i-l)}(t-\tau_1-t_{n-N_1}^0)^{r-l} \\
&+ \sum_{l=0}^{m}\frac{\hat{u}_{n-N_1,0}^{(l)}(t_{n-N_1}^0)}{l!}\int_{t_n^0-\tau_1}^{t-\tau_1}\partial_1^{(j)}k_2(t,s)\,(s-t_{n-N_1}^0)^l\,ds \\
&- \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m}\binom{i}{l}\frac{\hat{u}_{n-N_1,0}^{(r)}(t_{n-N_1}^0)}{(r-l)!}\Big[\partial_1^{(j-1-i)}k_1(t,t-\tau_1)\Big]^{(i-l)}(t-\tau_1-t_{n-N_1}^0)^{r-l} \\
&+ \sum_{l=0}^{m}\frac{\hat{u}_{n-N_1,0}^{(l)}(t_{n-N_1}^0)}{l!}\int_{t-\tau_1}^{t_{n+1}^0-\tau_1}\partial_1^{(j)}k_1(t,s)\,(s-t_{n-N_1}^0)^l\,ds \\
&+ \sum_{i=n+1-N_1}^{n-1}\sum_{l=0}^{m}\frac{\hat{u}_{i,0}^{(l)}(t_i^0)}{l!}\int_{t_i^0}^{t_{i+1}^0}\partial_1^{(j)}k_1(t,s)\,(s-t_i^0)^l\,ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\Big[\partial_1^{(j-1-i)}k_1(t,t)\Big]^{(i-l)}\hat{u}_{n,0}^{(l)}(t) + \int_{t_n^0}^{t}\partial_1^{(j)}k_1(t,s)\hat{u}_{n,0}(s)\,ds.
\end{aligned} \tag{3.7}$$

Evaluating the above expression at $t = t_n^0$, we obtain

$$\begin{aligned}
\hat{u}_{n,0}^{(j+1)}(t_n^0) &= g^{(j)}(t_n^0) + \sum_{l=0}^{j} \binom{j}{l} \hat{u}_{n-N_1,0}^{(l)}(t_{n-N_1}^0) A^{(j-l)}(t_n^0) \\
&+ \sum_{l=0}^{j} \binom{j}{l} B^{(j-l)}(t_n^0)\, \Phi^{(l)}(t_n^0 - \tau_2) \\
&+ \left(\int_0^{t-\tau_2} k_3(t,s)\Phi(s)\, ds + \int_{t-\tau_2}^{\tau_2} k_2(t,s)\Phi(s)\, ds \right)^{(j)} \Bigg|_{t=t_n^0} \\
&+ \sum_{i=0}^{n-N_1-1} \sum_{l=0}^{m} \frac{\hat{u}_{i,0}^{(l)}(t_i^0)}{l!} \int_{t_i^0}^{t_{i+1}^0} \partial_1^{(j)} k_2(t_n^0, s)\, (s - t_i^0)^l\, ds \\
&+ \sum_{i=0}^{j-1} \sum_{l=0}^{i} \binom{i}{l} \hat{u}_{n-N_1,0}^{(l)}(t_{n-N_1}^0) \Big[\partial_1^{(j-1-i)} k_2(t, t-\tau_1) \Big]^{(i-l)} (t_n^0) \\
&- \sum_{i=0}^{j-1} \sum_{l=0}^{i} \binom{i}{l} \hat{u}_{n-N_1,0}^{(l)}(t_{n-N_1}^0) \Big[\partial_1^{(j-1-i)} k_1(t, t-\tau_1) \Big]^{(i-l)} (t_n^0) \\
&+ \sum_{l=0}^{m} \frac{\hat{u}_{n-N_1,0}^{(l)}(t_{n-N_1}^0)}{l!} \int_{t_n^0-\tau_1}^{t_{n+1}^0-\tau_1} \partial_1^{(j)} k_1(t_n^0, s)\, (s - t_{n-N_1}^0)^l\, ds \\
&+ \sum_{i=n+1-N_1}^{n-1} \sum_{l=0}^{m} \frac{\hat{u}_{i,0}^{(l)}(t_i^0)}{l!} \int_{t_i^0}^{t_{i+1}^0} \partial_1^{(j)} k_1(t_n^0, s)\, (s - t_i^0)^l\, ds \\
&+ \sum_{i=0}^{j-1} \sum_{l=0}^{i} \binom{i}{l} \Big[\partial_1^{(j-1-i)} k_1(t,t) \Big]^{(i-l)} (t_n^0) \hat{u}_{n,0}^{(l)}(t_n^0).
\end{aligned}$$

Here, the continuity condition

$$\hat{u}_{n,0}(t_n^0) = u_{n-1}^0(t_n^0)$$

is imposed to ensure the consistency of the piecewise Taylor approximation.

Third, in order to approximate x by u_0^p $\Big(p \in \{1, 2, \dots, r-1\}\Big)$ on the interval σ_0^p, the function x must first be approximated by u_k^j $\Big(0 \le k \le N-1,\ 0 \le j < p\Big)$ on each interval σ_k^j. Accordingly, we define

$$u_0^p(t) = \sum_{j=0}^{m-1} \frac{\hat{u}_{0,p}^{(j)}(t_0^p)}{j!} (t - t_0^p)^j, \qquad t \in \sigma_0^p, \tag{3.8}$$

where $\hat{u}_{0,p}$ denotes the exact solution of the following integral equation:

$$
\begin{aligned}
\hat{u}'_{0,p}(t) = \; & g(t) + A(t)\, u^{p-1}_{N-N_1}(t-\tau_1) + B(t)\, u^{p-1}_0(t-\tau_2) + \int_0^{\tau_2} k_3(t,s)\, \phi(s)\, ds \\
& + \sum_{i=0}^{p-2} \sum_{d=0}^{N-1} \int_{t_d^i}^{t_{d+1}^i} k_3(t,s)\, u_d^i(s)\, ds \\
& + \int_{t_0^{p-1}}^{t-\tau_2} k_3(t,s)\, u_0^{p-1}(s)\, ds + \int_{t-\tau_2}^{t_1^{p-1}} k_2(t,s)\, u_0^{p-1}(s)\, ds \\
& + \sum_{d=1}^{N-N_1-1} \int_{t_d^{p-1}}^{t_{d+1}^{p-1}} k_2(t,s)\, u_d^{p-1}(s)\, ds + \int_{t_0^p-\tau_1}^{t-\tau_1} k_2(t,s)\, u^{p-1}_{N-N_1}(s)\, ds \\
& + \int_{t-\tau_1}^{t_1^p-\tau_1} k_1(t,s)\, u^{p-1}_{N-N_1}(s)\, ds + \sum_{d=N-N_1+1}^{N-1} \int_{t_d^{p-1}}^{t_{d+1}^{p-1}} k_1(t,s)\, u_d^{p-1}(s)\, ds \\
& + \int_{t_0^p}^{t} k_1(t,s)\, \hat{u}_{0,p}(s)\, ds.
\end{aligned}
\tag{3.9}
$$

The coefficients $\hat{u}^{(j)}_{0,p}(t^p_0)$ are given by the following formula:

$$
\begin{aligned}
\hat{u}^{(j+1)}_{0,p}(t) = {} & g^{(j)}(t) + \Big(A(t)\, u^{p-1}_{N-N_1}(t-\tau_1)\Big)^{(j)} + \Big(B(t)\, u^{p-1}_{0}(t-\tau_2)\Big)^{(j)} \\
& + \left(\int_0^{\tau_2} k_3(t,s)\,\Phi(s)\,ds\right)^{(j)}(t) \\
& + \sum_{i=0}^{p-2}\sum_{d=0}^{N-1}\sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{d,i}(t^i_d)}{l!} \int_{t^i_d}^{t^i_{d+1}} \partial_1^{(j)} k_3(t,s)\,(s-t^i_d)^l\,ds \\
& + \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m-1} \binom{i}{l} \frac{\hat{u}^{(r)}_{0,p-1}(t^{p-1}_0)}{(r-l)!} [\partial_1^{(j-1-i)} k_3(t,t-\tau_2)]^{(i-l)}(t)(t-\tau_2-t^{p-1}_0)^{r-l} \\
& + \sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{0,p-1}(t^{p-1}_0)}{l!} \int_{t^{p-1}_0}^{t-\tau_2} \partial_1^{(j)} k_3(t,s)\,(s-t^{p-1}_0)^l\,ds \\
& - \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m-1} \binom{i}{l} \frac{\hat{u}^{(r)}_{0,p-1}(t^{p-1}_0)}{(r-l)!} [\partial_1^{(j-1-i)} k_2(t,t-\tau_2)]^{(i-l)}(t)(t-\tau_2-t^{p-1}_0)^{r-l} \\
& + \sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{0,p-1}(t^{p-1}_0)}{l!} \int_{t-\tau_2}^{t^{p-1}_1} \partial_1^{(j)} k_2(t,s)\,(s-t^{p-1}_0)^l\,ds \\
& + \sum_{d=1}^{N-N_1-1}\sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{d,p-1}(t^{p-1}_d)}{l!} \int_{t^{p-1}_d}^{t^{p-1}_{d+1}} \partial_1^{(j)} k_2(t,s)\,(s-t^{p-1}_d)^l\,ds \\
& + \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m-1} \binom{i}{l} \frac{\hat{u}^{(r)}_{N-N_1,p-1}(t^{p-1}_{N-N_1})}{(r-l)!} [\partial_1^{(j-1-i)} k_2(t,t-\tau_1)]^{(i-l)}(t)(t-\tau_1-t^{p-1}_{N-N_1})^{r-l} \\
& + \sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{N-N_1,p-1}(t^{p-1}_{N-N_1})}{l!} \int_{t^p_0-\tau_1}^{t-\tau_1} \partial_1^{(j)} k_2(t,s)\,(s-t^{p-1}_{N-N_1})^l\,ds \\
& - \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m-1} \binom{i}{l} \frac{\hat{u}^{(r)}_{N-N_1,p-1}(t^{p-1}_{N-N_1})}{(r-l)!} [\partial_1^{(j-1-i)} k_1(t,t-\tau_1)]^{(i-l)}(t)(t-\tau_1-t^{p-1}_{N-N_1})^{r-l} \\
& + \sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{N-N_1,p}(t^{p}_{N-N_1})}{l!} \int_{t-\tau_1}^{t^{p-1}_{N-N_1+1}} \partial_1^{(j)} k_1(t,s)\,(s-t^{p}_{N-N_1})^l\,ds \\
& + \sum_{d=N-N_1+1}^{N-1}\sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{d,p-1}(t^{p-1}_d)}{l!} \int_{t^{p-1}_d}^{t^{p-1}_{d+1}} \partial_1^{(j)} k_1(t,s)\,(s-t^{p-1}_d)^l\,ds \\
& + \sum_{i=0}^{j-1}\sum_{l=0}^{i} \binom{i}{l} [\partial_1^{(j-1-i)} k_1(t,t)]^{(i-l)}(t)\,\hat{u}^{(l)}_{0,p}(t) \\
& + \int_{t^p_0}^{t} \partial_1^{(j)} k_1(t,s)\,\hat{u}_{0,p}(s)\,ds.
\end{aligned}
\tag{3.10}
$$

Hence, we obtain

$$\begin{aligned}
\hat{u}_{0,p}^{(j+1)}(t_0^p) &= g^{(j)}(t_0^p) + \sum_{l=0}^{j} \binom{j}{l} \hat{u}_{N-N_1,p}^{(l)}(t_{N-N_1}^p)\Big(A(t)\Big)^{(j-l)}(t_0^p) \\
&\quad + \sum_{l=0}^{j} \binom{j}{l} \hat{u}_{0,p-1}^{(l)}(t_0^{p-1})\Big(B(t)\Big)^{(j-l)}(t_0^p) \\
&\quad + \left(\int_0^{\tau_2} k_3(t,s)\,\Phi(s)\,ds\right)^{(j)}(t_0^p) \\
&\quad + \sum_{i=0}^{p-2}\sum_{d=0}^{N-1}\sum_{l=0}^{m-1} \frac{\hat{u}_{d,i}^{(l)}(t_d^i)}{l!} \int_{t_d^i}^{t_{d+1}^i} \partial_1^{(j)} k_3(t_0^p,s)\,(s-t_d^i)^l\,ds \\
&\quad + \sum_{i=0}^{j-1}\sum_{l=0}^{i} \binom{i}{l}\, \hat{u}_{0,p-1}^{(l)}(t_0^{p-1})\Big[\partial_1^{(j-1-i)} k_3(t,t-\tau_2)\Big]^{(i-l)}(t_0^p) \\
&\quad - \sum_{i=0}^{j-1}\sum_{l=0}^{i} \binom{i}{l}\, \hat{u}_{0,p-1}^{(l)}(t_0^{p-1})\Big[\partial_1^{(j-1-i)} k_2(t,t-\tau_2)\Big]^{(i-l)}(t_0^p) \\
&\quad + \sum_{l=0}^{m-1} \frac{\hat{u}_{0,p-1}^{(l)}(t_0^{p-1})}{l!} \int_{t_0^p-\tau_2}^{t_1^{p-1}} \partial_1^{(j)} k_2(t_0^p,s)\,(s-t_0^{p-1})^l\,ds \\
&\quad + \sum_{d=1}^{N-N_1-1}\sum_{l=0}^{m-1} \frac{\hat{u}_{d,p-1}^{(l)}(t_d^{p-1})}{l!} \int_{t_d^{p-1}}^{t_{d+1}^{p-1}} \partial_1^{(j)} k_2(t_0^p,s)\,(s-t_d^{p-1})^l\,ds \\
&\quad + \sum_{i=0}^{j-1}\sum_{l=0}^{i} \binom{i}{l}\, \hat{u}_{N-N_1,p-1}^{(l)}(t_{N-N_1}^{p-1})\Big[\partial_1^{(j-1-i)} k_2(t,t-\tau_1)\Big]^{(i-l)}(t_0^p) \\
&\quad - \sum_{i=0}^{j-1}\sum_{l=0}^{i} \binom{i}{l}\, \hat{u}_{N-N_1,p-1}^{(l)}(t_{N-N_1}^{p-1})\Big[\partial_1^{(j-1-i)} k_1(t,t-\tau_1)\Big]^{(i-l)}(t_0^p) \\
&\quad + \sum_{l=0}^{m-1} \frac{\hat{u}_{N-N_1,p}^{(l)}(t_{N-N_1}^p)}{l!} \int_{t_0^p-\tau_1}^{t_{N-N_1+1}^{p-1}} \partial_1^{(j)} k_1(t_0^p,s)\,(s-t_{N-N_1}^p)^l\,ds \\
&\quad + \sum_{d=N-N_1+1}^{N-1}\sum_{l=0}^{m-1} \frac{\hat{u}_{d,p-1}^{(l)}(t_d^{p-1})}{l!} \int_{t_d^{p-1}}^{t_{d+1}^{p-1}} \partial_1^{(j)} k_1(t_0^p,s)\,(s-t_d^{p-1})^l\,ds \\
&\quad + \sum_{i=0}^{j-1}\sum_{l=0}^{i} \binom{i}{l} \Big[\partial_1^{(j-1-i)} k_1(t,t)\Big]^{(i-l)}(t_0^p)\,\hat{u}_{0,p}^{(l)}(t_0^p).
\end{aligned}$$

Finally, in order to approximate x by u_n^p $\Big(n \in \{0,1,\ldots,N-1\},\ p \in \{1,2,\ldots,r-1\}\Big)$ on the interval σ_n^p, the function x must be approximated by u_k^j $\Big(0 \le k \le N-1,\ 0 \le j \le p\Big)$ on each interval σ_k^j. Accordingly, we define

$$u_n^p(t) = \sum_{j=0}^{m} \frac{\hat{u}_{n,p}^{(j)}(t_n^p)}{j!}\,(t-t_n^p)^j, \qquad t \in \sigma_n^p, \tag{3.11}$$

where $\hat{u}_{n,p}$ denotes the exact solution of the following integral equations.

For $t \in \sigma_n^p$ and $n \in \{1, 2, \ldots, N_1 - 1\}$, we have

$$\begin{aligned}
\hat{u}'_{n,p}(t) = \; & g(t) + A(t)\, u^{p-1}_{N-N_1+n}(t-\tau_1) + B(t)\, u^{p-1}_n(t-\tau_2) \\
& + \int_0^{\tau_2} k_3(t,s)\,\phi(s)\,ds + \sum_{i=0}^{p-2}\sum_{d=0}^{N-1} \int_{t^i_d}^{t^i_{d+1}} k_3(t,s)\, u^i_d(s)\,ds \\
& + \sum_{i=0}^{n-1} \int_{t^{p-1}_i}^{t^{p-1}_{i+1}} k_3(t,s)\, u^{p-1}_i(s)\,ds + \int_{t^{p-1}_n}^{t-\tau_2} k_3(t,s)\, u^{p-1}_n(s)\,ds \\
& + \int_{t-\tau_2}^{t^{p-1}_{n+1}} k_2(t,s)\, u^{p-1}_n(s)\,ds + \sum_{d=n+1}^{N-N_1+n-1} \int_{t^{p-1}_d}^{t^{p-1}_{d+1}} k_2(t,s)\, u^{p-1}_d(s)\,ds \\
& + \int_{t^p_n-\tau_1}^{t-\tau_1} k_2(t,s)\, u^{p-1}_{N-N_1+n}(s)\,ds + \int_{t-\tau_1}^{t^p_{n+1}-\tau_1} k_1(t,s)\, u^{p-1}_{N-N_1+n}(s)\,ds \\
& + \sum_{i=N-N_1+n+1}^{N-1} \int_{t^{p-1}_i}^{t^{p-1}_{i+1}} k_1(t,s)\, u^{p-1}_i(s)\,ds + \sum_{i=0}^{n-1} \int_{t^p_i}^{t^p_{i+1}} k_1(t,s)\, u^p_i(s)\,ds \\
& + \int_{t^p_n}^{t} k_1(t,s)\,\hat{u}_{n,p}(s)\,ds.
\end{aligned} \tag{3.12}$$

For $t \in \sigma_n^p$ and $n \in \{N_1, N_1+1, \ldots, N-1\}$, we obtain

$$\begin{aligned}
\hat{u}'_{n,p}(t) = \; & g(t) + A(t)\, u^{p}_{n-N_1}(t-\tau_1) + B(t)\, u^{p-1}_n(t-\tau_2) \\
& + \int_0^{\tau_2} k_3(t,s)\,\phi(s)\,ds + \sum_{i=0}^{p-2}\sum_{d=0}^{N-1} \int_{t^i_d}^{t^i_{d+1}} k_3(t,s)\, u^i_d(s)\,ds \\
& + \sum_{i=0}^{n-1} \int_{t^{p-1}_i}^{t^{p-1}_{i+1}} k_3(t,s)\, u^{p-1}_i(s)\,ds + \int_{t^{p-1}_n}^{t-\tau_2} k_3(t,s)\, u^{p-1}_n(s)\,ds \\
& + \int_{t-\tau_2}^{t^{p-1}_{n+1}} k_2(t,s)\, u^{p-1}_n(s)\,ds + \sum_{d=n+1}^{N-1} \int_{t^{p-1}_d}^{t^{p-1}_{d+1}} k_2(t,s)\, u^{p-1}_d(s)\,ds \\
& + \sum_{i=0}^{n-N_1-1} \int_{t^p_i}^{t^p_{i+1}} k_2(t,s)\, u^p_i(s)\,ds + \int_{t^p_n-\tau_1}^{t-\tau_1} k_2(t,s)\, u^p_{n-N_1}(s)\,ds \\
& + \int_{t-\tau_1}^{t^p_{n+1}-\tau_1} k_1(t,s)\, u^p_{n-N_1}(s)\,ds + \sum_{i=n+1-N_1}^{n-1} \int_{t^p_i}^{t^p_{i+1}} k_1(t,s)\, u^p_i(s)\,ds \\
& + \int_{t^p_n}^{t} k_1(t,s)\,\hat{u}_{n,p}(s)\,ds.
\end{aligned} \tag{3.13}$$

The coefficients $\hat{u}^{(j)}_{n,p}(t^p_n)$ are given by the following formula. For $t \in \sigma_n^p$ and $n \in$

$\{1, 2, \ldots, N_1 - 1\}$, we have

$$
\begin{aligned}
\hat{u}_{n,p}{}^{(j+1)}(t) =\; & g^{(j)}(t) + \Big(A(t)\, u^{p-1}_{N-N_1+n}(t-\tau_1)\Big)^{(j)} + \Big(B(t)\, u^{p-1}_{n}(t-\tau_2)\Big)^{(j)} \\
& + \left(\int_0^{\tau_2} k_3(t,s)\,\Phi(s)\,ds\right)^{(j)}(t) \\
& + \sum_{i=0}^{p-2}\sum_{d=0}^{N-1}\sum_{l=0}^{m} \frac{\hat{u}^{(l)}_{d,i}(t^i_d)}{l!} \int_{t^i_d}^{t^i_{d+1}} \partial_1^{(j)} k_3(t,s)\,(s-t^i_d)^l\,ds \\
& + \sum_{i=0}^{n-1}\sum_{l=0}^{m} \frac{\hat{u}^{(l)}_{i,p-1}(t^{p-1}_i)}{l!} \int_{t^{p-1}_i}^{t^{p-1}_{i+1}} \partial_1^{(j)} k_3(t,s)\,(s-t^{p-1}_i)^l\,ds \\
& + \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m} \binom{i}{l} \frac{\hat{u}^{(r)}_{n,p-1}(t^{p-1}_n)}{(r-l)!} \Big[\partial_1^{(j-1-i)} k_3(t,t-\tau_2)\Big]^{(i-l)}(t)(t-\tau_2-t^{p-1}_n)^{r-l} \\
& + \sum_{l=0}^{m} \frac{\hat{u}^{(l)}_{n,p-1}(t^{p-1}_n)}{l!} \int_{t^{p-1}_n}^{t-\tau_2} \partial_1^{(j)} k_3(t,s)\,(s-t^{p-1}_n)^l\,ds \\
& - \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m} \binom{i}{l} \frac{\hat{u}^{(r)}_{n,p-1}(t^{p-1}_n)}{(r-l)!} \Big[\partial_1^{(j-1-i)} k_2(t,t-\tau_2)\Big]^{(i-l)}(t)(t-\tau_2-t^{p-1}_n)^{r-l} \\
& + \sum_{l=0}^{m} \frac{\hat{u}^{(l)}_{n,p-1}(t^{p-1}_n)}{l!} \int_{t-\tau_2}^{t^{p-1}_{n+1}} \partial_1^{(j)} k_2(t,s)\,(s-t^{p-1}_n)^l\,ds \\
& + \sum_{d=n+1}^{N-N_1+n-1}\;\sum_{l=0}^{m} \frac{\hat{u}^{(l)}_{d,p-1}(t^{p-1}_d)}{l!} \int_{t^{p-1}_d}^{t^{p-1}_{d+1}} \partial_1^{(j)} k_2(t,s)\,(s-t^{p-1}_d)^l\,ds \\
& + \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m} \binom{i}{l} \frac{\hat{u}^{(r)}_{N-N_1+n,p-1}(t^{p-1}_{N-N_1+n})}{(r-l)!} \Big[\partial_1^{(j-1-i)} k_2(t,t-\tau_1)\Big]^{(i-l)}(t)(t-\tau_1-t^{p-1}_{N-N_1+n})^{r-l} \\
& + \sum_{l=0}^{m} \frac{\hat{u}^{(l)}_{N-N_1+n,p-1}(t^{p-1}_{N-N_1+n})}{l!} \int_{t^{p}_n-\tau_1}^{t-\tau_1} \partial_1^{(j)} k_2(t,s)\,(s-t^{p-1}_{N-N_1+n})^l\,ds \\
& - \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m} \binom{i}{l} \frac{\hat{u}^{(r)}_{N-N_1+n,p-1}(t^{p-1}_{N-N_1+n})}{(r-l)!} \Big[\partial_1^{(j-1-i)} k_1(t,t-\tau_1)\Big]^{(i-l)}(t)(t-\tau_1-t^{p-1}_{N-N_1+n})^{r-l} \\
& + \sum_{l=0}^{m} \frac{\hat{u}^{(l)}_{N-N_1+n,p-1}(t^{p-1}_{N-N_1+n})}{l!} \int_{t-\tau_1}^{t^{p}_{n+1}-\tau_1} \partial_1^{(j)} k_1(t,s)\,(s-t^{p-1}_{N-N_1+n})^l\,ds \\
& + \sum_{i=N-N_1+n+1}^{N-1}\;\sum_{l=0}^{m} \frac{\hat{u}^{(l)}_{i,p-1}(t^{p-1}_i)}{l!} \int_{t^{p-1}_i}^{t^{p-1}_{i+1}} \partial_1^{(j)} k_1(t,s)\,(s-t^{p-1}_i)^l\,ds \\
& + \sum_{i=0}^{n-1}\sum_{l=0}^{m} \frac{\hat{u}^{(l)}_{i,p}(t^{p}_i)}{l!} \int_{t^{p}_i}^{t^{p}_{i+1}} \partial_1^{(j)} k_1(t,s)\,(s-t^{p}_i)^l\,ds \\
& + \sum_{i=0}^{j-1}\sum_{l=0}^{i} \binom{i}{l} \Big[\partial_1^{(j-1-i)} k_1(t,t)\Big]^{(i-l)}(t)\,\hat{u}^{(l)}_{n,p}(t) \\
& + \int_{t^p_n}^{t} \partial_1^{(j)} k_1(t,s)\,\hat{u}_{n,p}(s)\,ds.
\end{aligned}
\tag{3.14}
$$

Hence, we obtain

$$
\begin{aligned}
\hat{u}^{(j+1)}_{n,p}(t^p_n) = g^{(j)}(t^p_n) &+ \sum_{l=0}^{j} \binom{j}{l} \hat{u}^{(l)}_{N-N_1+n,p-1}(t^{p-1}_{N-N_1+n}) \Big(A(t)\Big)^{(j-l)}(t^p_n) \\
&+ \sum_{l=0}^{j} \binom{j}{l} \hat{u}^{(l)}_{n,p-1}(t^{p-1}_n) \Big(B(t)\Big)^{(j-l)}(t^p_n) \\
&+ \Big(\int_0^{\tau_2} k_3(t^p_n, s)\, \Phi(s)\, ds\Big)^{(j)} (t^p_n) \\
&+ \sum_{i=0}^{p-2} \sum_{d=0}^{N-1} \sum_{l=0}^{m} \frac{\hat{u}^{(l)}_{d,i}(t^i_d)}{l!} \int_{t^i_d}^{t^i_{d+1}} \partial_1^{(j)} k_3(t^p_n, s)\, (s - t^i_d)^l\, ds \\
&+ \sum_{i=0}^{n-1} \sum_{l=0}^{m} \frac{\hat{u}^{(l)}_{i,p-1}(t^{p-1}_i)}{l!} \int_{t^{p-1}_i}^{t^{p-1}_{i+1}} \partial_1^{(j)} k_3(t^p_n, s)\, (s - t^{p-1}_i)^l\, ds \\
&+ \sum_{i=0}^{j-1} \sum_{l=0}^{i} \binom{i}{l} \hat{u}^{(l)}_{n,p-1}(t^{p-1}_n) \Big[\partial_1^{(j-1-i)} k_3(t, t-\tau_2)\Big]^{(i-l)}(t^p_n) \\
&- \sum_{i=0}^{j-1} \sum_{l=0}^{i} \binom{i}{l} \hat{u}^{(l)}_{n,p-1}(t^{p-1}_n) \Big[\partial_1^{(j-1-i)} k_2(t, t-\tau_2)\Big]^{(i-l)}(t^p_n) \\
&+ \sum_{l=0}^{m} \frac{\hat{u}^{(l)}_{n,p-1}(t^{p-1}_n)}{l!} \int_{t^p_n-\tau_2}^{t^{p-1}_{n+1}} \partial_1^{(j)} k_2(t^p_n, s)\, (s - t^{p-1}_n)^l\, ds \\
&+ \sum_{d=n+1}^{N-N_1+n-1} \sum_{l=0}^{m} \frac{\hat{u}^{(l)}_{d,p-1}(t^{p-1}_d)}{l!} \int_{t^{p-1}_d}^{t^{p-1}_{d+1}} \partial_1^{(j)} k_2(t^p_n, s)\, (s - t^{p-1}_d)^l\, ds \\
&+ \sum_{i=0}^{j-1} \sum_{l=0}^{i} \binom{i}{l} \hat{u}^{(l)}_{N-N_1+n,p-1}(t^{p-1}_{N-N_1+n}) \Big[\partial_1^{(j-1-i)} k_2(t, t-\tau_1)\Big]^{(i-l)}(t^p_n) \\
&- \sum_{i=0}^{j-1} \sum_{l=0}^{i} \binom{i}{l} \hat{u}^{(l)}_{N-N_1+n,p-1}(t^{p-1}_{N-N_1+n}) \Big[\partial_1^{(j-1-i)} k_1(t, t-\tau_1)\Big]^{(i-l)}(t^p_n) \\
&+ \sum_{l=0}^{m} \frac{\hat{u}^{(l)}_{N-N_1+n,p-1}(t^{p-1}_{N-N_1+n})}{l!} \int_{t^p_n-\tau_1}^{t^p_{n+1}-\tau_1} \partial_1^{(j)} k_1(t^p_n, s)\, (s - t^{p-1}_{N-N_1+n})^l\, ds \\
&+ \sum_{i=N-N_1+n+1}^{N-1} \sum_{l=0}^{m} \frac{\hat{u}^{(l)}_{i,p-1}(t^{p-1}_i)}{l!} \int_{t^{p-1}_i}^{t^{p-1}_{i+1}} \partial_1^{(j)} k_1(t^p_n, s)\, (s - t^{p-1}_i)^l\, ds \\
&+ \sum_{i=0}^{n-1} \sum_{l=0}^{m} \frac{\hat{u}^{(l)}_{i,p}(t^p_i)}{l!} \int_{t^p_i}^{t^p_{i+1}} \partial_1^{(j)} k_1(t^p_n, s)\, (s - t^p_i)^l\, ds \\
&+ \sum_{i=0}^{j-1} \sum_{l=0}^{i} \binom{i}{l} \Big[\partial_1^{(j-1-i)} k_1(t, t)\Big]^{(i-l)}(t^p_n)\, \hat{u}^{(l)}_{n,p}(t^p_n).
\end{aligned}
$$

and for $t \in \sigma_n^p$, $n \in \{N_1, N_1+1, \dots, N-1\}$,

$$
\begin{aligned}
\hat{u}_{n,p}^{(j+1)}(t) = \; & g^{(j)}(t) + \Big(A(t)\, u_{n-N_1}^{p}(t-\tau_1)\Big)^{(j)} + \Big(B(t)\, u_n^{p-1}(t-\tau_2)\Big)^{(j)} \\
& + \left(\int_0^{\tau_2} k_3(t,s)\,\Phi(s)\,ds\right)^{(j)}(t) \\
& + \sum_{i=0}^{p-2}\sum_{d=0}^{N-1}\sum_{l=0}^{m} \frac{\hat{u}_{d,i}^{(l)}(t_d^i)}{l!} \int_{t_d^i}^{t_{d+1}^i} \partial_1^{(j)} k_3(t,s)\,(s-t_d^i)^l\,ds \\
& + \sum_{i=0}^{n-1}\sum_{l=0}^{m} \frac{\hat{u}_{i,p-1}^{(l)}(t_i^{p-1})}{l!} \int_{t_i^{p-1}}^{t_{i+1}^{p-1}} \partial_1^{(j)} k_3(t,s)\,(s-t_i^{p-1})^l\,ds \\
& + \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m} \binom{i}{l} \frac{\hat{u}_{n,p-1}^{(r)}(t_n^{p-1})}{(r-l)!} \Big[\partial_1^{(j-1-i)} k_3(t,t-\tau_2)\Big]^{(i-l)}(t)\,(t-\tau_2-t_n^{p-1})^{r-l} \\
& + \sum_{l=0}^{m} \frac{\hat{u}_{n,p-1}^{(l)}(t_n^{p-1})}{l!} \int_{t_n^{p-1}}^{t-\tau_2} \partial_1^{(j)} k_3(t,s)\,(s-t_n^{p-1})^l\,ds \\
& - \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m} \binom{i}{l} \frac{\hat{u}_{n,p-1}^{(r)}(t_n^{p-1})}{(r-l)!} \Big[\partial_1^{(j-1-i)} k_2(t,t-\tau_2)\Big]^{(i-l)}(t)\,(t-\tau_2-t_n^{p-1})^{r-l} \\
& + \sum_{l=0}^{m} \frac{\hat{u}_{n,p-1}^{(l)}(t_n^{p-1})}{l!} \int_{t-\tau_2}^{t_{n+1}^{p-1}} \partial_1^{(j)} k_2(t,s)\,(s-t_n^{p-1})^l\,ds \\
& + \sum_{d=n+1}^{N-1}\sum_{l=0}^{m} \frac{\hat{u}_{d,p-1}^{(l)}(t_d^{p-1})}{l!} \int_{t_d^{p-1}}^{t_{d+1}^{p-1}} \partial_1^{(j)} k_2(t,s)\,(s-t_d^{p-1})^l\,ds \\
& + \sum_{i=0}^{n-N_1-1}\sum_{l=0}^{m} \frac{\hat{u}_{i,p}^{(l)}(t_i^{p})}{l!} \int_{t_i^{p}}^{t_{i+1}^{p}} \partial_1^{(j)} k_2(t,s)\,(s-t_i^{p})^l\,ds \\
& + \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m} \binom{i}{l} \frac{\hat{u}_{n-N_1,p}^{(r)}(t_{n-N_1}^{p})}{(r-l)!} \Big[\partial_1^{(j-1-i)} k_2(t,t-\tau_1)\Big]^{(i-l)}(t)\,(t-\tau_1-t_{n-N_1}^{p})^{r-l} \\
& + \sum_{l=0}^{m} \frac{\hat{u}_{n-N_1,p}^{(l)}(t_{n-N_1}^{p})}{l!} \int_{t_n^{p}-\tau_1}^{t-\tau_1} \partial_1^{(j)} k_2(t,s)\,(s-t_{n-N_1}^{p})^l\,ds \\
& - \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m} \binom{i}{l} \frac{\hat{u}_{n-N_1,p}^{(r)}(t_{n-N_1}^{p})}{(r-l)!} \Big[\partial_1^{(j-1-i)} k_1(t,t-\tau_1)\Big]^{(i-l)}(t)\,(t-\tau_1-t_{n-N_1}^{p})^{r-l} \\
& + \sum_{l=0}^{m} \frac{\hat{u}_{n-N_1,p}^{(l)}(t_{n-N_1}^{p})}{l!} \int_{t-\tau_1}^{t_{n+1}^{p}-\tau_1} \partial_1^{(j)} k_1(t,s)\,(s-t_{n-N_1}^{p})^l\,ds \\
& + \sum_{i=n+1-N_1}^{n-1}\sum_{l=0}^{m} \frac{\hat{u}_{i,p}^{(l)}(t_i^{p})}{l!} \int_{t_i^{p}}^{t_{i+1}^{p}} \partial_1^{(j)} k_1(t,s)\,(s-t_i^{p})^l\,ds \\
& + \sum_{i=0}^{j-1}\sum_{l=0}^{i} \binom{i}{l} \Big[\partial_1^{(j-1-i)} k_1(t,t)\Big]^{(i-l)}(t)\,\hat{u}_{n,p}^{(l)}(t) \\
& + \int_{t_n^p}^{t} \partial_1^{(j)} k_1(t,s)\,\hat{u}_{n,p}(s)\,ds.
\end{aligned}
\tag{3.15}
$$

Hence,

$$
\begin{aligned}
\hat{u}^{(j+1)}_{n,p}(t^p_n) = g^{(j)}(t^p_n) &+ \sum_{l=0}^{j} \binom{j}{l} \hat{u}^{(l)}_{n-N_1,p}(t^p_{n-N_1}) \Big(A(t)\Big)^{(j-l)}(t^p_n) \\
&+ \sum_{l=0}^{j} \binom{j}{l} \hat{u}^{(l)}_{n,p-1}(t^{p-1}_n) \Big(B(t)\Big)^{(j-l)}(t^p_n) \\
&+ \left(\int_0^{\tau_2} k_3(t,s)\,\Phi(s)\,ds\right)^{(j)}(t^p_n) \\
&+ \sum_{i=0}^{p-2} \sum_{d=0}^{N-1} \sum_{l=0}^{m} \frac{\hat{u}^{(l)}_{d,i}(t^i_d)}{l!} \int_{t^i_d}^{t^i_{d+1}} \partial_1^{(j)} k_3(t^p_n,s)\,(s-t^i_d)^l\,ds \\
&+ \sum_{i=0}^{n-1} \sum_{l=0}^{m} \frac{\hat{u}^{(l)}_{i,p-1}(t^{p-1}_i)}{l!} \int_{t^{p-1}_i}^{t^{p-1}_{i+1}} \partial_1^{(j)} k_3(t^p_n,s)\,(s-t^{p-1}_i)^l\,ds \\
&+ \sum_{i=0}^{j-1} \sum_{l=0}^{i} \binom{i}{l} \hat{u}^{(l)}_{n,p-1}(t^{p-1}_n) \Big[\partial_1^{(j-1-i)} k_3(t,t-\tau_2)\Big]^{(i-l)}(t^p_n) \\
&- \sum_{i=0}^{j-1} \sum_{l=0}^{i} \binom{i}{l} \hat{u}^{(l)}_{n,p-1}(t^{p-1}_n) \Big[\partial_1^{(j-1-i)} k_2(t,t-\tau_2)\Big]^{(i-l)}(t^p_n) \\
&+ \sum_{l=0}^{m} \frac{\hat{u}^{(l)}_{n,p-1}(t^{p-1}_n)}{l!} \int_{t^p_n-\tau_2}^{t^{p-1}_{n+1}} \partial_1^{(j)} k_2(t^p_n,s)\,(s-t^{p-1}_n)^l\,ds \\
&+ \sum_{d=n+1}^{N-1} \sum_{l=0}^{m} \frac{\hat{u}^{(l)}_{d,p-1}(t^{p-1}_d)}{l!} \int_{t^{p-1}_d}^{t^{p-1}_{d+1}} \partial_1^{(j)} k_2(t^p_n,s)\,(s-t^{p-1}_d)^l\,ds \\
&+ \sum_{i=0}^{n-N_1-1} \sum_{l=0}^{m} \frac{\hat{u}^{(l)}_{i,p}(t^p_i)}{l!} \int_{t^p_i}^{t^p_{i+1}} \partial_1^{(j)} k_2(t^p_n,s)\,(s-t^p_i)^l\,ds \\
&+ \sum_{i=0}^{j-1} \sum_{l=0}^{i} \binom{i}{l} \hat{u}^{(l)}_{n-N_1,p}(t^p_{n-N_1}) \Big[\partial_1^{(j-1-i)} k_2(t,t-\tau_1)\Big]^{(i-l)}(t^p_n) \\
&- \sum_{i=0}^{j-1} \sum_{l=0}^{i} \binom{i}{l} \hat{u}^{(l)}_{n-N_1,p}(t^p_{n-N_1}) \Big[\partial_1^{(j-1-i)} k_1(t,t-\tau_1)\Big]^{(i-l)}(t^p_n) \\
&+ \sum_{l=0}^{m} \frac{\hat{u}^{(l)}_{n-N_1,p}(t^p_{n-N_1})}{l!} \int_{t^p_n-\tau_1}^{t^p_{n+1}-\tau_1} \partial_1^{(j)} k_1(t^p_n,s)\,(s-t^p_{n-N_1})^l\,ds \\
&+ \sum_{i=n+1-N_1}^{n-1} \sum_{l=0}^{m} \frac{\hat{u}^{(l)}_{i,p}(t^p_i)}{l!} \int_{t^p_i}^{t^p_{i+1}} \partial_1^{(j)} k_1(t^p_n,s)\,(s-t^p_i)^l\,ds \\
&+ \sum_{i=0}^{j-1} \sum_{l=0}^{i} \binom{i}{l} \Big[\partial_1^{(j-1-i)} k_1(t,t)\Big]^{(i-l)}(t^p_n)\,\hat{u}^{(l)}_{n,p}(t^p_n).
\end{aligned}
\tag{3.16}
$$

where $\hat{u}_{n,p}(t^p_n) = u^p_{n-1}(t^p_n)$.

3.3 Convergence Analysis of the Proposed Scheme

In this section, we present the main analytical results required to establish the convergence of the proposed numerical method. The following lemmas provide uniform bounds on the approximate solutions and their derivatives, which play a fundamental role in the subsequent analysis.

Lemma 3.3.1 *Let g, A, B, and the kernels k_1, k_2, and k_3 be $m+1$ times continuously differentiable on their respective domains. Then there exists a positive constant $\alpha(m)$, depending only on m, such that for all*

$$n = 0, 1, \ldots, N-1, \qquad p = 0, 1, \ldots, r-1, \qquad j = 0, 1, \ldots, m+1,$$

the following uniform estimate holds:

$$\left\| \hat{u}_{n,p}^{(j)} \right\|_{L^\infty(\sigma_n^p)} \leq \alpha(m),$$

where the initial function satisfies $\hat{u}_{0,0}(t) = x(t)$ for all $t \in \sigma_0^0$.

Proof. The proof is divided into two main steps.

Claim 1 Let

$$a_n^j = \left\| \hat{u}_{n,0}^{(j)} \right\|_{L^\infty(\sigma_n^0)}, \qquad j = 0, 1, \ldots, m+1.$$

Then, for $n = 0$, we obtain

$$a_0^j \leq \max_{0 \leq j \leq m+1} \left\{ \left\| x^{(j)} \right\|_{L^\infty(\sigma_0^0)} \right\} =: \alpha_1(m). \tag{3.17}$$

From equation (3.6), it follows that for all $n = 1, 2, \ldots, N_1 - 1$ and $j = 1, 2, \ldots, m+1$,

$$a_n^{j+1} \leq c_1 + m + 1 b_1^1 \sum_{l=0}^{j-1} a_n^l + b_1^2 a_n^0 + h d_1^1 \sum_{i=0}^{n-1} \sum_{l=0}^{m+1} a_i^l,$$

and for all $n = N_1, N_1+1, \ldots, N-1$ and $j = 1, 2, \ldots, m+1$,

$$a_n^{j+1} \leq c_1 + (m+1)b_1^1 \sum_{l=0}^{j-1} a_n^l + 2(m+1)^2 d_1^2 \sum_{l=0}^{(m+1)} a_{n-N_1}^l + hd_1^1 \sum_{i=0}^{n-1} \sum_{l=0}^{(m+1)} a_i^l.$$

The constants c_1, b_1^1, b_1^2, d_1^1, and d_1^2 are positive and independent of N. Therefore, for all $n = 1, 2, \ldots, N_1 - 1$, we may write

$$a_n^{j+1} \leq c_1 + ((m+1)b_1^1 + b_1^2) \sum_{l=0}^{j-1} a_n^l + hd_1^1 \sum_{i=0}^{n-1} \sum_{l=0}^{(m+1)} a_i^l,$$

and for all $n = N_1, N_1+1, \ldots, N-1$,

$$a_n^{j+1} \leq c_1 + (m+1)b_1^1 \sum_{l=0}^{j-1} a_n^l + 2(m+1)^2 d_1^2 \sum_{l=0}^{(m+1)} a_{n-N_1}^l + hd_1^1 \sum_{i=0}^{n-1} \sum_{l=0}^{(m+1)} a_i^l.$$

Consequently, for all $j \geq 1$ and $n = 1, 2, \ldots, N-1$, there exist positive constants c, b_1, b_2, and d_1 such that

$$a_n^{j+1} \leq \ c + b_1 \sum_{k=0}^{j-1} a_n^k + b_2 \sum_{k=0}^{(m+1)} a_n^k + hd_1 \sum_{i=0}^{n-1} \sum_{l=0}^{(m+1)} a_i^l. \tag{3.18}$$

Moreover, for the case $j = 0$, it follows from (3.4) that for all $n = 0, 1, \ldots, N_1 - 1$,

$$a_n^1 \leq c_1 + b_1\, a_n^0 + hd_1^1 \sum_{i=0}^{n-1} \sum_{k=0}^{(m+1)} a_i^k. \tag{3.19}$$

Similarly, from (3.5), for all $n = N_1, N_1+1, \ldots, N-1$, we obtain

$$a_n^1 \leq c + b_1\, a_n^0 + b_1^1 \sum_{l=0}^{(m+1)} a_{n-N_1}^l + hd_1^1 \sum_{i=0}^{n-1} \sum_{l=0}^{(m+1)} a_i^l. \tag{3.20}$$

Next, we consider the sequence $\{a_n^j\}_{j=0}^{(m+1)}$. Combining the previous estimates, we deduce that for all $n = 1, 2, \ldots, N-1$ and $j = 0, 1, \ldots, (m+1)$, the following inequality holds:

$$a_n^j \le a_{n-1}^0 + c + b_1 \sum_{k=0}^{j-2} a_n^k + b_2 \sum_{k=0}^{m+1} a_n^k + hd_1 \sum_{i=0}^{n-1} \sum_{k=0}^{m+1} a_i^k. \tag{3.21}$$

where $c_1^1 = \max\{c, c_1\}$.

Using Lemma 1.7.2, we deduce that for all $j = 0, 1, \dots, (m+1)$ and $n = 1, 2, \dots, N-1$,

$$\begin{aligned}
a_n^{j+1} &\le c_1^1 + b_1 \sum_{k=0}^{j-1} a_n^k + b_2 \sum_{k=0}^{(m+1)} a_n^k + hd_1 \sum_{i=0}^{n-1} \sum_{l=0}^{(m+1)} a_i^l \\
&\le \left(c_1^1 + b_2 \sum_{k=0}^{(m+1)} a_n^k + hd_1 \sum_{i=0}^{n-1} \sum_{l=0}^{(m+1)} a_i^l \right) \exp\left(\sum_{k=1}^{j-1} b_1 \right) \\
&\le \underbrace{c_1^1 e^{(m+1)b_1}}_{=:c_2} + \underbrace{b_2 e^{(m+1)b_1}}_{=:b_2^2} \sum_{k=0}^{(m+1)} a_n^k + h \underbrace{d_1 e^{(m+1)b_1}}_{=:d_2} \sum_{i=0}^{n-1} \sum_{l=0}^{(m+1)} a_i^l \\
&\le c_2 + b_2^2 \sum_{k=0}^{(m+1)} a_n^k + hd_2 \sum_{i=0}^{n-1} \sum_{l=0}^{(m+1)} a_i^l.
\end{aligned} \tag{3.22}$$

Now, define the sequence

$$z_n := \sum_{l=0}^{(m+1)} a_n^l, \qquad n \ge 0.$$

Then, by inequality (3.22), we obtain

$$\begin{aligned}
z_n &\le \underbrace{mc_2}_{=:c_3^1} + \underbrace{mb_2^2}_{=:b_3} \sum_{k=0}^{(m+1)} a_n^k + h \underbrace{(m+1)d_2}_{=:d_3} \sum_{i=0}^{n-1} z_i \\
&\le c_3^1 + b_3 \sum_{k=0}^{(m+1)} a_n^k + hd_3 \sum_{i=0}^{n-1} a_i^0 + hd_3 \sum_{i=0}^{n-1} z_i.
\end{aligned} \tag{3.23}$$

Moreover, from (3.17), we obtain

$$z_0 \le (m+1)\, \alpha_1(m) =: c_3^2. \tag{3.24}$$

Let

$$c_3 := \max\{c_3^1,\, c_3^2\}.$$

Then, from (3.23) and (3.24), it follows that for all $n = 0, 1, \dots, N-1$,

$$z_n \leq c_3 + b_3 \sum_{i=0}^{n} a_i^0 + hd_3 \sum_{i=0}^{n-1} z_i.$$

Applying Lemma 1.7.3, we deduce that for all $n = 0, 1, \dots, N-1$,

$$\begin{aligned}
z_n &\leq c_3 + b_3 \sum_{i=0}^{n} a_i^0 + hd_3 \sum_{j=0}^{n-1} \left(c_3 + b_3 a_j^0 + hd_3 \sum_{i=0}^{j} a_i^0 \right) \exp\left(\sum_{k=0}^{n-1} hd_3 \right) \\
&\leq \underbrace{c_3\Big(1 + \tau_1 d_3\Big)}_{=:c_4} + b_3 a_n^0 + h \underbrace{\left(b_3 d_3 e^{\tau_1 d_3} + \tau_1 d_3^2 e^{\tau_1 d_3} \right)}_{=:d_4} \sum_{i=0}^{n-1} a_i^0 \qquad (3.25) \\
&\leq c_4 + b_3 a_n^0 + hd_4 \sum_{i=0}^{n-1} a_i^0.
\end{aligned}$$

On the other hand, by integrating equations (2.5) and (2.6) from t_0^0 to $t \in \sigma_n^0$, we obtain

$$\begin{aligned}
a_n^0 &\leq \left| u_{n-1}^0(t_n^0) \right| + hc_1 + hb_1 a_n^0 + h^2 d_1 \sum_{i=0}^{n-1} \sum_{k=0}^{(m+1)} a_i^k \\
&\leq \left| u_{n-1}^0(t_n^0) \right| + hc_1 + hb_1 a_n^0 + h^2 d_1 \left(\sum_{i=0}^{n-1} a_i^0 + \sum_{i=0}^{n-1} z_i \right) \\
&\leq \left| u_{n-1}^0(t_n^0) \right| + hc_1 + hb_1 a_n^0 + h^2 d_1 \sum_{i=0}^{n-1} a_i^0 + h^2 d_1 \sum_{i=0}^{n-1} z_i.
\end{aligned}$$

For $j = 1, \dots, (m+1)$, and using the relation $\hat{u}_{n,0}(t_n^0) = u_{n-1}^0(t_n^0)$ for $n \geq 0$, we obtain

$$u_{n-1}^0(t_n^0) \leq a_{n-1}^0 + h \sum_{j=1}^{(m+1)} a_{n-1}^j \leq a_{n-1}^0 + hz_{n-1}.$$

Consequently,

$$a_n^0 \leq a_{n-1}^0 + hz_{n-1} + hc_1 + hb_1 a_n^0 + h^2 d_1 \sum_{i=0}^{n-1} a_i^0 + h^2 d_1 \sum_{i=0}^{n-1} z_i.$$

Using estimate (3.25), we deduce

$$\begin{aligned} a_n^0 \le a_{n-1}^0 &+ h\Big(c_4 + b_4 a_{n-1}^0 + hd_4 \sum_{i=0}^{n-1} a_i^0\Big) + hc_1 + hb_1 a_n^0 \\ &+ h^2 d_1 \sum_{i=0}^{n-1} a_i^0 + h^2 d_1 \sum_{i=0}^{n-1}\Big(c_4 + b_4 a_{n-1}^0 + hd_4 \sum_{k=0}^{n-1} a_k^0\Big). \end{aligned}$$

After regrouping terms, we obtain

$$a_n^0 \le (1+hb_4)a_{n-1}^0 + h\,\underbrace{(c_4 + c_1 + \tau d_1 c_4)}_{=:c_5} + hb_1 a_n^0 + h^2\,\underbrace{(d_4 + d_1 + d_1 b_4 + \tau d_1 d_4)}_{=:d_5}\sum_{i=0}^{n-1} a_i^0.$$

Hence,

$$(1-hb_1)a_n^0 \le (1+hb_4)a_{n-1}^0 + hc_5 + h^2 d_5 \sum_{i=0}^{n-1} a_i^0,$$

and therefore,

$$a_n^0 \le \frac{1+hb_4}{1-hb_1}\, a_{n-1}^0 + \frac{hc_5}{1-hb_1} + \frac{h^2 d_5}{1-hb_1}\sum_{i=0}^{n-1} a_i^0.$$

Applying Lemma 1.7.4, we conclude that for all $n = 0, 1, \ldots, N-1$,

$$\begin{aligned} a_n^0 &\le \frac{a_0^0}{R_2 - R_1}\,[(R_2 - 1)R_2^n + (1 - R_1)R_1^n] + \frac{hc_5}{(R_2 - R_1)(1 - hb_1)}\,(R_2^n - R_1^n) \\ &\le \frac{\alpha_1(m)}{R_2 - R_1}\,[(R_2 - 1)R_2^n + (1 - R_1)R_1^n] + \frac{hc_5}{(R_2 - R_1)(1 - hb_1)}\,(R_2^n - R_1^n) \\ &\le \alpha_1^2(m), \qquad n = 0, 1, \ldots, N-1. \end{aligned}$$

Finally, it follows that

$$a_n^{j+1} \le z_n \le c_4 + b_4\alpha_1^2(m) + d_4\alpha_1^2(m) =: \alpha_1^3(m).$$

Hence, the first step of the proof is completed by defining

$$\alpha_1(m) := \max\Big\{\alpha_1^2(m),\, \alpha_1^3(m)\Big\}.$$

Claim 2 There exists a positive constant $\alpha(m)$ such that

$$\left\|\hat{u}_{n,p}^{(j)}\right\|_{L^\infty(\sigma_n^p)} \le \alpha(m), \qquad n = 0,1,\dots,N-1,\ p = 1,2,\dots,r-1,\ j = 0,1,\dots,(m+1).$$

Let

$$a_{n,p}^j := \left\|\hat{u}_{n,p}^{(j)}\right\|_{L^\infty(\sigma_n^p)}, \qquad \varepsilon_p := \max_{\substack{0\le i\le N-1\\ 0\le j\le (m+1)}} a_{i,p}^j, \qquad p = 1,\dots,r-1.$$

Proceeding as in Claim 1 and using equations (3.14) and (3.15), we obtain for all $n = 0,1,\dots,N-1$ and $j = 0,1,\dots,(m+1)$,

$$a_{n,p}^{j+1} \le c_5 + b_5 \sum_{i=0}^{p-1} \varepsilon_i + e_5 \sum_{l=0}^{j-1} a_{n,p}^l + d_5 h \sum_{i=0}^{n-1} \sum_{l=0}^{m} a_{i,p}^l, \tag{3.26}$$

where c_5, b_5, e_5, and d_5 are positive constants independent of n, p, and h.

Applying Lemma 1.7.2, it follows that for all $j = 0,1,\dots,(m+1)$,

$$\begin{aligned} a_{n,p}^{j+1} &\le \left(c_5 + b_5 \sum_{i=0}^{p-1} \varepsilon_i + d_5 h \sum_{i=0}^{n-1} \sum_{l=0}^{(m+1)} a_{i,p}^l\right) \exp\left(\sum_{k=0}^{j-1} e_5\right) \\ &\le c_6 + b_6 \sum_{i=0}^{p-1} \varepsilon_i + e_6 a_{n,p}^0 + d_6 h \sum_{i=0}^{n-1} \sum_{l=0}^{(m+1)} a_{i,p}^l, \end{aligned} \tag{3.27}$$

for suitable positive constants c_6, b_6, e_6, and d_6.

Next, define the sequence

$$y_n := \sum_{j=1}^{(m+1)} a_{n,p}^j, \qquad n = 0,1,\dots,N-1.$$

Then, by the previous inequality, the sequence $\{y_n\}$ satisfies, for all $n = 1,2,\dots,N-1$,

$$\begin{aligned} y_n &\le c_6 + b_6\, \varepsilon_{p-1} + e_6 a_{n,p}^0 + h d_6 \sum_{i=0}^{n-1} \Big(a_{i,p}^0 + \sum_{l=1}^{(m+1)} a_{i,p}^l\Big) \\ &\le c_6 + b_6\, \varepsilon_{p-1} + e_6 a_{n,p}^0 + h d_6 \sum_{i=0}^{n-1} a_{i,p}^0 + h d_6 \sum_{i=0}^{n-1} y_i. \end{aligned} \tag{3.28}$$

On the other hand, by integrating equations (2.8) and (2.9) from t_n^p to $t \in \sigma_n^p$, we obtain

$$\begin{aligned} a_{n,p}^0 &\leq \left| u_{n-1}^p(t_n^p) \right| + hc_7 + hb_7\varepsilon_{p-1} + he_7 a_{n,p}^0 + d_7 h^2 \sum_{i=0}^{n-1} \sum_{l=0}^{(m+1)} a_{i,p}^l \\ &\leq \left| u_{n-1}^p(t_n^p) \right| + hc_7 + hb_7\varepsilon_{p-1} + he_7 a_{n,p}^0 + d_7 h^2 \sum_{i=0}^{n-1} a_{i,p}^0 + d_7 h^2 \sum_{i=0}^{n-1} y_i. \end{aligned}$$

This inequality holds for all $n = 1, 2, \ldots, N-1$, where c_7, b_7, e_7, and d_7 are positive constants.

Moreover, from equation (2.8), we obtain

$$\begin{aligned} \left| u_{n-1}^p(t_n^p) \right| &\leq a_{n-1,p}^0 + h \sum_{j=1}^{(m+1)} a_{n-1,p}^j \\ &\leq a_{n-1,p}^0 + h y_{n-1}. \end{aligned} \tag{3.29}$$

Substituting this estimate into the previous inequality yields

$$\begin{aligned} a_{n,p}^0 \leq a_{n-1,p}^0 + h y_{n-1} + hc_7 + hb_7\varepsilon_{p-1} + he_7 a_{n,p}^0 \\ + d_7 h^2 \sum_{i=0}^{n-1} a_{i,p}^0 + d_7 h^2 \sum_{i=0}^{n-1} y_i. \end{aligned}$$

which implies, by using (3.28), that

$$\begin{aligned}
(1-he_7)\,a^0_{n,p} \le{}& a^0_{n-1,p} + h\Big(c_6 + b_6\varepsilon_{p-1} + e_6 a^0_{n-1,p} + hd_6\sum_{i=0}^{n-1} a^0_{i,p}\Big)\\
&+ hc_7 + hb_7\varepsilon_{p-1} + d_7h^2\sum_{i=0}^{n-1} a^0_{i,p}\\
&+ d_7h^2\sum_{i=0}^{n-1}\Big(c_7 + b_7\varepsilon_{p-1} + e_7 a^0_{i,p} + hd_7\sum_{k=0}^{n-1} a^0_{k,p}\Big)\\
\le{}& (1+he_7)a^0_{n-1,p} + h\,\underbrace{(c_6 + c_7 + \tau d_7 c_6)}_{=:c_8}\\
&+ h\,\underbrace{(b_6 + b_7 + \tau d_7 b_6)}_{=:b_9}\,\varepsilon_{p-1}\\
&+ h^2\,\underbrace{(d_6 + d_7 + d_7e_7 + \tau d_7 d_6)}_{=:d_8}\sum_{i=0}^{n-1} a^0_{i,p}.
\end{aligned}$$

Hence, there exists $h_3 \in (0, h_2]$ such that for all $h \in (0, h_3]$,

$$a^0_{n,p} \le \frac{1+he_7}{1-he_8}\,a^0_{n-1,p} + \frac{h\Big(c_9 + b_9\varepsilon_{p-1}\Big)}{1-he_8} + \frac{h^2 d_{10}}{1-he_8}\sum_{i=0}^{n-1} a^0_{i,p}. \tag{3.30}$$

Applying Lemma 1.7.4, we conclude that for all $n = 0, 1, \ldots, N-1$,

$$\begin{aligned}
a^0_{n,p} \le{}& \frac{a^0_{0,p}}{R_2 - R_1}\Big[(R_2-1)R_2^n + (1-R_1)R_1^n\Big]\\
&+ \frac{h\Big(c_9 + b_9\varepsilon_{p-1}\Big)}{(R_2-R_1)(1-he_8)}\Big(R_2^n - R_1^n\Big).
\end{aligned}$$

which implies, by using (3.30), that for all $p = 0, 1, \ldots, r-1$ and $n = 0, 1, \ldots, N-1$,

$$a^0_{n,p} \le c_{10} + b_{10}\varepsilon_{p-1}.$$

Then, from (3.28), it follows that for all $j = 1, \ldots, (m+1)$ and $n = 0, 1, \ldots, N-1$,

$$a^j_{n,p} \le y_n \le \underbrace{\Big(c_7 + e_7c_{10} + \tau d_8 c_{10}\Big)}_{=:c_{11}} + \underbrace{\Big(b_7 + e_7 b_{10} + \tau d_7 b_{10}\Big)}_{=:b_{11}}\,\varepsilon_{p-1}.$$

Let

$$c_{12} := \max\{c_{10}, c_{11}\}, \qquad b_{12} := \max\{b_{10}, b_{11}\}.$$

Then, by definition of ε_p, we obtain

$$\varepsilon_p \leq c_{12} + b_{12}\varepsilon_{p-1} \leq c_{12} + b_{12}\sum_{i=0}^{p-1} \varepsilon_i.$$

Applying Lemma 1.7.2, we conclude that

$$a_{n,p}^{j+1} \leq \varepsilon_p \leq c_{12}\exp(\tau b_{12}) =: \alpha(m),$$

for all $n = 0, 1, \ldots, N-1$, $p = 0, 1, \ldots, r-1$, and $j = 0, 1, \ldots, (m+1)$.

This completes the proof. $\square$

The following theorem establishes the convergence of the proposed numerical method.

Theorem 3.3.1 *Let g, k_1, k_2, and k_3 be $(m+1)$ times continuously differentiable on their respective domains. Then equations* (3.2), (3.3), (3.8) *and* (3.11) *define a unique approximation*

$$u \in S_m^{(0)}(\Pi_N),$$

and the associated error function $e := x - u$ satisfies the estimate

$$\|e\|_{L^\infty(\Pi_N)} \leq C\, h^{(m+1)},$$

provided that the step size h is sufficiently small, where C is a positive constant independent of h.

Proof. The proof is divided into two main steps.

Claim 1 There exists a positive constant C_1, independent of h, such that

$$\|e^0\|_{L^\infty(\sigma^0)} \leq C_1 h^{(m+1)},$$

where $e^0 := e|_{\sigma^0}$ denotes the restriction of the error to the initial strip σ^0. More precisely, for each subinterval σ_n^0, $n = 0, 1, \ldots, N-1$, the local error is defined by

$$e^0(t) = e_n^0(t) := \left|x'(t) - u_n^0(t)\right|, \qquad t \in \sigma_n^0.$$

Let $t \in \sigma_0^0$. By Theorem 3.3.1, and for sufficiently small h, we obtain

$$\left|e_0^0(t)\right| = \left|x'(t) - u_0^0(t)\right| \leq \frac{\|x^{((m+1))}\|_{L^\infty(\sigma_0^0)}}{(m+1)!}\, h^{(m+1)} \leq \frac{\alpha(m)}{(m+1)!}\, h^{(m+1)}.$$

From equation (2.3), for $n = 1, 2, \ldots, N_1 - 1$, we obtain

$$\|x' - \hat{u}'_{n,0}\|_{L^\infty(\sigma_n^0)} \leq hk \sum_{i=0}^{n-1} \|e_i^0\|_{L^\infty(\sigma_i^0)} + hk\|x' - \hat{u}'_{n,0}\|_{L^\infty(\sigma_n^0)}. \tag{3.31}$$

Similarly, from equation (2.4), for $n = N_1, \ldots, N-1$, we have

$$\begin{aligned} \|x' - \hat{u}'_{n,0}\|_{L^\infty(\sigma_n^0)} \leq{}& \|e_{n-N_1}^0\|_{L^\infty(\sigma_n^0)} + hk \sum_{i=0}^{n-N_1-1} \|e_i^0\|_{L^\infty(\sigma_i^0)} \\ &+ 2hk\|e_{n-N_1}^0\|_{L^\infty(\sigma_{n-N_1}^0)} + hk \sum_{i=n+1-N_1}^{n-1} \|e_i^0\|_{L^\infty(\sigma_i^0)} \\ &+ hk\|x' - \hat{u}'_{n,0}\|_{L^\infty(\sigma_n^0)}. \end{aligned} \tag{3.32}$$

Combining inequalities (3.31) and (3.32), we deduce that for all $n = 1, 2, \ldots, N-1$,

$$\|x' - \hat{u}'_{n,0}\|_{L^\infty(\sigma_n^0)} \leq \|e_{n-N_1}^0\|_{L^\infty(\sigma_n^0)} + 2hk \sum_{i=0}^{n-1} \|e_i^0\|_{L^\infty(\sigma_i^0)} + hk\|x' - \hat{u}'_{n,0}\|_{L^\infty(\sigma_n^0)}.$$

Here,

$$k := \max_{i=1,2,3} \|k_i\|_{L^\infty(\sigma_i^0)}.$$

Assuming $1 - hk > 0$, we obtain

$$\|x' - \hat{u}'_{n,0}\|_{L^\infty(\sigma_n^0)} \leq \frac{1}{1-hk}\|e_{n-N_1}^0\|_{L^\infty(\sigma_n^0)} + \frac{2hk}{1-hk} \sum_{i=0}^{n-1} \|e_i^0\|_{L^\infty(\sigma_i^0)}.$$

Therefore, by Theorem 3.3.1, for $n = 1, 2, \dots, N-1$, we have

$$\begin{aligned}
\|e_n^0\|_{L^\infty(\sigma_n^0)} &\le \|x' - \hat{u}'_{n,0}\|_{L^\infty(\sigma_n^0)} + \|\hat{u}'_{n,0} - u_n^0\|_{L^\infty(\sigma_n^0)} \\
&\le \|x' - \hat{u}'_{n,0}\|_{L^\infty(\sigma_n^0)} + \frac{\alpha(m)}{(m+1)!} h^{(m+1)} \\
&\le \frac{1}{1-hk}\|e_{n-N_1}^0\|_{L^\infty(\sigma_n^0)} + \frac{2hk}{1-hk}\sum_{i=0}^{n-1}\|e_i^0\|_{L^\infty(\sigma_i^0)} + \frac{\alpha(m)}{(m+1)!}h^{(m+1)}.
\end{aligned}$$

Finally, applying Lemma 1.7.2, we conclude that for all $n = 0, 1, \dots, N_1 - 1$,

$$\|e_n^0\|_{L^\infty(\sigma_n^0)} \le \underbrace{\left(\frac{\alpha(m)}{(m+1)!}\exp\left(\tau_1 \frac{2k}{1-hk}\right)\right)}_{=:C_1} h^{(m+1)}.$$

Claim 2 There exists a positive constant C, independent of the step size h, such that

$$\|e\|_{L^\infty(\Pi)} \le C h^{(m+1)}.$$

Define the error function e^p on σ^p by

$$e^p(t) := x'(t) - u^p(t),$$

and, more precisely, on each subinterval σ_n^p by

$$e^p(t) = e_n^p(t) := x'(t) - u_n^p(t), \qquad n = 0, \dots, N-1, \quad p = 0, \dots, r-1.$$

First, let $t \in \sigma_0^p$. From equation (3.9), we obtain

$$\begin{aligned}
\|x' - \hat{u}'_{0,p}\|_{L^\infty(\sigma_0^p)} \le{}& \|x' - \hat{u}_{N-N_1,p-1}\|_{L^\infty(\sigma_{N-N_1}^{p-1})} + hk\sum_{i=0}^{p-1}\sum_{d=0}^{N-1}\|e_d^i\|_{L^\infty(\sigma_d^i)} \\
&+ 2hk\|e_0^{p-1}\|_{L^\infty(\sigma_0^{p-1})} + 2hk\|e_{N-N_1}^{p-1}\|_{L^\infty(\sigma_{N-N_1}^{p-1})} \\
&+ hk\sum_{d=1}^{N-1}\|e_d^{p-1}\|_{L^\infty(\sigma_d^{p-1})} + hk\|x' - \hat{u}'_{0,p}\|_{L^\infty(\sigma_0^p)}.
\end{aligned}$$

Consequently,

$$\begin{aligned}\|x' - \hat{u}'_{0,p}\|_{L^\infty(\sigma_0^p)} \le \frac{\alpha(m)}{(m+1)!}h^{(m+1)} + 2hk\sum_{i=0}^{p-1}\sum_{d=0}^{N-1}\|e_d^i\|_{L^\infty(\sigma_d^i)} + 2hk\sum_{i=0}^{N-N_1}\|e_i^{p-1}\|_{L^\infty(\sigma_i^{p-1})}\\ + hk\|x' - \hat{u}'_{0,p}\|_{L^\infty(\sigma_0^p)} + hk\|\hat{u}'_{0,p} - u_0^p\|_{L^\infty(\sigma_0^p)}.\end{aligned}$$

Using the approximation estimate from Theorem 3.3.1,

$$\|\hat{u}'_{0,p} - u_0^p\|_{L^\infty(\sigma_0^p)} \le \frac{\alpha(m)}{(m+1)!}h^{(m+1)},$$

we obtain

$$\begin{aligned}\|x' - \hat{u}'_{0,p}\|_{L^\infty(\sigma_0^p)} \le \frac{2\alpha(m)}{(m+1)!}h^{(m+1)} + 2hk\sum_{i=0}^{p-1}\sum_{d=0}^{N-1}\|e_d^i\|_{L^\infty(\sigma_d^i)} + 2hk\sum_{i=0}^{N-N_1}\|e_i^{p-1}\|_{L^\infty(\sigma_i^{p-1})}\\ + hk\|x' - \hat{u}'_{0,p}\|_{L^\infty(\sigma_0^p)}.\end{aligned}$$

Here,

$$k := \max_{i=1,2,3}\|k_i\|_{L^\infty(\sigma_i^0)}.$$

Hence, we obtain

$$\begin{aligned}(1-hk)\,\|x' - \hat{u}'_{0,p}\|_{L^\infty(\sigma_0^p)} \le 2hk\sum_{i=0}^{p-1}\sum_{d=0}^{N-1}\|e_d^i\|_{L^\infty(\sigma_d^i)} + 2hk\sum_{i=0}^{N-N_1}\|e_i^{p-1}\|_{L^\infty(\sigma_i^{p-1})}\\ + 2\,\frac{\alpha(m)}{(m+1)!}\,h^{(m+1)}.\end{aligned}$$

Assuming that $1 - hk > 0$, it follows that

$$\begin{aligned}\|x' - \hat{u}'_{0,p}\|_{L^\infty(\sigma_0^p)} \le \frac{2hk}{1-hk}\sum_{i=0}^{p-1}\sum_{d=0}^{N-1}\|e_d^i\|_{L^\infty(\sigma_d^i)} + \frac{2hk}{1-hk}\sum_{i=0}^{N-N_1}\|e_i^{p-1}\|_{L^\infty(\sigma_i^{p-1})}\\ + \frac{2}{1-hk}\,\frac{\alpha(m)}{(m+1)!}\,h^{(m+1)}.\end{aligned}$$

Therefore, by Theorem 3.3.1, we have

$$\begin{aligned}\|e_0^p\|_{L^\infty(\sigma_0^p)} &\le \|x' - \hat{u}'_{0,p}\|_{L^\infty(\sigma_0^p)} + \|\hat{u}'_{0,p} - u_0^p\|_{L^\infty(\sigma_0^p)}\\ &\le \|x' - \hat{u}'_{0,p}\|_{L^\infty(\sigma_0^p)} + \frac{\alpha(m)}{(m+1)!}\,h^{(m+1)}.\end{aligned}$$

Consequently, for all $p = 1, 2, \ldots, r-1$, we obtain

$$\begin{aligned}\|e_0^p\|_{L^\infty(\sigma_0^p)} \le{}& \frac{2hk}{1-hk}\sum_{i=0}^{p-1}\sum_{d=0}^{N-1}\|e_d^i\|_{L^\infty(\sigma_d^i)} + \frac{2hk}{1-hk}\sum_{i=0}^{N-N_1}\|e_i^{p-1}\|_{L^\infty(\sigma_i^{p-1})}\\ &+\left(\frac{2}{1-hk}+1\right)\frac{\alpha(m)}{(m+1)!}\,h^{(m+1)}.\end{aligned}$$

Hence, by Lemma 1.7.2, for all $n = 0, 1, \ldots, N-1$, we obtain

$$\begin{aligned}\|e_0^p\|_{L^\infty(\sigma_0^p)} \le{}& \left(\frac{2}{1-hk}\,\frac{\alpha(m)}{(m+1)!}\,h^{(m+1)} + \frac{\alpha(m)}{(m+1)!}\,h^{(m+1)} + \frac{2k\tau_1}{1-hk}\sum_{i=0}^{p-1}\|e_i^{p-1}\|_{L^\infty(\sigma_i^{p-1})}\right)\\ &\times \exp\!\left(\frac{2k\tau_1}{1-hk}\right) \;\le\; c_2^1\,h^{(m+1)},\end{aligned}$$

where c_2^1 is a positive constant independent of h.

Next, let $t \in \sigma_n^p$ for $n = 1, 2, \ldots, N-1$. From (3.12) and (3.13), we have

$$\begin{aligned}\|x' - \hat{u}'_{n,p}\|_{L^\infty(\sigma_n^p)} \le{}& \|x' - \hat{u}_{n-N_1,p}\|_{L^\infty(\sigma_{n-N_1}^p)} + hk\sum_{i=0}^{p-1}\sum_{d=0}^{N-1}\|e_d^i\|_{L^\infty(\sigma_d^i)}\\ &+ hk\sum_{i=0}^{n-1}\|e_i^{p-1}\|_{L^\infty(\sigma_i^{p-1})} + 2hk\,\|e_n^{p-1}\|_{L^\infty(\sigma_n^{p-1})}\\ &+ hk\sum_{d=n+1}^{N-1}\|e_d^{p-1}\|_{L^\infty(\sigma_d^{p-1})} + hk\sum_{i=0}^{n-1}\|e_i^p\|_{L^\infty(\sigma_i^p)}\\ &+ hk\,\|x' - \hat{u}'_{n,p}\|_{L^\infty(\sigma_n^p)},\end{aligned}$$

where

$$k = \max_{i=1,2,3}\|k_i\|_{L^\infty(\sigma_i^0)}.$$

Rearranging terms yields

$$(1-hk)\,\|x'-\hat{u}'_{n,p}\|_{L^\infty(\sigma_n^p)} \leq \|e_{n-N_1}^{p-1}\|_{L^\infty(\sigma_{n-N_1}^{p-1})} + 2hk\sum_{i=0}^{p-1}\sum_{d=0}^{N-1}\|e_d^i\|_{L^\infty(\sigma_d^i)} + 2hk\sum_{i=0}^{n-1}\|e_i^{p-1}\|_{L^\infty(\sigma_i^{p-1})}.$$

Assuming $1-hk>0$, we deduce

$$\|x'-\hat{u}'_{n,p}\|_{L^\infty(\sigma_n^p)} \leq \frac{1}{1-hk}\,\frac{\alpha(m)}{(m+1)!}\,h^{(m+1)} + \frac{2hk}{1-hk}\sum_{i=0}^{p-1}\sum_{d=0}^{N-1}\|e_d^i\|_{L^\infty(\sigma_d^i)} + \frac{2hk}{1-hk}\sum_{i=0}^{n-1}\|e_i^{p-1}\|_{L^\infty(\sigma_i^{p-1})}.$$

Therefore, by Theorem 1.7.1, for all $n=1,2,\ldots,N-1$, we obtain

$$\begin{aligned}\|e_n^p\|_{L^\infty(\sigma_n^p)} &\leq \|x'-\hat{u}'_{n,p}\|_{L^\infty(\sigma_n^p)} + \|\hat{u}'_{n,p}-u_n^p\|_{L^\infty(\sigma_n^p)}\\ &\leq \|x'-\hat{u}'_{n,p}\|_{L^\infty(\sigma_n^p)} + \frac{\alpha(m)}{(m+1)!}\,h^{(m+1)}.\end{aligned}$$

Consequently, for $n=1,2,\ldots,N-1$, it follows that

$$\|e_n^p\|_{L^\infty(\sigma_n^p)} \leq \frac{2hkN}{1-hk}\sum_{i=0}^{p-1}\|e^i\|_{L^\infty(\sigma^i)} + \frac{2hk}{1-hk}\sum_{i=0}^{n-1}\|e_i^{p-1}\|_{L^\infty(\sigma_i^{p-1})} + \frac{\alpha(m)}{(m+1)!}\,h^{(m+1)} + \frac{1}{1-hk}\,\frac{\alpha(m)}{(m+1)!}\,h^{(m+1)}.$$

Applying Lemma 1.7.2, we deduce that for all $n=0,1,\ldots,N-1$,

$$\|e_n^p\|_{L^\infty(\sigma_n^p)} \leq \left(\frac{1}{1-hk}\,\frac{\alpha(m)}{(m+1)!}\,h^{(m+1)} + \frac{\alpha(m)}{(m+1)!}\,h^{(m+1)} + \frac{2k\tau_2}{1-hk}\sum_{i=0}^{p-1}\|e^i\|_{L^\infty(\sigma^i)}\right) \times \exp\left(\frac{2k\tau_1}{1-hk}\right) =: c_{23}\,h^{(m+1)}.$$

Using Lemma 1.7.2 once again, we conclude that for all $n=0,1,\ldots,N-1$,

$$\|e_n^p\|_{L^\infty(\sigma_n^p)} \leq C_2\,h^{(m+1)},$$

where C_2 is a positive constant independent of h.

Finally, the proof is completed by setting

$$C = \max\{C_1, C_2\}.$$

□

3.4 Numerical Experiments and Performance Evaluation

In this section, a series of numerical experiments is presented to evaluate the accuracy and efficiency of the Taylor collocation method (TCM) applied to the problem formulated in (3.1). Several representative test examples are considered in order to demonstrate the convergence behavior of the proposed scheme. The numerical results clearly confirm that the method achieves a convergence order equal to m, in agreement with the theoretical analysis.

In addition to accuracy, the computational efficiency of the method is assessed by reporting the CPU execution time (in seconds) for each test case. All numerical computations are carried out using MAPLE 17 on a personal computer equipped with an Intel Core i7–1165G7 processor running at 2.80 GHz and 16 GB of RAM, under the Microsoft Windows 7 operating system.

The numerical results indicate that increasing the discretization parameters N and m leads to a significant reduction in the absolute error, thereby confirming both the effectiveness and robustness of the proposed Taylor collocation method.

Example 3.4.1 *Consider the following Volterra integral equation with two constant de-*

lays:

$$x'(t) = g(t) + x(t-\tau_1) + x(t-\tau_2) + \int_0^{t-\tau_2} \cos(t-s)\, x(s)\, ds$$
$$+ \int_{t-\tau_2}^{t-\tau_1} \cos(t+s)\, x(s)\, ds,$$

defined for $t \in [0.9, 5.4]$, $\tau_1 = 0.45$, *and* $k_1 = 0$
The forcing function g *is chosen such that the exact solution is*

$$x(t) = 1 + \sin(t),$$

and the initial history function satisfies $x(t) = \Phi(t)$ *for* $t \in [0, 0.9]$.

The absolute errors obtained using the Taylor collocation method for the parameter sets

$$(N, m) \in \{(8,4), (8,5), (8,7), (8,8)\}$$

are reported in Table 3.1. Figure 3.1 illustrates the evolution of the absolute error for different values of the polynomial degree m *with a fixed number of subintervals* $N = 8$. *The results clearly show that increasing* m *significantly improves the accuracy, which confirms the theoretical convergence order established in Section 3.*

t	$N=8, m=4$	$N=8, m=5$	$N=8, m=7$	$N=8, m=8$
0.9	00	00	00	00
1.8	$3.43e-07$	$2.27e-08$	$2.98e-10$	$3.03e-10$
2.7	$9.77e-08$	$6.64e-08$	$1.02e-09$	$1.03e-09$
3.6	$6.02e-07$	$1.87e-07$	$3.22e-09$	$3.26e-09$
4.5	$2.20e-06$	$4.69e-07$	$8.74e-09$	$8.84e-09$
5.4	$5.03e-06$	$1.01e-06$	$1.97e-08$	$1.99e-08$

Table 3.1: Absolute errors for Example 3.4.1.

Example 3.4.2 *Consider the double-delay Volterra integral equation*

$$x'(t) = g(t) + (t+1)\, x(t-\tau_1) + t^2 x(t-\tau_2) + \int_0^{t-\tau_2} s\, x(s)\, ds$$
$$+ \int_{t-\tau_1}^{t} t\, x(s)\, ds,$$

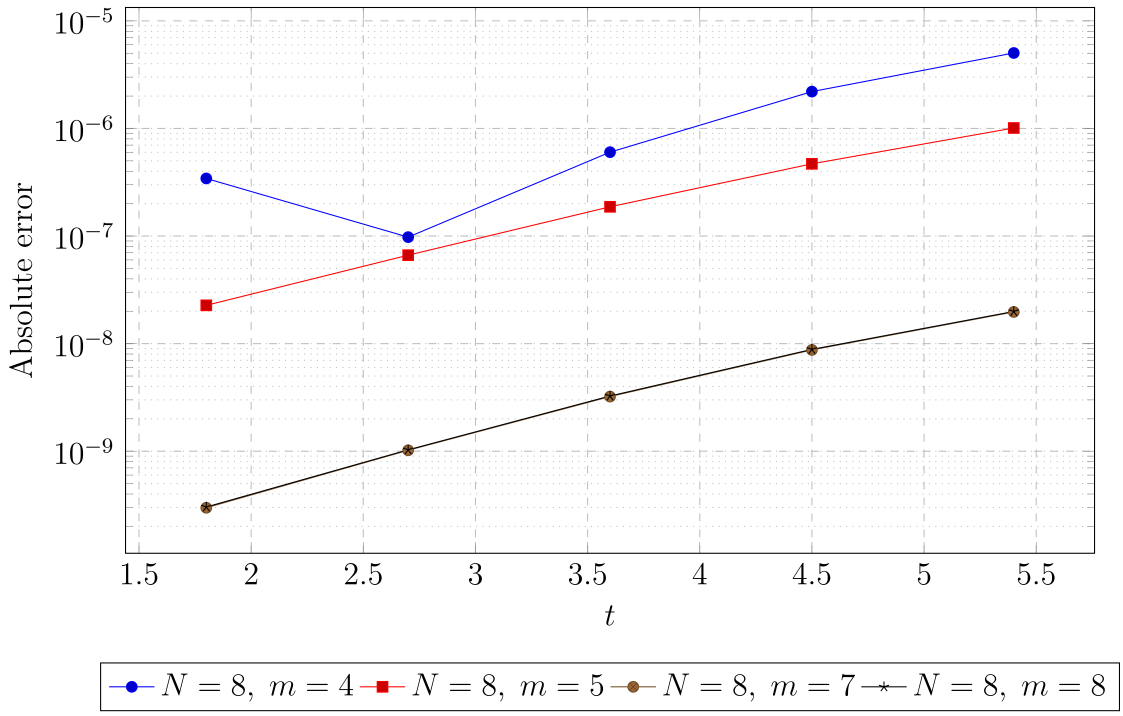

Figure 3.1: Logarithmic plot of the absolute errors for Example 3.4.1 with fixed $N = 8$ and varying polynomial degree m.

defined for $t \in [0.7,\, 4.2]$, *with* $k_2 = 0,\ \tau_1 = 0.35$.

The function $g(t)$ *is chosen such that the exact solution is* $x(t) = \sin(t)$, *while the history function is prescribed by* $x(t) = \Phi(t)$ *for* $t \in [0,\, 0.7]$.

The absolute errors corresponding to the parameter pairs

$$(N, m) \in \{(4,4), (6,6), (8,8), (10,10)\}$$

are reported in Table 3.2. As illustrated in Figure 3.2, increasing the discretization parameters (N, m) *leads to a significant reduction in the absolute error, thereby numerically confirming the convergence order predicted by the theoretical analysis.*

Example 3.4.3 *Consider the neutral double-delay Volterra integral equation*

$$x'(t) = g(t) + x(t-\tau_1) + x(t-\tau_2) + \int_{t-\tau_2}^{t-\tau_1} t\, e^{t}\, x(s)\, ds$$
$$+ \int_{t-\tau_1}^{t} (t+1)\, x(s)\, ds,$$

t	$(N=4, m=4)$	$(N=6, m=6)$	$(N=8, m=8)$	$(N=10, m=10)$
0.7	0	0	0	0
1.4	3.61×10^{-6}	1.24×10^{-9}	3.01×10^{-10}	9.69×10^{-11}
2.1	1.69×10^{-5}	2.33×10^{-9}	1.66×10^{-9}	5.49×10^{-10}
2.8	1.24×10^{-4}	3.43×10^{-3}	1.13×10^{-8}	3.89×10^{-9}
3.5	1.22×10^{-3}	9.03×10^{-3}	1.08×10^{-7}	3.81×10^{-8}
4.2	1.52×10^{-2}	3.24×10^{-2}	1.34×10^{-6}	4.74×10^{-7}

Table 3.2: Absolute errors for Example 3.4.2.

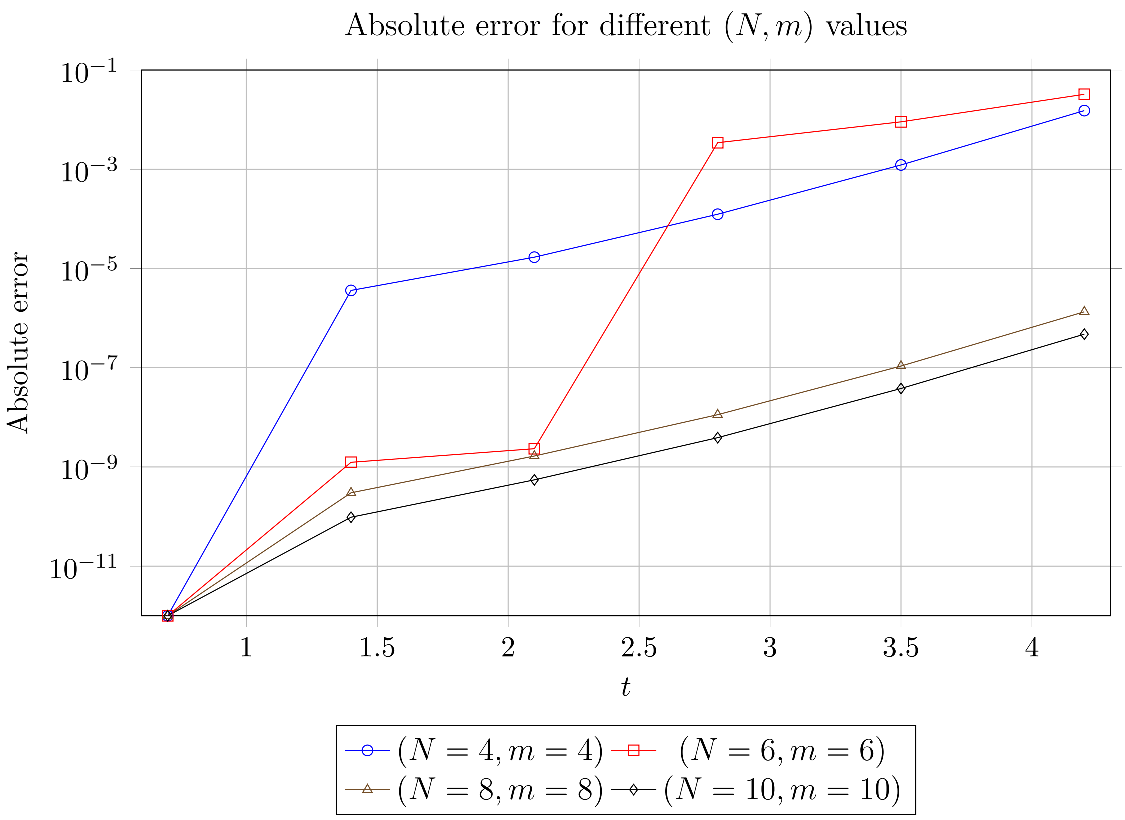

Figure 3.2: Logarithmic plot of absolute errors for Example 3.4.2.

defined for $t \in [0.5, 3]$*, with* $k_3 = 0$*,* $\tau_1 = 0.25$.

The source function g *is selected such that the exact solution is*

$$x(t) = \cos(t),$$

and the initial function satisfies $\Phi(t) = x(t)$ *for* $t \in [0, 0.5]$.

The absolute errors together with the corresponding computational times are reported *in Table 3.3 for the parameter pairs*

$$(N, m) \in \{(4,4), (4,6), (4,8), (6,4), (6,6)\}.$$

Figure 3.3 illustrates the evolution of the absolute error with respect to t for different values of (N, m).

	$N=4$			$N=6$	
t	$m=4$	$m=6$	$m=8$	$m=4$	$m=6$
0.5	$1.00e-15$	$1.00e-15$	$1.00e-15$	$1.00e-15$	$1.00e-15$
1.0	$7.05e-07$	$2.29e-11$	$3.10e-11$	$1.43e-07$	$3.00e-13$
1.5	$2.66e-06$	$8.51e-10$	$1.32e-10$	$5.48e-07$	$7.36e-11$
2.0	$9.92e-06$	$3.21e-09$	$4.43e-10$	$2.05e-06$	$1.45e-10$
2.5	$4.83e-05$	$1.59e-08$	$1.84e-09$	$1.00e-05$	$4.77e-10$
3.0	$3.55e-04$	$1.18e-07$	$1.27e-08$	$7.35e-05$	$2.69e-09$

Table 3.3: Absolute errors of Example 2.5.3.

Example 3.4.4 *Consider the double-delay Volterra integral equation*

$$\begin{aligned} x'(t) &= g(t) + \tfrac{1}{2}\, x(t-\tau_1) + \tfrac{1}{4}\, x(t-\tau_2) + \int_0^{t-\tau_2} s\, x(s)\, ds \\ &\quad + \int_{t-\tau_2}^{t-\tau_1} t\, \mathrm{e}^t\, x(s)\, ds + \int_{t-\tau_1}^{t} t^2\, x(s)\, ds, \end{aligned}$$

for $t \in [0.6, 3.6]$, $\tau_1 = 0.3$.

The function g is chosen such that the exact solution is

$$x(t) = 1 + \cos(t),$$

and the initial function satisfies $x(t) = \Phi(t)$ for $t \in [0, 0.6]$.

The absolute errors corresponding to the parameter pairs

$$(N, m) \in \{(4,4), (6,6), (8,8)\}$$

are reported in Table 3.4. In addition, the exact and approximate solutions obtained with

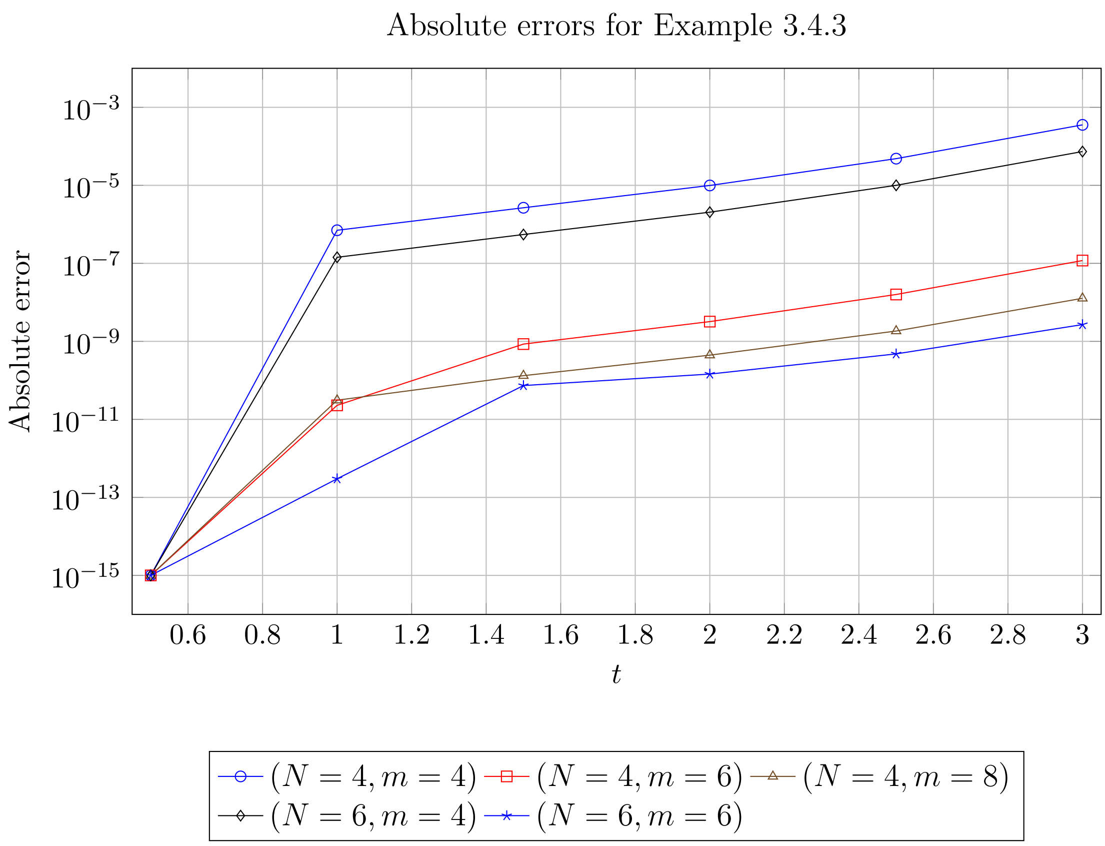

Figure 3.3: Logarithmic plot of absolute errors for Example 3.4.3.

$N = 20$ *and* $m = 20$ *are illustrated in Figure 3.4.*

Figure 3.4 demonstrates the decay of the absolute error as both the number of subintervals N *and the polynomial degree* m *increase. For a fixed discretization level, higher polynomial degrees result in a substantial reduction of the error, confirming the high-order convergence of the Taylor collocation method. Moreover, mesh refinement through larger values of* N *further enhances the numerical accuracy, in full agreement with the theoretical convergence results established in the previous section.*

Example 3.4.5 *Consider the neutral double-delay Volterra integral equation*

$$x'(t) = g(t) + 2t\,x(t-\tau_1) + t\,x(t-\tau_2) + \int_{t-\tau_2}^{t-\tau_1} \mathrm{e}^{t-s}\,x(s)\,ds,$$

defined for $t \in [1, 6]$*, with* $k_1 = 0$ *and* $k_3 = 0$*,* $\tau_1 = 0.5$*.*

t	$(N=4, m=4)$	$(N=6, m=6)$	$(N=8, m=8)$
0.6	00	00	00
1.2	$1.99e-06$	$7.36e-10$	$6.37e-10$
1.8	$8.39e-06$	$2.27e-09$	$1.82e-09$
2.4	$5.00e-05$	$1.34e-08$	$1.06e-08$
3.0	$5.64e-04$	$1.52e-07$	$1.21e-07$
3.6	$1.29e-02$	$3.52e-06$	$2.81e-06$

Table 3.4: Absolute errors for Example 2.5.4

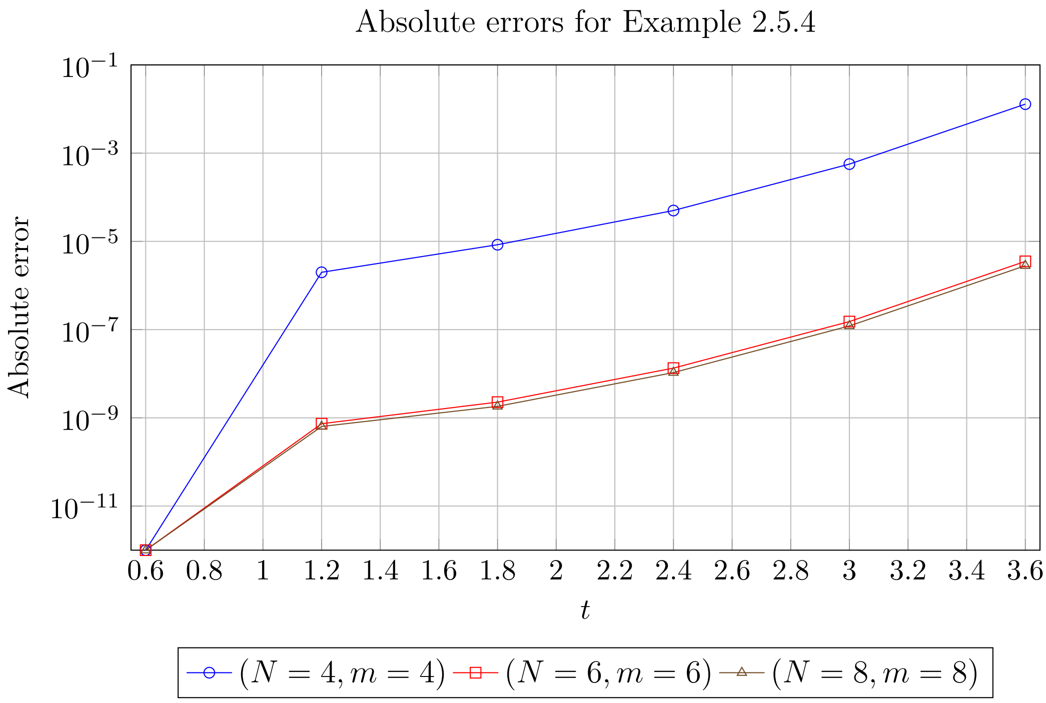

Figure 3.4: Logarithmic plot of absolute errors for Example 2.5.4.

The function g is chosen such that the exact solution is

$$x(t) = 1 + \mathrm{e}^t,$$

and the initial function satisfies $\Phi(t) = x(t)$ for $t \in [0, 0.5]$.

The absolute errors together with the corresponding computational times obtained for the parameter sets

$$(N, m) \in \{(8,4), (8,6), (10,4), (10,6), (10,8)\}$$

are reported in Table 3.5. The graphical representation of the absolute error as a function of t is shown in Figure 3.5.

Figure 3.5 illustrates the convergence characteristics of the Taylor collocation method when applied to the considered neutral problem. For a fixed number of subintervals N, increasing the polynomial degree m leads to a pronounced reduction in the absolute error, thereby confirming the high-order accuracy of the proposed scheme. In addition, refining the temporal discretization by increasing N further enhances the numerical accuracy and improves stability across the entire computational interval. These numerical findings are in full agreement with the theoretical convergence results established earlier and clearly demonstrate the reliability and effectiveness of the method for solving neutral Volterra integral equations with multiple delays.

	$N = 8$		$N = 10$	
t	$m = 4$	$m = 6$	$m = 4$	$m = 6$
1.0	00	00	00	00
2.0	$1.14e-05$	$3.55e-10$	$4.78e-06$	$2.72e-09$
3.0	$1.25e-04$	$4.68e-09$	$5.31e-05$	$3.85e-08$
4.0	$1.63e-03$	$9.97e-08$	$6.90e-04$	$5.38e-07$
5.05	$2.62e-02$	$1.76e-06$	$1.10e-02$	$8.80e-06$
6.0	$5.14e-01$	$3.51e-05$	$2.17e-01$	$1.73e-04$

Table 3.5: Absolute errors for Example 3.4.5.

Example 3.4.6 *Consider the double-delay Volterra integral equation*

$$x'(t) = g(t) + \frac{3}{2}\,x(t-\tau_1) + \frac{1}{2}\,x(t-\tau_2) + \int_{t-\tau_1}^{t} \exp(t)\cos(t)\,x(s)\,ds,$$

defined for $t \in [0.5, 3]$, with $K(2) = 0$ and $K(3) = 0$, $\tau_1 = 0.25$.
The function g is chosen such that the exact solution is

$$x(t) = 1 + t^3,$$

and the history function satisfies $\Phi(t) = x(t)$ for $t \in [0, 0.5]$.

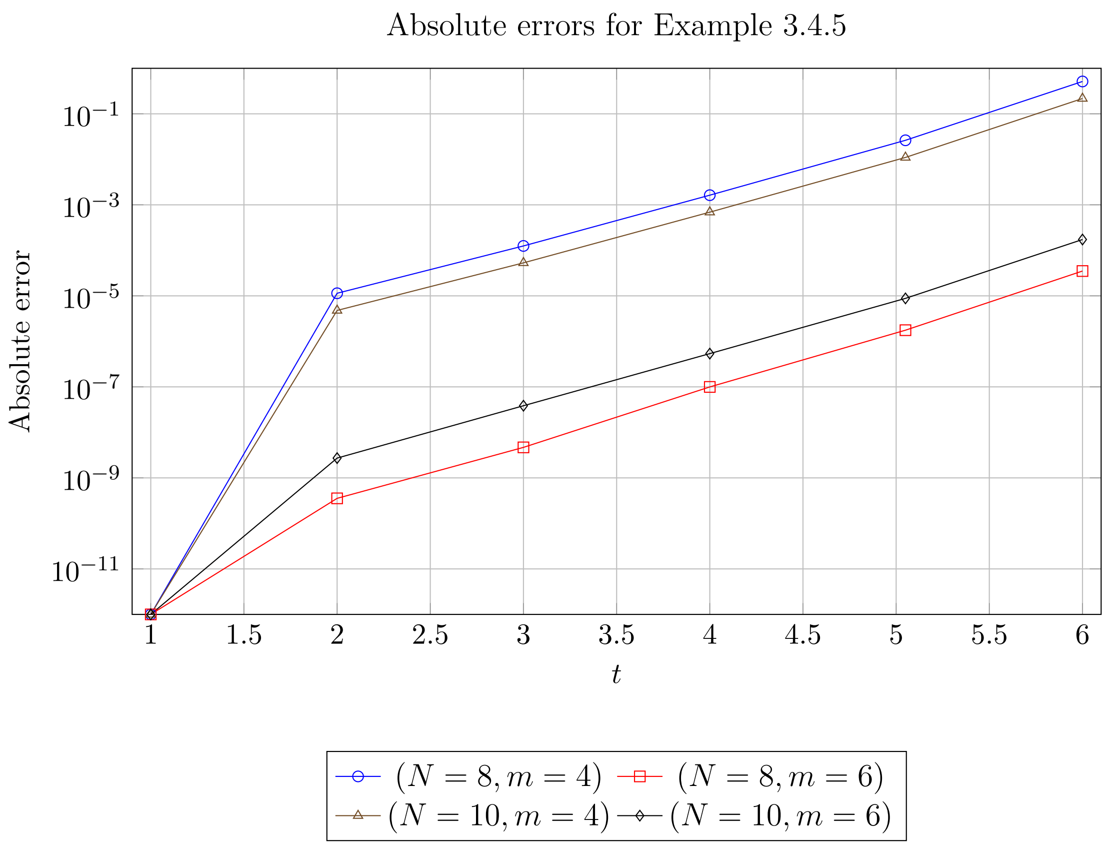

Figure 3.5: Logarithmic plot of absolute errors for Example 3.4.5.

The absolute errors obtained for the parameter pairs

$$(N, m) \in \{(6, 6), (12, 12), (16, 16)\}$$

are reported in Table 3.6. The corresponding error curves are depicted in Figure 3.6, which clearly illustrate the rapid decay of the numerical error as both the number of subintervals N *and the polynomial degree* m *increase, thereby confirming the high-order convergence of the proposed Taylor collocation method.*

t	$(N = 6, m = 6)$	$(N = 12, m = 12)$	$(N = 16, m = 16)$
0.5	00	00	00
1.0	$1.92e - 12$	$1.04e - 12$	$1.20e - 13$
1.5	$4.76e - 11$	$4.19e - 11$	$2.30e - 11$
2.0	$7.06e - 11$	$6.56e - 11$	$3.00e - 13$
2.5	$7.88e - 11$	$7.31e - 11$	$8.00e - 13$
3.0	$3.40e - 11$	$3.16e - 11$	$4.00e - 13$

Table 3.6: Absolute errors for Example 3.4.6

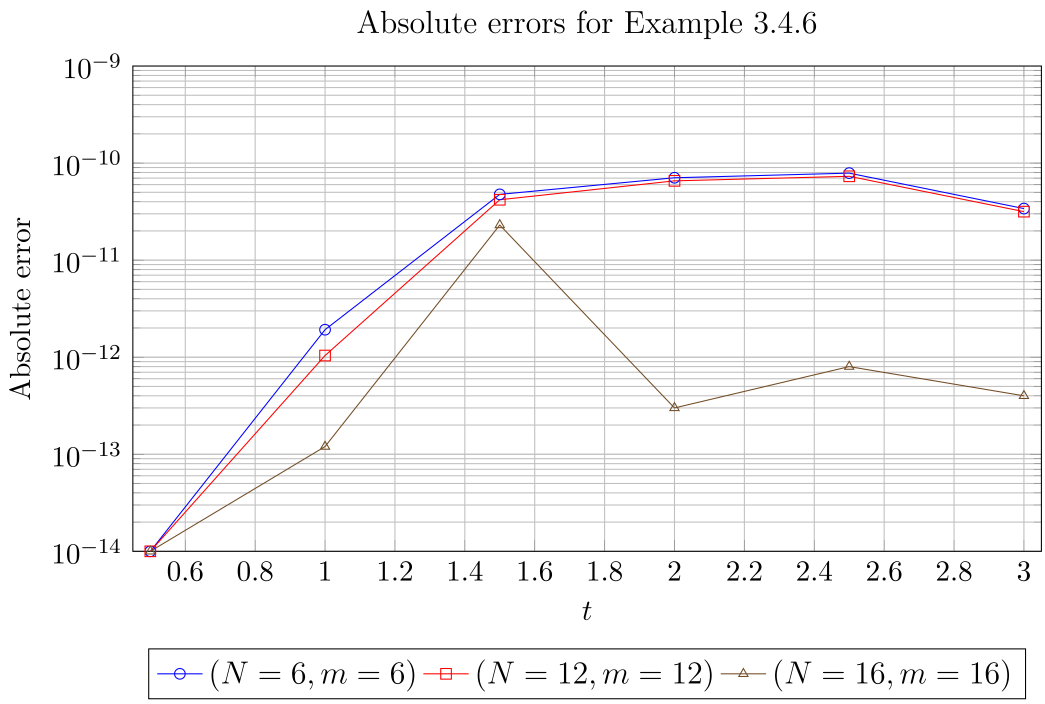

Figure 3.6: Logarithmic plot of absolute errors for Example 3.4.6.

General Conclusion and Perspectives

This thesis has been devoted to the development, analysis, and validation of a Taylor collocation method for the numerical solution of Volterra integral and integro-differential equations with constant delays. The studied models include linear Volterra integral equations, delay Volterra integral equations, and Volterra integro-differential equations with one or two constant delays in the following models

Model 1: Linear Volterra integral equation (second kind).

Find $x : [0, T] \to \mathbb{R}$ such that

$$x(t) = g(t) + \int_0^t K(t,s)\, x(s)\, ds, \qquad t \in [0, T].$$

Model 2: Linear delay Volterra integral equation (constant delays).

Let $0 < \tau_1 \leq \tau_2$ and let Φ be a given history. Find $x : [0, T] \to \mathbb{R}$ such that

$$x(t) = g(t) + A(t)\, x(t - \tau_1) + B(t)\, x(t - \tau_2) + \int_0^t K(t,s)\, x(s)\, ds, \qquad t \in [\tau_2, T],$$

with

$$x(t) = \Phi(t), \qquad t \in [0, \tau_2].$$

Model 3: Volterra integro-differential equation with one constant delay.

Let $\tau > 0$ and let Φ be given. Find $x : [0, T] \to \mathbb{R}$ such that

$$x'(t) = g(t) + A(t)\, x(t - \tau) + \int_0^t K(t, s)\, x(s)\, ds, \qquad t \in [\tau, T],$$

with

$$x(t) = \Phi(t), \qquad t \in [0, \tau].$$

Model 4: Volterra integro-differential equation with two constant delays.

Let $0 < \tau_1 \leq \tau_2$ and let Φ be given. Find $x : [0, T] \to \mathbb{R}$ such that

$$\begin{aligned} x'(t) = g(t) + A(t)\, x(t - \tau_1) + B(t)\, x(t - \tau_2) + \int_0^{t-\tau_2} k_3(t, s)\, x(s)\, ds \\ + \int_{t-\tau_2}^{t-\tau_1} k_2(t, s)\, x(s)\, ds + \int_{t-\tau_1}^{t} k_1(t, s)\, x(s)\, ds, \qquad t \in [\tau_2, T], \end{aligned}$$

with

$$x(t) = \Phi(t), \qquad t \in [0, \tau_2].$$

Which naturally arise in many scientific and engineering applications involving memory and hereditary effects.

The proposed numerical method is based on piecewise Taylor polynomial approximations constructed on a uniform temporal partition. By enforcing the governing equations at carefully selected collocation points and incorporating continuity conditions at the interfaces of subintervals, the method yields a fully explicit and direct scheme. A key strength of the approach is that it avoids solving large algebraic systems, which significantly reduces computational cost and makes the method easy to implement.

A rigorous convergence analysis has been established. Under standard smoothness assumptions on the data, the method is shown to converge with order $\mathcal{O}(h^m)$, where m is the degree of the Taylor polynomial. The proof relies on sharp bounds for the Taylor coefficients and discrete Grönwall-type inequalities adapted to delayed terms. Numerical experiments confirm the theoretical results and demonstrate that high accuracy can be achieved with relatively low computational effort.

From both a theoretical and practical point of view, this work shows that Taylor collocation methods provide an efficient, accurate, and flexible framework for solving delay Volterra-type equations.

Several extensions of this work are possible. A natural continuation is the treatment of nonlinear Volterra integro-differential equations with delays, where the interaction between nonlinearity and memory effects poses additional analytical challenges. Another important direction is the extension to variable or state-dependent delays, which are frequently encountered in biological and control systems.

The method can also be generalized to systems of equations and to higher-dimensional problems, including partial integro-differential equations with memory terms. Adaptive strategies in time, as well as hybrid approaches combining Taylor collocation with spectral or spline-based methods, may further improve efficiency.

The proposed method is well suited for applications where delay and memory effects play a central role, such as:

- population dynamics and epidemiological models with incubation or maturation delays,
- control systems with delayed feedback,
- viscoelasticity and material models with hereditary behavior,
- biological and chemical processes governed by integro-differential equations.

Because the method is explicit and computationally inexpensive, it is particularly attractive for real-time simulations and parameter studies.

An important practical perspective of this work is the development of a user-friendly computational tool based on the proposed Taylor collocation method. Such a tool could provide:

- a graphical interface allowing users to define delay parameters, kernels, and initial functions,
- automatic generation of Taylor coefficients and collocation grids,
- real-time visualization of numerical solutions and error evolution,
- interactive comparison between different polynomial orders and step sizes.

This would make the method accessible not only to numerical analysts, but also to engineers and applied scientists who may not be specialists in numerical analysis. Integrating the method into scientific computing environments such as MATLAB, Python, or web-based platforms would further enhance its usability and impact.

In conclusion, the Taylor collocation framework developed in this thesis provides a solid theoretical foundation and a practical numerical tool for the study of delay Volterra-type equations. The combination of mathematical rigor, computational efficiency, and implementation simplicity makes it a promising approach for both further research and real-world applications.

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