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الملخـــــص

تُطوّر هذه الأطروحة طريقةً عدديةً لحل معادلات فولتيرا التكاملية وبعض امتداداتها، بالاعتماد على طريقة التجميع باستخدام كثيرات حدود تايلور، كما تقدّم تحليلًا نظريًا لخصائصها العددية. يبدأ العمل بدراسة معادلات فولتيرا التكاملية ذات التأخيرات النسبية، حيث تُثبت نتائج التقارب وتُوضّح دقة الطريقة من خلال مجموعة من الأمثلة العددية. ثم تُوسَّع الطريقة لتشمل أنظمة معادلات فولتيرا التكاملية ثنائية الأبعاد، مع تقديم تحليل نظري للتقارب والتحقق من النتائج حسابيًا. وأخيرًا، تُعمَّم الطريقة على أنظمة معادلات فولتيرا التكاملية ثنائية الأبعاد التي تتضمن تأخيرات نسبية. وفي كل مساهمة، تُقدَّم براهين دقيقة للتقارب، وتُجرى تجارب عددية تُبرز فعالية الخوارزميات المقترحة ودقتها.

الكلمات المفتاحية : المعادلات التكاملية، المعادلات التكاملية التفاضلية، طريقة التجميع، كثيرات حدود تايلور، تحليل الخطأ.

ABSTRACT

This thesis develops and analyzes Taylor collocation methods for the numerical solution of Volterra integral equations and their extensions. The work begins with a Taylor collocation approach for Volterra integral equations with proportional delays, establishing convergence properties and demonstrating accuracy through numerical examples. The methodology is then extended to systems of two-dimensional Volterra integral equations, providing both theoretical convergence analysis and computational validation. Finally, the method is further generalized to systems of two-dimensional Volterra integral equations incorporating proportional delays. Throughout each contribution, rigorous convergence proofs are provided, and numerical experiments are included to illustrate the effectiveness and precision of the proposed algorithms.

Key Words: Integral equation, Integro-differential equations, Collocation method, Taylor polynomials, Error analysis.

RÉSUMÉ

Cette thèse développe et analyse les méthodes de collocation de Taylor pour la résolution numérique des équations intégrales de Volterra et de leurs extensions. Elle débute par une approche de collocation de Taylor pour les équations intégrales de Volterra à retards proportionnels, établissant leurs propriétés de convergence et démontrant leur précision par des exemples numériques. La méthodologie est ensuite étendue aux systèmes d'équations intégrales de Volterra bidimensionnelles, avec une analyse théorique de la convergence et une validation numérique. Enfin, la méthode est généralisée aux systèmes d'équations intégrales de Volterra bidimensionnelles intégrant des retards proportionnels. Chaque contribution est étayée par des preuves rigoureuses de convergence et des expériences numériques illustrant l'efficacité et la précision des algorithmes proposés.

Mots-clés: Équation intégrale, Équations intégro-différentielles, Méthode de collocation, Polynômes de Taylor, Analyse d'erreur.

CONTENTS

GENERAL INTRODUCTION

Volterra integral equations are fundamental mathematical tools for modeling phenomena characterized by memory effects and hereditary properties. Unlike ordinary differential equations, which describe systems based solely on their current state, Volterra integral equations incorporate the entire history of the system through integral operators. This makes them particularly suitable for applications in physics, biology, engineering, economics, and population dynamics, where past states influence present and future behavior.

The inclusion of delays in Volterra integral equations extends their modeling capabilities even further. Delay arguments, particularly proportional delays, arise naturally in problems where the system's evolution depends on both current and past configurations scaled in time. Such equations appear in control theory with transmission lags [36], epidemiology considering incubation phases [13], population biology incorporating maturation intervals [27], and materials science characterizing memory phenomena [47]. The presence of delays introduces significant mathematical and computational challenges, as the solution must be determined simultaneously on multiple time scales.

Systems of two-dimensional Volterra integral equations represent another important extension, allowing the modeling of phenomena with both spatial and temporal dependencies. These systems arise in fluid dynamics, signal processing, image analysis, environmental modeling, and electromagnetic wave propagation [1, 12, 22]. When combined with delay arguments, they provide a powerful framework for describing complex physical and biological processes where memory effects and spatial variations interact.

Despite their importance, analytical solutions to delay Volterra integral equations and systems of two-dimensional Volterra integral equations are rarely attainable. This motivates the development of efficient and accurate numerical methods. Over the past decades, numerous approaches have been proposed, each with its own advantages and limitations.

Among the most prominent methods are spectral techniques, which have been successfully applied to pantograph-type differential and integral equations with multiple delays [3]. These methods exploit the global approximation properties of orthogonal polynomials to achieve high accuracy for smooth solutions. Collocation methods, in their various forms, have proven particularly versatile. Iterated collocation frameworks offer enhanced convergence through iterative refinement of approximate solutions [9, 10]. Geometric mesh strategies adapt the discretization to handle solution singularities near the origin [10], while moving least squares collocation provides flexibility for irregular domains [14].

Chebyshev spectral collocation combines the efficiency of spectral methods with the simplicity of collocation, offering exponential convergence for sufficiently smooth problems [55]. Multilevel correction approaches improve accuracy through hierarchical refinement strategies without excessive computational cost [49].

For systems of equations, researchers have developed specialized techniques including modified homotopy perturbation methods [23], which extend classical perturbation ideas to handle nonlinear systems through series expansions. Semi-orthogonal B-spline wavelet collocation provides a multiresolution approach that captures both global and local solution features efficiently [38]. Hybrid Legendre block-pulse functions combine the advantages of orthogonal polynomials and piecewise constant functions for solving systems of Fredholm-Hammerstein integral equations [39].

In the context of two-dimensional problems, operational matrix methods using hybrid functions have been developed to reduce integral equations to algebraic systems through matrix representations [32]. Legendre wavelet methods offer adaptive refinement capabilities while maintaining computational efficiency [25]. Modified Newton methods and adaptive iterative regularization schemes have been explored for nonlinear and two-dimensional systems [16, 17], providing robust tools for handling ill-posed problems. Extensions of modified moving least squares methods offer meshless alternatives for problems on irregular domains [33].

For systems with delay arguments, differential transformation methods provide an alternative approach based on Taylor series expansions in a transformed domain. These methods have been successfully applied to Volterra integral and integro-differential equations with proportional delays [52].

Despite this rich variety of numerical approaches, the Taylor collocation method, which combines the high-order approximation properties of Taylor series with the pointwise satisfaction of collocation conditions, offers distinct advantages for the classes of equations considered in this thesis. By constructing local polynomial approximations on each subinterval and determining their coefficients through collocation, this method avoids complex mesh refinement near delay points and eliminates the need for solving large global systems. The method's conceptual simplicity and computational efficiency make it an appealing alternative to existing numerical schemes [28, 29, 4, 5].

This thesis is devoted to the development, analysis, and numerical validation of Taylor collocation methods for various classes of Volterra integral equations with delays. The work is organized into four chapters, each addressing a specific type of equation and contributing to the overall understanding and applicability of the method.

Chapter 1 presents essential preliminary concepts and mathematical tools that form the foundation for the subsequent chapters. It introduces fundamental Gronwall-type inequalities, including discrete Gronwall inequalities, Pachpatte's inequality for two variables, and Wendroff's inequality [7, 35, 34, 48]. These inequalities are crucial tools for establishing stability and convergence results throughout the thesis. The chapter also presents Taylor's theorem for functions of two independent variables [41], which is the cornerstone of the numerical methods developed in later chapters. These basic lemmas and theorems provide the analytical framework necessary for rigorous error analysis.

Chapter 2 develops a Taylor collocation method for solving Volterra integral equations with proportional delays. The method constructs piecewise polynomial approximations using Taylor series expansions on each subinterval of a uniform partition. A detailed derivation of the numerical scheme is provided, including the determination of initial derivatives from the original equation and the recursive computation of Taylor coefficients on subsequent intervals. The convergence analysis establishes that the numerical solution achieves an order of accuracy equal to the degree of the approximating polynomials, provided the data functions are sufficiently smooth. Numerical experiments confirm the theoretical predictions and demonstrate the method's effectiveness.

Chapter 3 extends the Taylor collocation method to systems of two-dimensional Volterra integral equations without delays. The method employs bivariate Taylor polynomial expansions on each rectangular subdomain, with coefficients determined by solving auxiliary integral equations. A rigorous convergence analysis proves that the error satisfies an optimal bound in terms of the step sizes. Numerical examples illustrate the method's accuracy and confirm the theoretical convergence rates.

Chapter 4 combines and generalizes the previous developments by considering systems of two-dimensional Volterra integral equations with proportional delays. The general form includes matrix coefficients and both standard and delayed integral terms. The Taylor collocation method is extended to handle the added complexity of delay parameters and the matrix structure [11]. A comprehensive convergence analysis demonstrates that the method maintains the same order of accuracy as in the simpler cases. Numerical experiments validate the theoretical findings and showcase the method's applicability to problems with both spatial and temporal memory effects.

Throughout this thesis, emphasis is placed on rigorous mathematical analysis, including existence and uniqueness of solutions, stability of the numerical scheme, and precise error estimates. The theoretical results are complemented by numerical experiments that demonstrate the practical performance of the proposed methods. The Taylor collocation method presented herein offers a unified framework for solving a wide range of Volterra integral equations with delays, providing high accuracy, computational efficiency, and conceptual simplicity.

The contributions of this thesis are both theoretical and practical. On the theoretical side, we provide complete convergence analyses establishing the order of accuracy for each class of equations considered. On the practical side, we present fully implementable algorithms and validate them through extensive numerical testing. The methods developed are suitable for applications in science and engineering where high precision is required.

Future research directions, discussed in the concluding sections of each chapter, include extensions to nonlinear equations, development of adaptive strategies, and applications to more complex functional equations arising in real-world problems. The work presented here lays a solid foundation for these future investigations and contributes to the growing body of knowledge on numerical methods for functional equations with memory and delay.

CHAPTER 1

PRELIMINARIES AND FOUNDATIONAL TOOLS

In this chapter, we present the fundamental concepts and foundational principles that underpin the discussions to follow. These preliminary ideas are essential for building a clear understanding of the topic and establish the key definitions and theorems required. By outlining these core elements, we aim to ensure a logical and coherent progression throughout the text.

1.1 Taylor Series and Taylor's Theorem

The Taylor series, sometimes called the Taylor expansion, is a way of writing a function as an infinite sum built from the values of its derivatives evaluated at one particular point. It serves as a powerful tool for approximating functions that would otherwise be difficult or impossible to compute exactly. For many common functions, this series converges to the actual function near the chosen point. It bears the name of Brook Taylor, who first presented the idea in 1715. When the expansion is centered at zero, it becomes a special case called the Maclaurin series, a form studied in depth by Colin Maclaurin during the eighteenth century.

1.1.1 Taylor's Theorem for Functions of One Variable

Definition 1.1.1 *For a function ψ that is differentiable up to order $r+1$ at the point x_0, Taylor's theorem provides the following approximation of $\psi(x)$ near x_0:*

$$\psi(x) = \psi(x_0) + \psi'(x_0)(x - x_0) + \frac{\psi''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{\psi^{(r)}(x_0)}{r!}(x - x_0)^r + R_r(x),$$

where $R_r(x)$ represents the remainder term that quantifies the error incurred by truncating the series after r terms. The remainder can be written in various forms, with one common expression being the Lagrange form:

$$R_r(x) = \frac{\psi^{(r+1)}(\xi)}{(r+1)!}(x - x_0)^{r+1},$$

where ξ is a point between x_0 and x.

1.1.2 Taylor's Theorem for Functions of Two Variables

For functions depending on two independent variables, Taylor's theorem extends naturally as follows.

Theorem 1.1.1 ([41]) *Let H be m times continuously differentiable on $\Omega = [\alpha, \beta] \times [\gamma, \delta]$ and let $(s_0, t_0) \in \Omega$. Then for all $(s, t) \in \Omega$, we have*

$$\begin{aligned} H(s,t) &= \sum_{k+l=0}^{m-1} \frac{1}{k!\,l!} \frac{\partial^{k+l} H(s_0,t_0)}{\partial s^k \partial t^l} (s-s_0)^k (t-t_0)^l \\ &\quad + \sum_{k+l=m} \frac{1}{k!\,l!} \frac{\partial^{k+l} H(\bar{s},\bar{t})}{\partial s^k \partial t^l} (s-s_0)^k (t-t_0)^l, \end{aligned}$$

where

$$\begin{cases} \bar{s} = \lambda s + (1-\lambda) s_0 \in [\alpha, \beta], \\ \bar{t} = \lambda t + (1-\lambda) t_0 \in [\gamma, \delta], \end{cases} \qquad \lambda \in (0,1).$$

This theorem provides the mathematical foundation for the Taylor collocation methods developed in subsequent chapters. The remainder term allows us to estimate the error when approximating a function by its Taylor polynomial.

1.2 Integral Equations

An integral equation is one in which the unknown function $w(z)$ appears under an integral sign. Such equations serve as a powerful and versatile tool across pure and applied mathematics, and play an important role in modeling a wide range of physical phenomena. Numerous initial-value and boundary-value problems involving ordinary or partial differential equations can be reformulated instead as equivalent integral equation problems.

1.2.1 General Form

A common linear form of an integral equation is expressed as:

$$\lambda(z)w(z) = g(z) + \mu \int_{\alpha(z)}^{\beta(z)} \mathcal{K}(z,\tau)w(\tau)d\tau,$$

Here μ denotes a constant parameter, while $\mathcal{K}(z,\tau)$ is referred to as the kernel of the equation. The unknown function $w(z)$ occurs under the integral sign, and depending on the equation, it may also appear outside the integral simultaneously.

Note that $\lambda(z)$, $g(z)$, and $\mathcal{K}(z,\tau)$ are given functions, whereas the integration bounds $\alpha(z)$ and $\beta(z)$ may be constants, variables, or some mixture of the two.

1.2.2 Classification of Integral Equations

Integral equations can be categorized into a wide range of types, each with distinct theoretical and numerical characteristics. One key distinction is between one-dimensional and multi-dimensional integral equations, and within each of these categories, integral equations can be classified as either linear or nonlinear.

Classification by Integration Limits

A further classification depends on the limits of integration, with two main types:

- **Fredholm integral equations**: If the limits of integration are constant, the equation is referred to as a Fredholm integral equation, which has the form:

$$\lambda(z)w(z) = g(z) + \mu \int_{a}^{b} \mathcal{K}(z,\tau)w(\tau)d\tau,$$

  where a and b are constants.

- **Volterra integral equations**: If at least one limit is variable, the equation is known as a Volterra integral equation, with the form:

$$\lambda(z)w(z) = g(z) + \mu \int_a^z \mathcal{K}(z,\tau)w(\tau)d\tau.$$

Both Fredholm and Volterra integral equations can be either homogeneous or in-homogeneous. If $g(z) = 0$, the equation is classified as a homogeneous Fredholm or Volterra integral equation.

Classification by the Coefficient Function

Additionally, integral equations are also classified based on how the function $\lambda(z)$ appears, as outlined below:

- **First kind**: If $\lambda(z) = 0$, the equation is called a first-kind Fredholm or Volterra integral equation.
- **Second kind**: If $\lambda(z) = 1$, the equation is called a second-kind Fredholm or Volterra integral equation.
- **Third kind**: If $\lambda(z) \neq 0$ and $\lambda(z) \neq 1$, the equation is referred to as a third-kind Fredholm or Volterra integral equation.

1.3 Leibniz Rule for Differentiation of Integrals

One effective method for solving integral equations consists of transforming them into equivalent differential equations. The Leibniz integral rule offers a formula for differentiating under the integral sign and is named after Gottfried Wilhelm Leibniz, who formulated it in the seventeenth century.

Theorem 1.3.1 *(Leibniz integral rule [46]) Consider $\phi(z,\tau)$ as a continuous function, and* *assume that $\partial_z \phi$ is continuous within the domain $[z_0, z_1] \times [\tau_0, \tau_1]$. For the integral expression*

$$\Phi(z) = \int_{\alpha(z)}^{\beta(z)} \phi(z, \tau) d\tau,$$

the derivative of $\Phi(z)$ with respect to z is given by:

$$\Phi'(z) = \phi(z, \beta(z))\beta'(z) - \phi(z, \alpha(z))\alpha'(z) + \int_{\alpha(z)}^{\beta(z)} \partial_z \phi(z, \tau) d\tau.$$

1.4 Piecewise Polynomial Spaces

Piecewise polynomial spaces are mathematical constructs used to approximate functions over an interval by partitioning the domain into smaller subintervals and defining a polynomial on each of them. These spaces serve as a cornerstone in areas such as numerical analysis, the finite element method, computer-aided design, and data interpolation, where they provide a flexible and localized means of capturing complex behavior across a domain.

1.4.1 One-Dimensional Piecewise Polynomial Spaces

Let $\Delta_N = \{x_\ell : 0 = x_0 < x_1 < \cdots < x_N = L\}$ represent a grid (or mesh) on the interval $[0, L]$, with the stepsize defined as $h = \frac{L}{N}$. Define the subintervals $\delta_\ell = [x_\ell, x_{\ell+1}]$ for $\ell = 0, \ldots, N-1$.

Definition 1.4.1 *For the grid Δ_N, the piecewise polynomial space $\mathcal{S}_\rho^{(d)}(\Delta_N)$ is defined for parameters $\rho \geq 0$ and $-1 \leq d \leq \rho$ as follows:*

$$\mathcal{S}_\rho^{(d)}(\Delta_N) := \{u \in C^d([0, L]) : u|_{\delta_\ell} \in \mathbb{P}_\rho,\ 0 \leq \ell \leq N-1\},$$

where $\mathbb{P}_\rho$ represents the space of real polynomials whose degree does not exceed ρ.

It can be shown that $\mathcal{S}_{\rho}^{(d)}(\Delta_N)$ is a (real) linear vector space, and its dimension is given by:

$$\dim \mathcal{S}_{\rho}^{(d)}(\Delta_N) = N(\rho - d) + d + 1.$$

Remark 1.4.1 *The specific piecewise polynomial space $\mathcal{S}_{\rho+d}^{(d)}(\Delta_N)$, where $\rho \geq 1$ and $d \geq -1$, has dimension $N\rho + d + 1$. It can be regarded as the natural collocation space for approximating solutions to initial-value problems associated with ordinary differential equations or Volterra equations.*

The choice of the regularity parameter d is dictated by the number of prescribed initial conditions, while the term $N\rho$ reflects that ρ distinct collocation points should be allocated to each of the N subintervals δ_ℓ. Consequently, the appropriate selection for d is as follows:

- *For Volterra integral equations with no initial conditions, we take $d = -1$.*
- *For first-order ordinary differential equations or Volterra integro-differential equations subject to one initial condition, we set $d = 0$.*
- *For ordinary differential equations or Volterra integro-differential equations of order $k \geq 2$, which involve k initial conditions, we choose $d = k - 1$.*

1.4.2 Two-Dimensional Piecewise Polynomial Spaces

For problems involving two independent variables, we construct tensor-product spline spaces.

Definition 1.4.2 *Let $\Delta_N = \{x_n : 0 = x_0 < x_1 < \cdots < x_N = L\}$ and $\Delta_M = \{y_m : 0 = y_0 < y_1 < \cdots < y_M = K\}$ be partitions of the intervals $[0, L]$ and $[0, K]$, respectively. Let $\Theta_{N,M} = \Delta_N \times \Delta_M$ denote the resulting rectangular grid on $\Omega = [0, L] \times [0, K]$.*

The bivariate piecewise polynomial space $\mathcal{S}_{p-1,q-1}^{(-1)}(\Theta_{N,M})$ is defined as the tensor-product space:

$$\mathcal{S}_{p-1,q-1}^{(-1)}(\Theta_{N,M}) = \mathcal{S}_{p-1}^{(-1)}(\Delta_N) \otimes \mathcal{S}_{q-1}^{(-1)}(\Delta_M),$$

where $\mathcal{S}_{p-1}^{(-1)}(\Delta_N)$ *and* $\mathcal{S}_{q-1}^{(-1)}(\Delta_M)$ *are the univariate spline spaces with discontinuity at the grid points.*

Elements of this space may exhibit jump discontinuities across the interior grid lines $x = x_n$ $(n = 1, \dots, N-1)$ *and* $y = y_m$ $(m = 1, \dots, M-1)$*, and the space itself has dimension* $NMpq$*.*

These spaces will be used throughout this thesis to construct approximate solutions to Volterra integral equations in both one and two dimensions.

1.5 The Collocation Method

The collocation method offers an effective numerical means of approximating solutions to differential and integral equations. It works by choosing a finite-dimensional space of candidate solutionsusually polynomials of some fixed degreetogether with a set of points in the domain known as collocation points, and then determining the solution that satisfies the equation exactly at those chosen points.

1.5.1 General Principles

The collocation method can be summarized in the following steps:

1. Choose a finite-dimensional space $\mathcal{V}$ of approximating functions (e.g., polynomials, splines).

2. Select a set of collocation points $\{t_i\}_{i=1}^{M}$ in the domain.

3. Find $v \in \mathcal{V}$ such that the equation is satisfied exactly at each collocation point.

This approach transforms the original infinite-dimensional problem into a finite system of algebraic equations that can be solved numerically.

1.5.2 Variants of the Collocation Method

The collocation method is highly versatile and can be adapted to a wide range of equation types:

- **For ordinary differential equations**: Prominent variants include the Chebyshev collocation method, which builds solutions from Chebyshev series; the Legendre-Gauss collocation method, tailored to initial value problems for second-order ordinary differential equations via Legendre-Gauss interpolation; the Bernstein collocation method, which relies on Bernstein polynomials to tackle nonlinear ordinary differential equations; and the block hybrid collocation method, formulated specifically for third-order ordinary differential equations.

- **For partial differential equations**: Important approaches include the sparse grid stochastic collocation method, which draws on a Smolyak-type sparse grid to estimate statistical properties of partial differential equation solutions; the wavelet collocation method, grounded in the autocorrelation functions of Daubechies wavelets; and the spline-collocation method, built on B-spline representations.

- **For integral equations**: Specialized techniques encompass the Galerkin collocation method, commonly used for first-kind integral equations, and the iterated collocation method, which is especially well suited to nonlinear second-kind Volterra integral equations.

1.5.3 Advantages of Collocation Methods

The performance of the collocation method depends on several factors, including the choice of collocation points, the selection of basis functions, and the characteristics of the equation under consideration. Proper attention to these elements is essential for ensuring convergence and accuracy in the approximated solution.

A major strength of the collocation method lies in its flexibility. In some implementations, such as node-based techniques, it can be applied without the need for meshing, thereby significantly reducing computational complexity. This ease of implementation, combined with its capacity to accommodate complex boundary conditions and irregular domains, establishes it as a powerful and adaptable tool in numerical analysis. Moreover, with an appropriate choice of basis functions and collocation points, the method can achieve highly accurate approximations across a broad spectrum of problems.

1.6 The Taylor Collocation Method

The Taylor collocation method is a numerical technique employed to solve various types of differential, integral, and integro-differential equations by representing the solution as a finite Taylor series. The approach involves selecting specific pointscalled collocation pointsand determining the coefficients of the Taylor series so that the equation is satisfied exactly at those points.

1.6.1 Historical Development and Literature Review

Initially applied to Volterra integral equations, the method has since been refined and extended by numerous researchers:

- Karamete and Sezer [24] proposed a spectral Taylor matrix collocation approach for linear integro-differential equations, based on truncating the Taylor series. Their technique converted the integro-differential equation into a system of algebraic equations through Taylor matrix representations.
- Laib et al. [28, 29] formulated numerical methods grounded in Taylor polynomials for approximating solutions of delay integral and integro-differential equations. Their Taylor collocation method was applied to systems of nonlinear Volterra

delay integro-differential equations, demonstrating its effectiveness for problems involving delay terms.

- Birem et al. [4, 5] extended the method to two-dimensional Volterra integral equations and Goursat problems, demonstrating that the Taylor collocation approach can be successfully adapted to higher-dimensional settings.

These contributions have significantly broadened the applicability of the Taylor collocation method to a diverse array of complex mathematical problems across various disciplines.

1.6.2 Scope of the Present Work

In this thesis, we aim to build upon these foundational contributions by extending the Taylor collocation method to a wider class of Volterra integral equations, with a particular focus on two-dimensional systems involving proportional delays. By addressing both linear and nonlinear formulations, this research seeks to enhance the method's applicability and to provide more efficient numerical solutions for a broader range of integral equation problems.

The Taylor collocation method offers several key features that make it especially well-suited to these problems:

- High-order accuracy for smooth solutions, with convergence rates matching the degree of the approximating polynomials.
- Conceptual simplicity, grounded in the well-established framework of Taylor series expansions.
- Computational efficiency, as it avoids the need to solve large global systems by constructing approximations locally.
- Flexibility in handling diverse equation types, including those with delay arguments.

1.7 Comparison Theorems and Inequalities

In this section, we review several fundamental theorems concerning discrete and continuous inequalities. These theorems are essential for constructing the proofs required to establish the convergence of the approximate solution to the exact solution, a topic that will be addressed in the chapters that follow.

1.7.1 One-Dimensional Discrete Gronwall Inequality

The following discrete Gronwall inequality is fundamental for analyzing the stability and convergence of numerical methods for one-dimensional problems.

Lemma 1.7.1 ([7]) *Let $(\alpha_k)_{k=0}^n$ be a non-negative sequence and suppose the sequence η_n satisfies*

$$\eta_n \leq \beta_0 + \sum_{i=0}^{n-1} \alpha_i \eta_i, \quad n \geq 0,$$

with $\beta_0 \geq 0$. Then,

$$\eta_n \leq \beta_0 \exp\left(\sum_{j=0}^{n-1} \alpha_j\right), \quad n \geq 0.$$

1.7.2 Continuous Gronwall Inequality

For problems involving continuous functions, the following continuous version of Gronwall's inequality is essential.

Lemma 1.7.2 ([15]) *Let $\rho(t)$ be a positive, monotonic nondecreasing function, and let $u(t) \geq 0$, $v(t) \geq 0$ be continuous. If*

$$u(t) \leq \rho(t) + \int_a^t v(s)u(s), ds, \quad a \leq t \leq b, \tag{1.1}$$

then,

$$u(t) \leq \rho(t) \exp\left(\int_a^t v(s), ds \right), \quad a \leq t \leq b. \tag{1.2}$$

1.7.3 Pachpatte's Inequality for Two Variables

For problems involving sequences depending on two indices, Pachpatte's inequality provides a two-dimensional discrete analogue of Gronwall's inequality.

Lemma 1.7.3 ([35]) *Let $\gamma(i,j)$, $\lambda(i,j)$ and $\mu(i,j)$ be non-negative sequences such that $\lambda(i,j)$ is nondecreasing in each variable i and j. If*

$$\gamma(i,j) \leq \lambda(i,j) + \sum_{p=0}^{i-1} \sum_{q=0}^{j-1} \mu(p,q)\gamma(p,q),$$

for all $i, j \in \mathbb{N}$, then

$$\gamma(i,j) \leq \lambda(i,j) \prod_{p=0}^{i-1} \left[1 + \sum_{q=0}^{j-1} \mu(p,q) \right], \quad i, j \in \mathbb{N}.$$

1.7.4 Two-Dimensional Discrete Gronwall Inequality

A more specialized two-dimensional discrete Gronwall inequality, suitable for certain recursive structures, is given below.

Lemma 1.7.4 ([34]) *Let $\omega_{n,m}$ be a given non-negative sequence, and let $c_i, (i = 1, 2, 3)$ and Θ be strictly positive. If the sequence $\omega_{n,m}$ satisfies*

$$\omega_{n,m} \leq hc_1 \sum_{\xi=0}^{n-1} \omega_{\xi,m} + kc_2 \sum_{\eta=0}^{m-1} \omega_{n,\eta} + hkc_3 \sum_{\xi=0}^{n-1} \sum_{\eta=0}^{m-1} \omega_{\xi,\eta} + \Theta \tag{1.3}$$

for all $n = 0, 1, \ldots, N$, $m = 0, 1, \ldots, M$, then

$$\omega_{n,m} \leq \Theta \exp\left(\kappa(Nh + Mk)\right) \tag{1.4}$$

where $\kappa = \frac{1}{2}\left(c_1 + c_2 + \sqrt{(c_1 + c_2)^2 + 4c_3}\right)$.

1.7.5 Wendroff's Inequality

For continuous functions of two variables, Wendroff's inequality provides a powerful tool for bounding solutions to two-dimensional integral inequalities.

Lemma 1.7.5 ([48]) *Consider ϕ and ψ, two continuous, non-negative functions on $[p,q] \times [r,s]$, together with χ, a non-negative function that is twice continuously differentiable on $[p,q] \times [r,s] \times [p,q] \times [r,s]$. Under these conditions, the inequality*

$$\psi(u,v) \leq \phi(u,v) + \int_p^u \int_r^v \chi(u,v,\tau,\sigma)\, \psi(\tau,\sigma)\, d\sigma\, d\tau, \quad (u,v) \in [p,q] \times [r,s], \tag{1.5}$$

leads to the bound

$$\psi(u,v) \leq \phi(u,v) \exp\left(\int_p^u \int_r^v \Gamma(\tau,\sigma)\, d\sigma\, d\tau\right), \quad (u,v) \in [p,q] \times [r,s], \tag{1.6}$$

where

$$\begin{aligned}\Gamma(u,v) = \chi(u,v,u,v) &+ \int_p^u \partial_1 \chi(u,v,\tau,v)\, d\tau \\ &+ \int_r^v \partial_2 \chi(u,v,u,\sigma)\, d\sigma \\ &+ \int_p^u \int_r^v \partial_2 \partial_1 \chi(u,v,\tau,\sigma)\, d\sigma\, d\tau.\end{aligned}$$

These comparison theorems form the analytical backbone of this thesis. They will be invoked repeatedly in the convergence analyses presented in the following chapters, providing the necessary tools to establish rigorous error bounds for the proposed numerical methods.

CHAPTER 2

TAYLOR COLLOCATION METHOD FOR SOLVING VOLTERRA INTEGRAL EQUATIONS WITH PROPORTIONAL DELAYS

2.1 Introduction

This chapter develops a computational approach for constructing approximate solutions of Volterra integral equations with a proportionally scaled argument, formulated generally as:

$$u(t) = g(t) + \int_0^t \mathcal{M}_1(t,s)u(s)ds + \int_0^{\alpha t} \mathcal{M}_2(t,s)u(s)ds, \quad t \in \Omega := [0, \mathcal{T}], \tag{2.1}$$

where the delay parameter α satisfies $0 < \alpha < 1$. It is assumed throughout that $g(t)$, $\mathcal{M}_1(t,s)$, and $\mathcal{M}_2(t,s)$ are sufficiently smooth on their respective domains $\Omega, \mathcal{R}_1 = \{(t,s) : 0 \leq s \leq t \leq \mathcal{T}\}$, and $\mathcal{R}_2 = \{(t,s) : 0 \leq s \leq \alpha t, t \in \Omega\}$. Questions of existence and uniqueness for solutions of (2.1) have already been addressed in earlier work, such as [26].

A particular instance of equation (2.1) is the delay integral equation:

$$u(t) = g(t) + \int_{\alpha t}^t \mathcal{N}(t,s)u(s)ds, \quad t \in \Omega,$$

which emerges by selecting $\mathcal{M}_1 = -\mathcal{M}_2 =: \mathcal{N}$.

Equations of this type offer a robust mathematical structure for representing systems whose current state $u(t)$ depends upon its complete history and a particular prior state $u(\alpha t)$. Such formulations prove essential across diverse scientific disciplines, including control theory with transmission lags [36], epidemiology considering incubation phases [13], population biology incorporating maturation intervals [27], and materials science characterizing memory phenomena [47].

The advancement of numerical schemes for these equations continues as an active research domain. Multiple strategies have been effectively implemented, including Legendre spectral techniques [3], collocation procedures [43], iterated collocation frameworks [9, 10], least-squares collocation [14], multilevel correction approaches [49], and Chebyshev spectral collocation [55].

This contribution focuses on developing and examining a **Taylor Collocation Method** (TCM) for the equation class represented by (2.1). Our methodology approximates the unknown solution $u(t)$ locally using Taylor expansions, whose coefficients are established by enforcing the equation at a collection of collocation points. The proposed technique offers several distinct advantages, most notably its capacity for high-order accuracy when the true solution exhibits smoothness, its conceptual simplicity that circumvents intricate mesh refinement near the delay argument αt, and its computational efficiency stemming from a direct determination of the solution's series representation.

While Taylor expansions have been utilized in various numerical contexts, their implementation as a direct collocation technique for this particular category of delay Volterra equations remains comparatively unexplored. This gap in the literature establishes the novelty of our contribution. We substantiate the proposed methodology with a rigorous convergence analysis and demonstrate its practical effectiveness through a sequence of computational experiments. The results indicate that the Taylor Collocation Method constitutes a highly accurate and efficient solver, positioning it as a compelling alternative to existing numerical schemes for this significant class of problems.

2.2 Description of the Method

For a fixed positive integer N, let $\mathcal{P}_N$ denote a uniform partition of the solution domain $\Omega = [0, \mathcal{T}]$, determined by the nodal points $t_\nu = \nu\Delta t$, for $\nu = 0, 1, \ldots, N-1$, with step size $\Delta t = \mathcal{T}/N$. The subintervals are given by $\Omega_\nu = [t_\nu, t_{\nu+1})$ for $\nu = 0, 1, \ldots, N-2$, and $\Omega_{N-1} = [t_{N-1}, t_N]$. Let $\mathbb{P}_{r-1}$ denote the set of all real polynomials of degree at most $r-1$. The piecewise polynomial space of degree $r-1$, with no continuity requirements imposed at the partition points, is defined as:

$$\mathcal{S}_{r-1}^{(-1)}(\mathcal{P}_N) = \{w : w_\nu = w|_{\Omega_\nu} \in \mathbb{P}_{r-1},\ \nu = 0, \ldots, N-1\}.$$

This space $\mathcal{S}_{r-1}^{(-1)}(\mathcal{P}_N)$ forms a real vector space of dimension Nr.

The goal is to construct an approximate solution $w \in \mathcal{S}_{r-1}^{(-1)}(\mathcal{P}_N)$ to the exact solution u of (2.1), where the coefficients of the local polynomials w_ν, for $\nu = 0, \ldots, N-1$, are obtained via Taylor series expansions on each subinterval Ω_ν.

The exact solution u is first approximated on the initial interval Ω_0 by the polynomial:

$$w_0(t) = \sum_{j=0}^{r-1} \frac{u^{(j)}(0)}{j!} t^j, \quad t \in \Omega_0. \tag{2.2}$$

The derivatives $u^{(j)}(0)$ are found by differentiating the original equation (2.1) j times:

$$\begin{aligned}
u^{(j)}(t) &= g^{(j)}(t) + \sum_{i=0}^{j-1} \left[\partial_t^{(i)} \mathcal{M}_1(t,t) u(t)\right]^{(j-1-i)} + \int_0^t \partial_t^{(j)} \mathcal{M}_1(t,s) u(s)\, ds \\
&\quad + \alpha \sum_{i=0}^{j-1} \left[\partial_t^{(i)} \mathcal{M}_2(t,\alpha t) u(\alpha t)\right]^{(j-1-i)} + \int_0^{\alpha t} \partial_t^{(j)} \mathcal{M}_2(t,s) u(s)\, ds \\
&= g^{(j)}(t) + \sum_{i=0}^{j-1} \sum_{\ell=0}^{j-1-i} \binom{j-1-i}{\ell} \left[\partial_t^{(i)} \mathcal{M}_1(t,t)\right]^{(j-1-i-\ell)} u^{(\ell)}(t) + \int_0^t \partial_t^{(j)} \mathcal{M}_1(t,s) u(s)\, ds \\
&\quad + \alpha \sum_{i=0}^{j-1} \sum_{\ell=0}^{j-1-i} \binom{j-1-i}{\ell} \left[\partial_t^{(i)} \mathcal{M}_2(t,\alpha t)\right]^{(j-1-i-\ell)} \alpha^\ell u^{(\ell)}(\alpha t) + \int_0^{\alpha t} \partial_t^{(j)} \mathcal{M}_2(t,s) u(s)\, ds,
\end{aligned}$$

which, upon evaluation at $t = 0$, yields:

$$\begin{aligned}
u^{(j)}(0) = g^{(j)}(0) &+ \sum_{i=0}^{j-1} \sum_{\ell=0}^{j-1-i} \binom{j-1-i}{\ell} \left[\partial_t^{(i)} \mathcal{M}_1(t,t)\right]^{(j-1-i-\ell)} (0)\, u^{(\ell)}(0) \\
&+ \alpha \sum_{i=0}^{j-1} \sum_{\ell=0}^{j-1-i} \binom{j-1-i}{\ell} \alpha^\ell \left[\partial_t^{(i)} \mathcal{M}_2(t,\alpha t)\right]^{(j-1-i-\ell)} (0)\, u^{(\ell)}(0),
\end{aligned}$$

for $j = 0, 1, \ldots, r-1$.

Subsequently, for $\nu \in \{1, 2, \ldots, N-1\}$, we approximate u on the interval Ω_ν by the polynomial:

$$w_\nu(t) = \sum_{j=0}^{r-1} \frac{\tilde{w}_\nu^{(j)}(t_\nu)}{j!} (t - t_\nu)^j, \quad t \in \Omega_\nu, \tag{2.3}$$

where $\tilde{w}_\nu$ is the exact solution of the local integral equation:

$$
\begin{aligned}
\tilde{w}_\nu(t) = g(t) &+ \sum_{\kappa=0}^{\nu-1} \int_{t_\kappa}^{t_{\kappa+1}} \mathcal{M}_1(t,s) w_\kappa(s)\, ds + \int_{t_\nu}^{t} \mathcal{M}_1(t,s)\tilde{w}_\nu(s)\, ds \\
&+ \sum_{\kappa=0}^{\nu_0-1} \int_{t_\kappa}^{t_{\kappa+1}} \mathcal{M}_2(t,s) w_\kappa(s)\, ds + \int_{t_{\nu_0}}^{\alpha t} \mathcal{M}_2(t,s) w_{\nu_0}(s)\, ds,
\end{aligned}
\tag{2.4}
$$

with $\alpha t \in \Omega_{\nu_0}$.

The derivatives $\tilde{w}_\nu^{(j)}(t_\nu)$ are obtained by differentiating (2.4) j times:

$$
\begin{aligned}
\tilde{w}_\nu^{(j)}(t) = g^{(j)}(t) &+ \sum_{\kappa=0}^{\nu-1} \int_{t_\kappa}^{t_{\kappa+1}} \partial_t^{(j)} \mathcal{M}_1(t,s) w_\kappa(s)\, ds + \sum_{i=0}^{j-1} \left[\partial_t^{(i)} \mathcal{M}_1(t,t)\tilde{w}_\nu(t)\right]^{(j-1-i)} \\
&+ \int_{t_\nu}^{t} \partial_t^{(j)} \mathcal{M}_1(t,s)\tilde{w}_\nu(s)\, ds + \sum_{\kappa=0}^{\nu_0-1} \int_{t_\kappa}^{t_{\kappa+1}} \partial_t^{(j)} \mathcal{M}_2(t,s) w_\kappa(s)\, ds \\
&+ \alpha \sum_{i=0}^{j-1} \left[\partial_t^{(i)} \mathcal{M}_2(t,\alpha t) w_{\nu_0}(\alpha t)\right]^{(j-1-i)} + \int_{t_{\nu_0}}^{\alpha t} \partial_t^{(j)} \mathcal{M}_2(t,s) w_{\nu_0}(s)\, ds,
\end{aligned}
\tag{2.5}
$$

leading to the following expression at $t = t_\nu$:

$$
\begin{aligned}
\tilde{w}_\nu^{(j)}(t_\nu) = g^{(j)}(t_\nu) &+ \sum_{\kappa=0}^{\nu-1} \int_{t_\kappa}^{t_{\kappa+1}} \partial_t^{(j)} \mathcal{M}_1(t_\nu,s) w_\kappa(s)\, ds \\
&+ \sum_{i=0}^{j-1} \sum_{\ell=0}^{j-1-i} \binom{j-1-i}{\ell} \left[\partial_t^{(i)} \mathcal{M}_1(t,t)\right]^{(j-1-i-\ell)}(t_\nu)\, \tilde{w}_\nu^{(\ell)}(t_\nu) \\
&+ \sum_{\kappa=0}^{\nu_0-1} \int_{t_\kappa}^{t_{\kappa+1}} \partial_t^{(j)} \mathcal{M}_2(t_\nu,s) w_\kappa(s)\, ds \\
&+ \alpha \sum_{i=0}^{j-1} \sum_{\ell=0}^{j-1-i} \binom{j-1-i}{\ell} \alpha^\ell \left[\partial_t^{(i)} \mathcal{M}_2(t,\alpha t)\right]^{(j-1-i-\ell)}(t_\nu)\, w_{\nu_0}^{(\ell)}(\alpha t_\nu),
\end{aligned}
$$

for $j \in \{0,1,\ldots,r-1\}$.

2.3 Convergence Analysis

This section is dedicated to a rigorous convergence analysis of the proposed Taylor Collocation Method. Our primary objective is to establish that the numerical solution w converges to the exact solution u of (2.1) at a rate of $O((\Delta t)^r)$, where r is the degree of the local Taylor polynomials, provided the data is sufficiently smooth. The analysis proceeds in two main stages.

First, we prove a lemma establishing uniform bounds on the derivatives of the local solutions $\tilde{w}_\nu$. This step is essential, as these bounds directly control the local approximation error of the Taylor polynomial.

Second, building upon these preliminary results, we prove the main convergence theorem. The error analysis is conducted separately on the initial interval and the subsequent subintervals.

Lemma 2.3.1 *Let $g, \mathcal{M}_1$, and $\mathcal{M}_2$ be r-times continuously differentiable on their respective domains. Then, there exists a constant $\beta(r) > 0$ such that for all $\nu = 0, 1, \ldots, N-1$ and $j = 0, 1, \ldots, r$,*

$$\|\tilde{w}_\nu^{(j)}\|_{L^\infty(\Omega_\nu)} \leq \beta(r),$$

provided Δt is sufficiently small, where $\tilde{w}_0(t) = u(t)$ for $t \in \Omega_0$.

Proof. The argument proceeds in two parts.

Part 1. There exists $\beta_1(r) > 0$ such that $\|\tilde{w}_\nu^{(j)}\|_{L^\infty(\Omega_\nu)} \leq \beta_1(r)$.

Let $b_\nu^j = \|\tilde{w}_\nu^{(j)}\|_{L^\infty(\Omega_\nu)}$. For all $j = 0, 1, \ldots, r$, we have

$$b_0^j \leq \max\left\{\|u^{(j)}\|_{L^\infty(\Omega_0)},\ j = 0, 1, \ldots, r\right\} = \beta_1(r). \tag{2.6}$$

Part 2. For $\nu = 1, \ldots, N-1$, since $0 < \alpha < 1$, it follows that $\nu_0 \leq \nu$.

Define $\Gamma_\nu = \max\left\{b_\nu^j : j = 0, \ldots, r\right\}$. From (2.5), for all $j = 1, \ldots, r$,

$$b_\nu^j \le C_1 + 2C_2\Delta t \sum_{\kappa=0}^{\nu-1} \Gamma_\kappa + (C_3 + \alpha C_3) \sum_{\ell=0}^{j-1} b_\nu^\ell + 2C_2\Delta t\, b_\nu^0,$$

where

$$\begin{aligned}
C_1 &= \max\left\{\|g^{(j)}\|_{L^\infty(\Omega)} : j = 0, \ldots, r\right\}, \\
C_2 &= \max\left\{\|\partial_t^{(j)}\mathcal{M}_1\|_{L^\infty(\mathcal{R}_1)},\ \|\partial_t^{(j)}\mathcal{M}_2\|_{L^\infty(\mathcal{R}_2)} : j = 0, \ldots, r\right\}, \\
C_3 &= \max\left\{\binom{j-1-i}{\ell}\|[\partial_t^{(i)}\mathcal{M}_1(t,t)]^{(j-1-i-\ell)}\|_{L^\infty(\Omega)},\right. \\
&\left.\binom{j-1-i}{\ell}\|[\partial_t^{(i)}\mathcal{M}_2(t,\alpha t)]^{(j-1-i-\ell)}\|_{L^\infty(\Omega)} : j = 1, \ldots, r;\ i = 0, \ldots, j-1;\ \ell = 0, \ldots, j-1-i\right\}.
\end{aligned}$$

Let $C_4 = C_3 + \alpha C_3 + 2C_2\mathcal{T}$. Then,

$$b_\nu^j \le C_1 + 2C_2\Delta t \sum_{\kappa=0}^{\nu-1} \Gamma_\kappa + C_4 \sum_{\ell=0}^{j-1} b_\nu^\ell. \tag{2.7}$$

Fixing $\nu \in \{1, \ldots, N-1\}$ and applying Lemma 1.7.1 to (2.7) yields:

$$\begin{aligned}
b_\nu^j &\le \left(C_1 + 2C_2\Delta t \sum_{\kappa=0}^{\nu-1} \Gamma_\kappa\right) \exp\left(\sum_{\ell=0}^{j-1} C_4\right) \le \underbrace{C_1 \exp(rC_4)}_{C_5} + \underbrace{2C_2 \exp(rC_4)}_{C_6} \Delta t \sum_{\kappa=0}^{\nu-1} \Gamma_\kappa \\
&\le C_5 + C_6\Delta t \sum_{\kappa=0}^{\nu-1} \Gamma_\kappa.
\end{aligned}$$

Applying Lemma 1.7.1 once more gives, for all $\nu = 1, \ldots, N-1$:

$$\Gamma_\nu \le C_5 \exp\left(C_6 \sum_{\kappa=0}^{\nu-1} \Delta t\right) \le C_5 \exp(C_6\mathcal{T}).$$

The result follows by taking $\beta(r) = \max\{\beta_1(r),\ C_5\exp(C_6\mathcal{T})\}$. ■

Theorem 2.3.1 *Let $g, \mathcal{M}_1$, and $\mathcal{M}_2$ be r-times continuously differentiable on their respective domains. Then, equations* (2.2) *and* (2.3) *define a unique approximation $w \in \mathcal{S}_{r-1}^{(-1)}(\mathcal{P}_N)$, and* *the error* $\mathcal{E} := u - w$ *satisfies*

$$\|\mathcal{E}\|_{L^\infty(\Omega)} \le C(\Delta t)^r$$

for sufficiently small Δt*, where* C *is a constant independent of* Δt*.*

Proof. We again proceed in two steps.

Part 1. There exists $\widetilde{C}_1 > 0$, independent of Δt, such that $\|\mathcal{E}_0\|_{L^\infty(\Omega_0)} \le \widetilde{C}_1(\Delta t)^r$, where $\mathcal{E}_0(t) = |u(t) - w_0(t)|$ for $t \in \Omega_0$.

For $t \in \Omega_0$, Lemma 2.3.1 implies that for sufficiently small Δt,

$$|\mathcal{E}_0(t)| = |u(t) - w_0(t)| \le \frac{\|u^{(r)}\|_{L^\infty(\Omega_0)}}{r!}(\Delta t)^r \le \frac{\beta(r)}{r!}(\Delta t)^r.$$

Part 2. There exists $\widetilde{C}_2 > 0$, independent of Δt, such that $\|\mathcal{E}_\nu\|_{L^\infty(\Omega_\nu)} \le \widetilde{C}_2(\Delta t)^r$, where $\mathcal{E}_\nu(t) = u(t) - w_\nu(t)$ for $\nu \in \{1, \ldots, N-1\}$.

For $t \in \Omega_\nu$, it follows from (2.4) that

$$\begin{aligned} u(t) - \tilde{w}_\nu(t) = &\sum_{\kappa=0}^{\nu-1} \int_{t_\kappa}^{t_{\kappa+1}} \mathcal{M}_1(t,s)\mathcal{E}_\kappa(s)\,ds + \int_{t_\nu}^{t} \mathcal{M}_1(t,s)\left(u(s) - \tilde{w}_\nu(s)\right)ds \\ &+ \sum_{\kappa=0}^{\nu_0-1} \int_{t_\kappa}^{t_{\kappa+1}} \mathcal{M}_2(t,s)\mathcal{E}_\kappa(s)\,ds + \int_{t_{\nu_0}}^{\alpha t} \mathcal{M}_2(t,s)\mathcal{E}_{\nu_0}(s)\,ds. \end{aligned}$$

Since $\nu_0 \le \nu$ and $\alpha t \le t$, we obtain

$$\|u - \tilde{w}_\nu\| \le \sum_{\kappa=0}^{\nu-1} \Delta t\,\overline{\mathcal{K}}\,\|\mathcal{E}_\kappa\| + \overline{\mathcal{K}} \int_{t_\nu}^{t} \|u - \tilde{w}_\nu\|\,ds,$$

where $\overline{\mathcal{K}} = 2\max\left\{\|\mathcal{M}_1\|_{L^\infty(\mathcal{R}_1)},\ \|\mathcal{M}_2\|_{L^\infty(\mathcal{R}_2)}\right\}$.

Applying Lemma 1.7.2:

$$\|u - \tilde{w}_\nu\|_{L^\infty(\Omega_\nu)} \le \left(\sum_{\kappa=0}^{\nu-1} \Delta t\, \overline{\mathcal{K}}\, \|\mathcal{E}_\kappa\|\right) \exp\left(\int_{t_\nu}^{t} \overline{\mathcal{K}}\, ds\right) \le \sum_{\kappa=0}^{\nu-1} \Delta t\, \underbrace{\overline{\mathcal{K}} \exp(\mathcal{T}\overline{\mathcal{K}})}_{\Lambda}\, \|\mathcal{E}_\kappa\|$$

$$\le \sum_{\kappa=0}^{\nu-1} \Delta t\, \Lambda\, \|\mathcal{E}_\kappa\|.$$

Consequently,

$$\|\mathcal{E}_\nu\| \le \|u - \tilde{w}_\nu\| + \|\tilde{w}_\nu - w_\nu\| \le \sum_{\kappa=0}^{\nu-1} \Delta t\, \Lambda\, \|\mathcal{E}_\kappa\| + \frac{\|\tilde{w}_\nu^{(r)}\|_{L^\infty(\Omega_\nu)}}{r!} (\Delta t)^r.$$

Invoking Lemma 2.3.1 and then Lemma 1.7.1, we find:

$$\|\mathcal{E}_\nu\| \le \frac{\beta(r)}{r!} (\Delta t)^r \exp\left(\sum_{\kappa=0}^{\nu-1} \Delta t\, \Lambda\right) \le \frac{\beta(r)}{r!} \exp(\mathcal{T}\Lambda)(\Delta t)^r.$$

Taking $\widetilde{C}_2 = \frac{\beta(r)}{r!} \exp(\mathcal{T}\Lambda)$ completes this part.

The theorem is proved by setting $C = \max\left\{\widetilde{C}_1, \widetilde{C}_2\right\}$. ■

2.4 Numerical Experiments

This section presents the numerical validation of the Taylor Collocation Method. To test its performance, the scheme is applied to several Volterra integral equations involving proportional delays, each with a known exact solution available for comparison. All calculations reported below were carried out using Maple 17.

Example 2.4.1 ([54]) *Consider the following pantograph-type Volterra integral equation:*

$$u(t) = (-t^2 + t + 1)e^t + t(\alpha t - 1)e^{\alpha t} + \int_{\alpha t}^{t} ts\, u(s)\, ds,$$

where the kernels are $\mathcal{M}_1(t,s) = -\mathcal{M}_2(t,s) = ts$, and the exact solution is $u(t) = e^t$.

Table 2.1 and Figure 2.1 display the absolute errors between the exact and approximate solutions for $N = 10$ and various values of polynomial degree r and α. Table 2.2 shows a comparison between the sinc-collocation method [54] and TCM for $r = 7$.

Table 2.1: Absolute errors for Example 2.4.1

t	$\alpha = 0.1$		$\alpha = 0.5$		$\alpha = 0.99$	
	$r = 5$	$r = 7$	$r = 5$	$r = 7$	$r = 5$	$r = 7$
0.1	4.64×10^{-10}	1.87×10^{-11}	4.67×10^{-10}	1.87×10^{-11}	7.40×10^{-11}	2.06×10^{-11}
0.2	4.11×10^{-10}	6.40×10^{-11}	4.14×10^{-10}	6.40×10^{-11}	3.24×10^{-10}	1.10×10^{-9}
0.3	2.77×10^{-10}	4.67×10^{-10}	3.82×10^{-10}	4.37×10^{-10}	4.56×10^{-10}	8.33×10^{-10}
0.4	1.21×10^{-9}	4.89×10^{-10}	1.24×10^{-9}	4.83×10^{-10}	1.84×10^{-10}	1.96×10^{-10}
0.5	1.62×10^{-9}	5.75×10^{-10}	1.67×10^{-9}	7.85×10^{-10}	1.32×10^{-9}	2.08×10^{-10}
0.6	2.87×10^{-9}	3.22×10^{-10}	2.00×10^{-9}	8.20×10^{-10}	1.70×10^{-11}	8.31×10^{-10}
0.7	4.86×10^{-9}	1.10×10^{-9}	3.54×10^{-9}	4.71×10^{-10}	4.19×10^{-10}	6.39×10^{-10}
0.8	8.00×10^{-9}	1.98×10^{-10}	4.60×10^{-9}	4.58×10^{-10}	1.34×10^{-9}	6.85×10^{-10}
0.9	1.17×10^{-8}	1.29×10^{-10}	7.78×10^{-9}	1.29×10^{-9}	1.10×10^{-9}	1.24×10^{-9}
1.0	2.22×10^{-7}	2.63×10^{-10}	2.17×10^{-7}	1.43×10^{-9}	1.15×10^{-7}	1.52×10^{-9}

Table 2.2: Maximum error comparison between TCM and Sinc-Collocation for Example 2.4.1

α	TCM		Sinc-Collocation [54]	
	$N = 10$	$N = 30$	$N = 10$	$N = 30$
0.01	1.11×10^{-9}	1.84×10^{-9}	3.73×10^{-6}	5.50×10^{-10}
0.09	1.11×10^{-9}	1.92×10^{-9}	4.57×10^{-6}	3.61×10^{-9}
0.1	1.10×10^{-9}	1.91×10^{-9}	4.19×10^{-6}	5.25×10^{-9}
0.5	1.43×10^{-9}	1.87×10^{-9}	8.84×10^{-5}	4.09×10^{-7}
0.99	1.52×10^{-9}	1.52×10^{-9}	1.75×10^{-5}	7.55×10^{-8}

Example 2.4.2 *To provide a comprehensive evaluation of the proposed Taylor Collocation Method (TCM), we compare its performance with two classical quadrature methods: the repeated trapezoid quadrature method and the repeated Simpson quadrature method with variable step [37]. For this comparison, we consider the following Volterra integral equation:*

$$u(t) = t + \int_0^t \mathcal{M}_1(t,s)u(s)\,ds, \quad t \in \mathcal{I} := [0, \Theta] \tag{2.1}$$

with $\Theta = 2$ and the kernel $\mathcal{M}_1(t,s) = \frac{1}{5}ts$. The exact solution of this equation is $u(t) = te^{t^3/15}$. We compute the absolute errors at equally spaced points $t = 0, 0.2, \ldots, 2$ for each method.

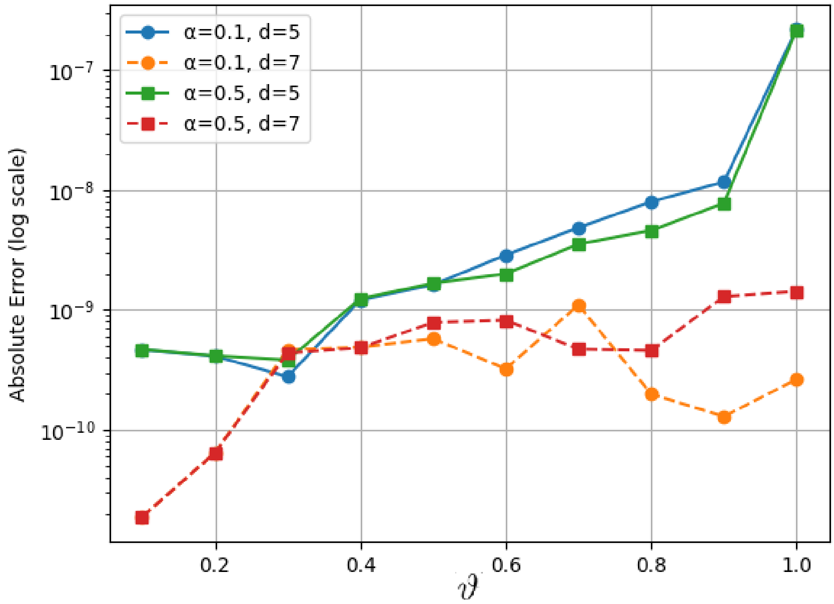


Figure 2.1: Absolute error for $\alpha = 0.1$ and $\alpha = 0.5$ for Example 2.4.1.

Table 2.3 and Figure 2.2 present the comparison of the TCM ($r = 10$) with the repeated trapezoid method and the repeated Simpson method.

Table 2.3: Comparison of absolute errors for Example 2.4.2.

t	TCM ($r = 10, N = 10$)	Repeated Trapezoid	Repeated Simpson
0.0	0	0	0
0.2	2.93×10^{-10}	1.30×10^{-5}	3.56×10^{-9}
0.4	1.48×10^{-10}	5.40×10^{-5}	2.89×10^{-8}
0.6	3.59×10^{-10}	1.26×10^{-4}	1.01×10^{-7}
0.8	1.02×10^{-9}	2.38×10^{-4}	2.57×10^{-7}
1.0	1.35×10^{-10}	4.11×10^{-4}	5.61×10^{-7}
1.2	2.32×10^{-10}	6.83×10^{-4}	1.14×10^{-6}
1.4	3.20×10^{-9}	1.13×10^{-3}	2.23×10^{-6}
1.6	3.29×10^{-8}	1.87×10^{-3}	4.35×10^{-6}
1.8	4.50×10^{-8}	3.15×10^{-3}	8.55×10^{-6}
2.0	8.27×10^{-8}	5.43×10^{-3}	1.71×10^{-5}

2.5 Conclusion

This chapter presented a Taylor Collocation Method (TCM) for solving linear Volterra integral equations with a proportional delay. The method constructs local Taylor poly-

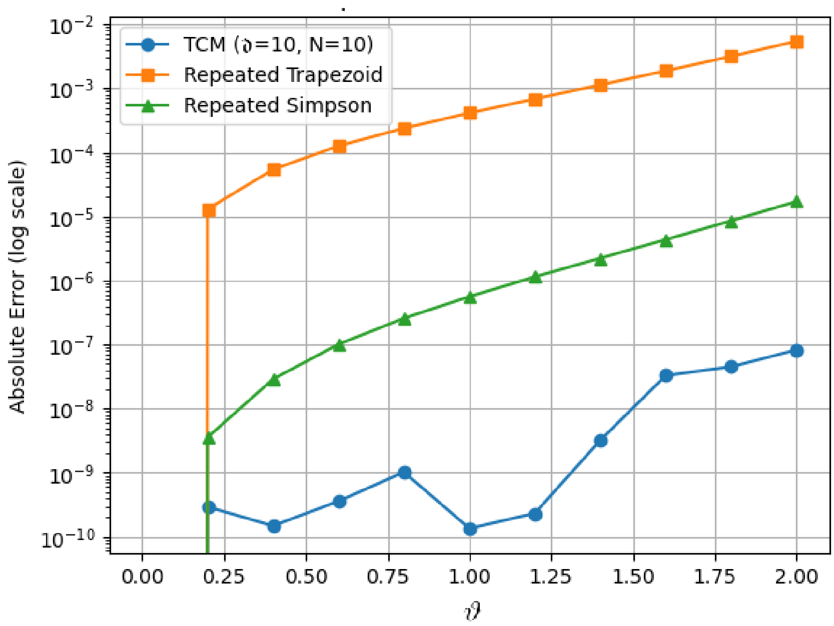


Figure 2.2: Comparison of absolute errors for Example 2.4.2.

nomial approximations on a uniform mesh, determining coefficients through collocation. This approach avoids complex mesh refinement near the delay point and circumvents solving large global systems.

A rigorous convergence analysis established that the method achieves $O((\Delta t)^r)$ accuracy for sufficiently smooth solutions. Numerical experiments confirmed this high-order convergence across various delay parameters, demonstrating the method's robustness and reliability.

Future work may extend this approach to nonlinear problems, more complex functional equations, and adaptive step-size strategies.

CHAPTER 3

TAYLOR COLLOCATION METHOD FOR SOLVING A SYSTEM OF TWO-DIMENSIONAL VOLTERRA INTEGRAL EQUATIONS

3.1 Introduction

Systems of two-dimensional Volterra integral equations (2D-VIEs) are fundamental mathematical tools for modeling complex phenomena in diverse scientific and engineering domains. These systems arise naturally in fluid dynamics, signal processing, control theory, mathematical biology, image analysis, environmental modeling, and electromagnetic wave propagation [1, 12, 22]. A typical linear system of 2D-VIEs has the form:

$$\begin{cases} \psi_1(x,y) = f_1(x,y) + \int_0^x \int_0^y \left[H_{1,1}(x,y,t,s)\psi_1(t,s) + H_{1,2}(x,y,t,s)\psi_2(t,s)\right] dsdt, \\ \psi_2(x,y) = f_2(x,y) + \int_0^x \int_0^y \left[H_{2,1}(x,y,t,s)\psi_1(t,s) + H_{2,2}(x,y,t,s)\psi_2(t,s)\right] dsdt, \end{cases} \tag{3.1}$$

where $(x,y) \in \Omega = [0,\alpha] \times [0,\beta]$, and the functions f_d and $H_{d,\delta}$ $(d,\delta = 1,2)$ are given sufficiently smooth functions defined on appropriate domains.

While substantial progress has been made in developing numerical methods for one-dimensional integral equations—including the modified homotopy perturbation method [23], semi-orthogonal B-spline collocation [38], and Hybrid Legendre Block-Pulse functions [39]—computational techniques for systems of two-dimensional integral equations remain relatively scarce. Notable existing approaches include operational matrices of hybrid functions [32], Legendre wavelet methods [25], modified Newton methods for nonlinear problems, adaptive iterative regularization schemes [16, 17], extensions of modified moving least squares [33], and spectral collocation methods [53].

This chapter develops and analyzes the Taylor Collocation Method (TCM) for solving systems of linear two-dimensional Volterra integral equations of the form (3.1), extending foundational work in [4, 5, 28, 29]. The method constructs approximations directly through Taylor series expansions within a polynomial spline framework, preserving the integral structure and leveraging solution regularity for accurate results.

The existence and uniqueness of solutions for such systems are established in [11, 41]. Our primary contributions are: (i) a complete algorithmic description of TCM for 2D-VIE systems, (ii) rigorous convergence analysis with error bounds, and (iii) numerical verification of theoretical predictions.

The chapter is organized as follows: Section 2 details the method construction using Taylor polynomial approximations. Section 3 presents the complete convergence analysis. Section 4 demonstrates the method's effectiveness through numerical examples. Section 5 offers conclusions.

3.2 Method Description

We seek numerical approximations to the solution pair (ψ_1, ψ_2) of system (3.1) over the rectangular domain $\Omega = [0, \alpha] \times [0, \beta]$. Define uniform partitions:

$$\mathcal{P}_N = \{x_i = i\Delta x,\ i = 0, \ldots, N\},$$
$$\mathcal{P}_M = \{y_j = j\Delta y,\ j = 0, \ldots, M\},$$

with step sizes $\Delta x = \frac{\alpha}{N}$ and $\Delta y = \frac{\beta}{M}$.

The resulting grid is:

$$\mathcal{G}_{N,M} = \mathcal{P}_N \times \mathcal{P}_M = \{(x_n, y_m) : 0 \leq n \leq N,\ 0 \leq m \leq M\}.$$

Define subintervals:

$$I_n = [x_n, x_{n+1}),\ n = 0, \ldots, N-2; \quad I_{N-1} = [x_{N-1}, x_N],$$
$$J_m = [y_m, y_{m+1}),\ m = 0, \ldots, M-2; \quad J_{M-1} = [y_{M-1}, y_M],$$

and sub-rectangles:

$$R_{n,m} := I_n \times J_m, \quad n = 0, \ldots, N-1;\ m = 0, \ldots, M-1.$$

Let $\mathcal{Q}_{p-1,p-1}$ denote the space of bivariate polynomials of degree at most $p-1$ in each variable. The polynomial spline space is:

$$\mathcal{S}^{(-1)}_{p-1,p-1}(\mathcal{G}_{N,M}) = \{w : w_{n,m} = w|_{R_{n,m}} \in \mathcal{Q}_{p-1,p-1},\ n = 0,\dots,N-1;\ m = 0,\dots,M-1\}.$$

This space has dimension NMp^2. The solution ψ satisfies boundary conditions:

$$\psi(x,0) = f(x,0), \quad 0 \le x \le \alpha,$$
$$\psi(0,y) = f(0,y), \quad 0 \le y \le \beta.$$

Approximation on Initial Rectangle $R_{0,0}$

On $R_{0,0}$, approximate ψ_d by:

$$w_{d,0,0}(x,y) = \sum_{i+j=0}^{p-1} \frac{1}{i!j!} \frac{\partial^{i+j}\psi_d(0,0)}{\partial x^i \partial y^j} x^i y^j, \quad (x,y) \in R_{0,0},\ d = 1,2. \tag{3.2}$$

The derivatives $\frac{\partial^{i+j}\psi_d(0,0)}{\partial x^i \partial y^j}$ are obtained by differentiating (3.1). Differentiating j times with respect to y and then i times with respect to x yields:

$$\begin{aligned}
\frac{\partial^{i+j}\psi_d(x,y)}{\partial x^i \partial y^j} &= \partial_1^{(i)}\partial_2^{(j)} f_d(x,y) + \sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q} \binom{r}{l}\binom{q}{\eta} \times \\
&\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\bigg|_{t=x} \left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)} H_{d,1}(x,y,t,y)\right]\right)\right] \frac{\partial^{\eta+l}\psi_1(x,y)}{\partial x^\eta \partial y^l} \\
&+ \frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\bigg|_{t=x} \left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)} H_{d,2}(x,y,t,y)\right]\right)\right] \frac{\partial^{\eta+l}\psi_2(x,y)}{\partial x^\eta \partial y^l} \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_0^x \frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)} H_{d,1}(x,y,t,y)]\right] \frac{\partial^l \psi_1(t,y)}{\partial y^l} \\
&+ \frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)} H_{d,2}(x,y,t,y)]\right] \frac{\partial^l \psi_2(t,y)}{\partial y^l} dt
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_0^y \frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}H_{d,1}(x,y,x,s)\right]\frac{\partial^\eta\psi_1(x,s)}{\partial x^\eta}\\
&+\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}H_{d,2}(x,y,x,s)\right]\frac{\partial^\eta\psi_2(x,s)}{\partial x^\eta}ds\\
&+\int_0^x\int_0^y \partial_1^{(i)}\partial_2^{(j)}H_{d,1}(x,y,t,s)\psi_1(t,s)+\partial_1^{(i)}\partial_2^{(j)}H_{d,2}(x,y,t,s)\psi_2(t,s)dsdt.
\end{aligned}
$$

Setting $(x, y) = (0, 0)$ gives:

$$
\begin{aligned}
&\frac{\partial^{i+j}\psi_d(0,0)}{\partial x^i\partial y^j} = \partial_1^{(i)}\partial_2^{(j)}f_d(0,0)\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\times\\
&\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\left.\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\right|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}H_{d,1}(x,y,t,y)\right]\right)\right]_{x=y=0}\frac{\partial^{\eta+l}\psi_1(0,0)}{\partial x^\eta\partial y^l}\\
&+\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\left.\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\right|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}H_{d,2}(x,y,t,y)\right]\right)\right]_{x=y=0}\frac{\partial^{\eta+l}\psi_2(0,0)}{\partial x^\eta\partial y^l}.
\end{aligned}
$$

Approximation on Rectangles $R_{n,0}$ for $n \geq 1$

For $n = 1, \ldots, N-1$, we approximate ψ by the polynomials:

$$
w_{d,n,0}(x,y) = \sum_{i+j=0}^{p-1}\frac{1}{i!j!}\frac{\partial^{i+j}\tilde{w}_{d,n,0}(x_n,0)}{\partial x^i\partial y^j}(x-x_n)^i y^j, \quad d=1,2;\ (x,y)\in R_{n,0}, \tag{3.3}
$$

where $\tilde{w}_{d,n,0}$ solves:

$$
\begin{aligned}
\tilde{w}_{d,n,0}(x,y) &= f_d(x,y)+\sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\int_0^y\left[H_{d,1}(x,y,t,s)w_{1,\xi,0}(t,s)+H_{d,2}(x,y,t,s)w_{2,\xi,0}(t,s)\right]dsdt\\
&+\int_{x_n}^x\int_0^y\left[H_{d,1}(x,y,t,s)\tilde{w}_{1,n,0}(t,s)+H_{d,2}(x,y,t,s)\tilde{w}_{2,n,0}(t,s)\right]dsdt. \qquad (3.4)
\end{aligned}
$$

Differentiating (3.4) j times with respect to y and i times with respect to x, we obtain:

$$
\begin{aligned}
&\frac{\partial^{i+j}\tilde{w}_{d,n,0}(x,y)}{\partial x^i \partial y^j} = \partial_1^{(i)}\partial_2^{(j)} f_d(x,y) \\
&+ \sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\xi}^{x_{\xi+1}} \frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}H_{d,1}(x,y,t,y)]\right]\frac{\partial^l w_{1,\xi,0}(t,y)}{\partial y^l} \\
&+ \frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}H_{d,2}(x,y,t,y)]\right]\frac{\partial^l w_{2,\xi,0}(t,y)}{\partial y^l}dt \\
&+ \sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\int_0^y \partial_1^{(i)}\partial_2^{(j)}H_{d,1}(x,y,t,s)w_{1,\xi,0}(t,s) + \partial_1^{(i)}\partial_2^{(j)}H_{d,2}(x,y,t,s)w_{2,\xi,0}(t,s)dsdt \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\times \\
&\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\left.\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\right|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}H_{d,1}(x,y,t,y)\right]\right)\right]\frac{\partial^{\eta+l}\tilde{w}_{1,n,0}(x,y)}{\partial x^\eta \partial y^l} \\
&+ \frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\left.\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\right|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}H_{d,2}(x,y,t,y)\right]\right)\right]\frac{\partial^{\eta+l}\tilde{w}_{2,n,0}(x,y)}{\partial x^\eta \partial y^l} \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_n}^{x}\frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}H_{d,1}(x,y,t,y)]\right]\frac{\partial^l \tilde{w}_{1,n,0}(t,y)}{\partial y^l} \\
&+ \frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}H_{d,2}(x,y,t,y)]\right]\frac{\partial^l \tilde{w}_{2,n,0}(t,y)}{\partial y^l}dt \\
&+ \sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_0^y \frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}H_{d,1}(x,y,x,s)\right]\frac{\partial^\eta \tilde{w}_{1,n,0}(x,s)}{\partial x^\eta} \\
&+ \frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}H_{d,2}(x,y,x,s)\right]\frac{\partial^\eta \tilde{w}_{2,n,0}(x,s)}{\partial x^\eta}ds \\
&+ \int_{x_n}^{x}\int_0^y \partial_1^{(i)}\partial_2^{(j)}H_{d,1}(x,y,t,s)\tilde{w}_{1,n,0}(t,s) + \partial_1^{(i)}\partial_2^{(j)}H_{d,2}(x,y,t,s)\tilde{w}_{2,n,0}(t,s)dsdt \qquad (3.5)
\end{aligned}
$$

then evaluating at $(x_n, 0)$,

$$
\begin{aligned}
&\frac{\partial^{i+j}\tilde{w}_{d,n,0}(x_n,0)}{\partial x^i \partial y^j} = \partial_1^{(i)}\partial_2^{(j)} f_d(x_n,0) \\
&+ \sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\xi}^{x_{\xi+1}} \frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}H_{d,1}(x,y,t,y)]\right]_{x=x_n,y=0}\frac{\partial^l w_{1,\xi,0}(t,0)}{\partial y^l} \\
&+ \frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}H_{d,2}(x,y,t,y)]\right]_{x=x_n,y=0}\frac{\partial^l w_{2,\xi,0}(t,0)}{\partial y^l}dt
\end{aligned}
$$

$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\times$$
$$\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\bigg|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}H_{d,1}(x,y,t,y)\right]\right)\right]_{x=x_n,y=0}\frac{\partial^{\eta+l}\tilde{w}_{1,n,0}(x_n,0)}{\partial x^{\eta}\partial y^{l}}$$
$$+\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\bigg|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}H_{d,2}(x,y,t,y)\right]\right)\right]_{x=x_n,y=0}\frac{\partial^{\eta+l}\tilde{w}_{2,n,0}(x_n,0)}{\partial x^{\eta}\partial y^{l}}.$$

Approximation on Rectangles $R_{n,m}$ for $m \geq 1$

we approximate ψ by $w_{n,m}$ in the rectangles $R_{n,m}, n = 0, \ldots, N-1$ and $m = 1, \ldots, M-1$ such that

$$w_{d,n,m}(x,y) = \sum_{i+j=0}^{p-1}\frac{1}{i!j!}\frac{\partial^{i+j}\tilde{w}_{d,n,m}(x_n,y_m)}{\partial x^i\partial y^j}(x-x_n)^i(y-y_m)^j, \quad d=1,2;\ (x,y)\in R_{n,m}, \tag{3.6}$$

where $\tilde{w}_{d,n,m}$ solves:

$$\begin{aligned}\tilde{w}_{d,n,m}(x,y) &= f_d(x,y)\\ &+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{x_\xi}^{x_{\xi+1}}\int_{y_\rho}^{y_{\rho+1}}\left[H_{d,1}(x,y,t,s)w_{1,\xi,\rho}(t,s)+H_{d,2}(x,y,t,s)w_{2,\xi,\rho}(t,s)\right]dsdt\\ &+\sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\int_{y_m}^{y}\left[H_{d,1}(x,y,t,s)w_{1,\xi,m}(t,s)+H_{d,2}(x,y,t,s)w_{2,\xi,m}(t,s)\right]dsdt\\ &+\sum_{\rho=0}^{m-1}\int_{x_n}^{x}\int_{y_\rho}^{y_{\rho+1}}\left[H_{d,1}(x,y,t,s)w_{1,n,\rho}(t,s)+H_{d,2}(x,y,t,s)w_{2,n,\rho}(t,s)\right]dsdt\\ &+\int_{x_n}^{x}\int_{y_m}^{y}\left[H_{d,1}(x,y,t,s)\tilde{w}_{1,n,m}(t,s)+H_{d,2}(x,y,t,s)\tilde{w}_{2,n,m}(t,s)\right]dsdt.\end{aligned} \tag{3.7}$$

Differentiating (3.7) j-times with respect to y then i-times with respect to x, we obtain

$$\frac{\partial^{i+j}\tilde{w}_{d,n,m}(x,y)}{\partial x^i\partial y^j} = \partial_1^{(i)}\partial_2^{(j)}f_d(x,y)$$

$$
\begin{aligned}
&+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{x_\xi}^{x_{\xi+1}}\int_{y_\rho}^{y_{\rho+1}}\partial_1^{(i)}\partial_2^{(j)}H_{d,1}(x,y,t,s)w_{1,\xi,\rho}(t,s)+\partial_1^{(i)}\partial_2^{(j)}H_{d,2}(x,y,t,s)w_{2,\xi,\rho}(t,s)dsdt\\
&+\sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\xi}^{x_{\xi+1}}\frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}H_{d,1}(x,y,t,y)]\right]\frac{\partial^l w_{1,\xi,m}(t,y)}{\partial y^l}\\
&+\frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}H_{d,2}(x,y,t,y)]\right]\frac{\partial^l w_{2,\xi,m}(t,y)}{\partial y^l}dt\\
&+\sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\int_{y_m}^{y}\partial_1^{(i)}\partial_2^{(j)}H_{d,1}(x,y,t,s)w_{1,\xi,m}(t,s)+\partial_1^{(i)}\partial_2^{(j)}H_{d,2}(x,y,t,s)w_{2,\xi,m}(t,s)dsdt\\
&+\sum_{\rho=0}^{m-1}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{y_\rho}^{y_{\rho+1}}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}H_{d,1}(x,y,x,s)\right]\frac{\partial^\eta w_{1,n,\rho}(x,s)}{\partial x^\eta}\\
&+\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}H_{d,2}(x,y,x,s)\right]\frac{\partial^\eta w_{2,n,\rho}(x,s)}{\partial x^\eta}ds\\
&+\sum_{\rho=0}^{m-1}\int_{x_n}^{x}\int_{y_\rho}^{y_{\rho+1}}\partial_1^{(i)}\partial_2^{(j)}H_{d,1}(x,y,t,s)w_{1,n,\rho}(t,s)+\partial_1^{(i)}\partial_2^{(j)}H_{d,2}(x,y,t,s)w_{2,n,\rho}(t,s)dsdt\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\times\\
&\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\bigg|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}H_{d,1}(x,y,t,y)\right]\right)\right]\frac{\partial^{\eta+l}\tilde{w}_{1,n,m}(x,y)}{\partial x^\eta\partial y^l}\\
&+\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\bigg|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}H_{d,2}(x,y,t,y)\right]\right)\right]\frac{\partial^{\eta+l}\tilde{w}_{2,n,m}(x,y)}{\partial x^\eta\partial y^l}\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_n}^{x}\frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}H_{d,1}(x,y,t,y)]\right]\frac{\partial^l \tilde{w}_{1,n,m}(t,y)}{\partial y^l}\\
&+\frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}H_{d,2}(x,y,t,y)]\right]\frac{\partial^l \tilde{w}_{2,n,m}(t,y)}{\partial y^l}dt\\
&+\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{y_m}^{y}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}H_{d,1}(x,y,x,s)\right]\frac{\partial^\eta \tilde{w}_{1,n,m}(x,s)}{\partial x^\eta}\\
&+\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}H_{d,2}(x,y,x,s)\right]\frac{\partial^\eta \tilde{w}_{2,n,m}(x,s)}{\partial x^\eta}ds\\
&+\int_{x_n}^{x}\int_{y_m}^{y}\partial_1^{(i)}\partial_2^{(j)}H_{d,1}(x,y,t,s)\tilde{w}_{1,n,m}(t,s)+\partial_1^{(i)}\partial_2^{(j)}H_{d,2}(x,y,t,s)\tilde{w}_{2,n,m}(t,s)dsdt
\end{aligned}
\tag{3.8}
$$

and evaluating at (x_n, y_m) gives:

$$
\begin{aligned}
&\frac{\partial^{i+j}\tilde{w}_{d,n,m}(x_n,y_m)}{\partial x^i \partial y^j} = \partial_1^{(i)}\partial_2^{(j)} f_d(x_n,y_m) \\
&+ \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{x_\xi}^{x_{\xi+1}}\int_{y_\rho}^{y_{\rho+1}} \partial_1^{(i)}\partial_2^{(j)} H_{d,1}(x_n,y_m,t,s) w_{1,\xi,\rho}(t,s) \\
&+ \partial_1^{(i)}\partial_2^{(j)} H_{d,2}(x_n,y_m,t,s) w_{2,\xi,\rho}(t,s) ds dt \\
&+ \sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_\xi}^{x_{\xi+1}} \frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)} H_{d,1}(x,y,t,y)]\right]_{x=x_n,y=y_m} \frac{\partial^l w_{1,\xi,m}(t,y_m)}{\partial y^l} \\
&+ \frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)} H_{d,2}(x,y,t,y)]\right]_{x=x_n,y=y_m} \frac{\partial^l w_{2,\xi,m}(t,y_m)}{\partial y^l} dt \\
&+ \sum_{\rho=0}^{m-1}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{y_\rho}^{y_{\rho+1}} \frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)} H_{d,1}(x,y,x,s)\right]_{x=x_n,y=y_m} \frac{\partial^\eta w_{1,n,\rho}(x_n,s)}{\partial x^\eta} \\
&+ \frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)} H_{d,2}(x,y,x,s)\right]_{x=x_n,y=y_m} \frac{\partial^\eta w_{2,n,\rho}(x_n,s)}{\partial x^\eta} ds \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta} \\
&\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\left.\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\right|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)} H_{d,1}(x,y,t,y)\right]\right)\right]_{x=x_n,y=y_m} \frac{\partial^{\eta+l}\tilde{w}_{1,n,m}(x_n,y_m)}{\partial x^\eta \partial y^l} \\
&+ \frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\left.\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\right|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)} H_{d,2}(x,y,t,y)\right]\right)\right]_{x=x_n,y=y_m} \frac{\partial^{\eta+l}\tilde{w}_{2,n,m}(x_n,y_m)}{\partial x^\eta \partial y^l}
\end{aligned}
$$

for $n = 0, ..., N-1$ and $m = 1, ..., M-1$.

3.3 Convergence Analysis

We establish the convergence properties of the proposed method. Consider $L^\infty(\Omega)$ with the norm:

$$||\varphi||_{L^\infty(\Omega)} = \inf\{C > 0 : |\varphi(x,y)| \leq C \text{ for a.e. } (x,y) \in \Omega\}.$$

Boundedness of Derivatives

We need the following lemma to prove the convergence of the presented method.

Lemma 3.3.1 *Let f_d and $H_{d,\delta}$, $d,\delta = 1,2$ be p times continuously differentiable on their respective domains. Then, there exists a positive number $\Lambda(p)$ such that for all $n = 0,\ldots,N-1$, $m = 0,\ldots,M-1$, $d = 1,2$ and $i+j = 0,1,\ldots,p$, we have,*

$$\left\|\frac{\partial^{i+j}\tilde{w}_{d,n,m}}{\partial x^i \partial y^j}\right\|_{L^\infty(R_{n,m})} \leq \Lambda(p),$$

where $\tilde{w}_{d,0,0}(x,y) = \psi_d(x,y)$ for $(x,y) \in R_{0,0}$.

Proof. Let $a^{i,j}_{d,n,m} = \left\|\frac{\partial^{i+j}\tilde{w}_{d,n,m}}{\partial x^i \partial y^j}\right\|_{L^\infty(R_{n,m})}$, we have for all $i+j = 0,1,\ldots,p$,

$$a^{i,j}_{d,0,0} \leq \max\left\{\left\|\frac{\partial^{i+j}\psi_d}{\partial x^i \partial y^j}\right\|_{L^\infty(R_{0,0})}, i+j = 0,1,\ldots,p; d = 1,2\right\} = \Lambda_1(p). \tag{3.9}$$

From (3.5), we have for all $n = 1,\ldots,N-1$; $i+j = 0,1,\ldots,p$ and $d = 1,2$

$$\begin{aligned}
a^{i,j}_{d,n,0} \leq\ & c_{d,1} + c_{d,2}\sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\int_{x_\xi}^{x_{\xi+1}}(a^{0,l}_{1,\xi,0} + a^{0,l}_{2,\xi,0})dt \\
&+ c_{d,3}\sum_{\xi=0}^{n-1}\int_{x_\xi}^{x_{\xi+1}}\int_0^y (a^{0,0}_{1,\xi,0} + a^{0,0}_{2,\xi,0})dsdt \\
&+ c_{d,4}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}(a^{\eta,l}_{1,n,0} + a^{\eta,l}_{2,n,0}) + c_{d,2}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\int_{x_n}^{x}(a^{0,l}_{1,n,0} + a^{0,l}_{2,n,0})dt \\
&+ c_{d,5}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\int_0^y (a^{\eta,0}_{1,n,0} + a^{\eta,0}_{2,n,0})ds + c_{d,3}\int_{x_n}^{x}\int_0^y (a^{0,0}_{1,n,0} + a^{0,0}_{2,n,0})dsdt,
\end{aligned}$$

where, for all $i+j = 0,1,\ldots,p$

$c_{d,1} = \max\left\{\left\|\partial_1^{(i)}\partial_2^{(j)} f_d\right\|_{L^\infty(\Omega)}\right\},$

$$c_{d,2}=\max\left\{\binom{r}{l}\left\|\frac{\partial^i}{\partial x^i}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}H_{d,\delta}(x,y,t,y)]\right]\right\|_{L^\infty(\Omega)}, r=0,\dots,j-1; l=0,\dots,r; \delta=1,2\right\},$$

$$c_{d,3}=\max\left\{\left\|\partial_1^{(i)}\partial_2^{(j)}H_{d,\delta}(x,y,t,s)\right\|_{L^\infty(\Omega)}, \delta=1,2\right\},$$

$$c_{d,4}=\max\left\{\begin{array}{c}\binom{r}{l}\binom{q}{\eta}\left\|\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\Big|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}H_{d,\delta}(x,y,t,y)\right]\right)\right]\right\|_{L^\infty(\Omega)}, r=0,\dots,j-1;\\ l=0,\dots,r; q=0,\dots,i-1; \eta=0,\dots,q; \delta=1,2\end{array}\right\},$$

and

$$c_{d,5}=\max\left\{\binom{q}{\eta}\left\|\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}H_{d,\delta}(x,y,x,s)\right]\right\|_{L^\infty(\Omega)}, q=0,\dots,i-1; \eta=0,\dots,q; \delta=1,2\right\},$$

the constants $c_{d,i}, d=1,2; i=1,\dots,5$ are positive and independent of N and M, hence,

$$\begin{aligned}
a_{n,0}^{i,j} &\le c_{d,1}+c_{d,2}\Delta x\sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}(a_{1,\xi,0}^{0,l}+a_{2,\xi,0}^{0,l})+c_{d,3}\Delta x\Delta y\sum_{\xi=0}^{n-1}(a_{1,\xi,0}^{0,0}+a_{2,\xi,0}^{0,0})\\
&+c_{d,4}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}(a_{1,n,0}^{\eta,l}+a_{2,n,0}^{\eta,l})+c_{d,2}\Delta x\sum_{r=0}^{j-1}\sum_{l=0}^{r}(a_{1,n,0}^{0,l}+a_{2,n,0}^{0,l})\\
&+c_{d,5}\Delta y\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}(a_{1,n,0}^{\eta,0}+a_{2,n,0}^{\eta,0})+c_{d,3}\Delta x\Delta y(a_{1,n,0}^{0,0}+a_{2,n,0}^{0,0})\\
&\le c_{d,1}+c_{d,2}\Delta xp\sum_{\xi=0}^{n-1}\sum_{l=0}^{j-1}(a_{1,\xi,0}^{0,l}+a_{2,\xi,0}^{0,l})+c_{d,3}\Delta x\Delta y\sum_{\xi=0}^{n-1}(a_{1,\xi,0}^{0,0}+a_{2,\xi,0}^{0,0})\\
&+c_{d,4}p^2\sum_{l=0}^{j-1}\sum_{\eta=0}^{i-1}(a_{1,n,0}^{\eta,l}+a_{2,n,0}^{\eta,l})+c_{d,2}\Delta xp\sum_{l=0}^{j-1}(a_{1,n,0}^{0,l}+a_{2,n,0}^{0,l})\\
&+c_{d,5}\Delta yp\sum_{\eta=0}^{i-1}(a_{1,n,0}^{\eta,0}+a_{2,n,0}^{\eta,0})+c_{d,3}\Delta x\Delta y(a_{1,n,0}^{0,0}+a_{2,n,0}^{0,0}),
\end{aligned}$$

which implies that,

$$a_{d,n,0}^{i,j}\le c_{d,1}+c_{d,6}\Delta x\sum_{\xi=0}^{n-1}\sum_{l=0}^{j-1}(a_{1,\xi,0}^{0,l}+a_{2,\xi,0}^{0,l})+c_{d,7}\sum_{l=0}^{j-1}\sum_{\eta=0}^{i-1}(a_{1,n,0}^{\eta,l}+a_{2,n,0}^{\eta,l}),\qquad d=1,2 \tag{3.10}$$

where $c_{d,6}=c_{d,2}p+c_{d,3}\Delta y$ and $c_{d,7}=c_{d,4}p^2+c_{d,2}p\Delta x+c_{d,5}p\Delta y+c_{d,3}\Delta x\Delta y.; d=1,2$

Now, we consider the sequences $\Gamma_{d,n}=\max\{a_{d,n,0}^{i,j}, i+j=0,\dots,p\}$, $n=0,1,\dots,N-1.$,

$d = 1, 2$

$$a^{i,j}_{d,n,0} \leq c_{d,1} + c_{d,6} p \Delta x \sum_{\xi=0}^{n-1} (\Gamma_{1,\xi} + \Gamma_{2,\xi}) + c_{d,7} \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} (a^{\eta,l}_{1,n,0} + a^{\eta,l}_{2,n,0}), \qquad d = 1, 2 \tag{3.11}$$

Let $\alpha^{i,j}_{n,0} = a^{i,j}_{1,n,0} + a^{i,j}_{2,n,0}$, for $n = 0, 1, \ldots, N-1.$, $i + j = 0, \ldots, p.$ Adding the inequalities of (3.11) we get

$$\alpha^{i,j}_{n,0} \leq c_1 + c_6 p \Delta x \sum_{\xi=0}^{n-1} (\Gamma_{1,\xi} + \Gamma_{2,\xi}) + c_7 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} \alpha^{\eta,l}_{n,0}, \tag{3.12}$$

where $c_1 = c_{1,1} + c_{1,2}$, $c_6 = c_{6,1} + c_{6,2}$, $c_7 = c_{7,1} + c_{7,2}$

Using the notations of Lemma 1.7.3, we put for each fixed $n \in \mathbb{N}$

$$\gamma(i, j) = \alpha^{i,j}_{n,0}, \quad \lambda(i, j) = c_1 + c_6 p \Delta x \sum_{\xi=0}^{n-1} (\Gamma_{1,\xi} + \Gamma_{2,\xi}), \quad \mu(i, j) = c_7.$$

It is clear that the sequence $\lambda(i, j)$ is nondecreasing in each variable i and j.

Then, by Lemma 1.7.3, we obtain from (3.12)

$$\begin{aligned} \alpha^{i,j}_{n,0} &\leq \left(c_1 + c_6 p \Delta x \sum_{\xi=0}^{n-1} (\Gamma_{1,\xi} + \Gamma_{2,\xi}) \right) \prod_{s=0}^{i-1} \left[1 + \sum_{t=0}^{j-1} c_7 \right] \\ &\leq \underbrace{c_1 \left[1 + p c_7 \right]^p}_{c_8} + \underbrace{c_6 \left[1 + p c_7 \right]^p}_{c_9} p \Delta x \sum_{\xi=0}^{n-1} (\Gamma_{1,\xi} + \Gamma_{2,\xi}), \end{aligned} \tag{3.13}$$

it is clear that $a^{i,j}_{d,n,0} \leq \alpha^{i,j}_{n,0}$; $d = 1, 2$, $i + j = 0, \ldots, p$,

Let $\Gamma_n = \Gamma_{1,n} + \Gamma_{2,n}$, $n = 0, 1, \ldots, N-1.$, it follows that for $d = 1, 2$

$$a^{i,j}_{d,n,0} \leq c_8 + c_9 p \Delta x \sum_{\xi=0}^{n-1} \Gamma_\xi \quad \Rightarrow \quad \Gamma_n \leq c_8 + c_9 p \Delta x \sum_{\xi=0}^{n-1} \Gamma_\xi \tag{3.14}$$

It follows, by Lemma 1.7.1, for all $n = 0, 1, \dots, N-1$, and $d = 1, 2$

$$\Gamma_{d,n} \le c_8 \exp(pc_9\alpha). \tag{3.15}$$

Consider, the sequence $\Gamma_{d,n,m} = \max\{a^{i,j}_{d,n,m}, i+j = 0, \dots, p\}$,
$n = 0, 1, \dots, N-1; m = 0, \dots, M-1; d = 1, 2$
On the other hand, we have, from (3.8), for all $n = 0, \dots, N-1$, $m = 1, \dots, M-1$ and $i + j = 0, \dots, p$

$$\begin{aligned} a^{i,j}_{d,n,m} \le c_{d,1} &+ c_{d,3}\Delta x\Delta y \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}(\Gamma_{1,\xi,\rho} + \Gamma_{2,\xi,\rho}) + c_{d,2}p^2\Delta x \sum_{\xi=0}^{n-1}(\Gamma_{1,\xi,m} + \Gamma_{2,\xi,m}) \\ &+ c_{d,3}\Delta x\Delta y \sum_{\xi=0}^{n-1}(\Gamma_{1,\xi,m} + \Gamma_{2,\xi,m}) + c_{d,5}p^2\Delta y \sum_{\rho=0}^{m-1}(\Gamma_{1,n,\rho} + \Gamma_{2,n,\rho}) \\ &+ c_{d,3}\Delta x\Delta y \sum_{\rho=0}^{m-1}(\Gamma_{1,n,\rho} + \Gamma_{2,n,\rho}) + c_{d,4}p^2 \sum_{l=0}^{j-1}\sum_{\eta=0}^{i-1}(a^{\eta,l}_{1,n,m} + a^{\eta,l}_{2,n,m}) \\ &+ c_{d,2}p\Delta x \sum_{l=0}^{j-1}(a^{0,l}_{1,n,m} + a^{0,l}_{2,n,m}) + c_{d,5}p\Delta y \sum_{\eta=0}^{i-1}(a^{\eta,0}_{1,n,m} + a^{\eta,0}_{2,n,m}) \\ &+ c_{d,3}\Delta x\Delta y(a^{0,0}_{1,n,m} + a^{0,0}_{2,n,m}), \end{aligned}$$

which implies that,

$$\begin{aligned} a^{i,j}_{d,n,m} \le c_{d,1} &+ b_{d,1}\Delta x\Delta y \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}(\Gamma_{1,\xi,\rho} + \Gamma_{2,\xi,\rho}) + b_{d,2}\Delta x \sum_{\xi=0}^{n-1}(\Gamma_{1,\xi,m} + \Gamma_{2,\xi,m}) \\ &+ b_{d,3}\Delta y \sum_{\rho=0}^{m-1}(\Gamma_{1,n,\rho} + \Gamma_{2,n,\rho}) + b_{d,4} \sum_{l=0}^{j-1}\sum_{\eta=0}^{i-1}(a^{\eta,l}_{1,n,m} + a^{\eta,l}_{2,n,m}), \qquad d = 1, 2 \end{aligned} \tag{3.16}$$

where $b_{d,1} = c_{d,3}$, $b_{d,2} = c_{d,2}p^2 + c_{d,3}\Delta y$, $b_{d,3} = c_{d,5}p^2 + c_{d,3}\Delta x$ and
$b_{d,4} = c_{d,4}p^2 + c_{d,2}p\Delta x + c_{d,5}p\Delta y + c_{d,3}\Delta x\Delta y$.

Let $\alpha^{i,j}_{n,m} = a^{i,j}_{1,n,m} + a^{i,j}_{2,n,m}$, for $n = 0, 1, \dots, N-1.$,$m = 0, 1, \dots, M-1.$, $i + j = 0, \dots, p$.

Adding the inequalities of (3.16), we get

$$\begin{aligned}\alpha_{n,m}^{i,j} \leq c_1 + b_1\Delta x\Delta y\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}(\Gamma_{1,\xi,\rho}+\Gamma_{2,\xi,\rho}) + b_2\Delta x\sum_{\xi=0}^{n-1}(\Gamma_{1,\xi,m}+\Gamma_{2,\xi,m})\\ + b_3\Delta y\sum_{\rho=0}^{m-1}(\Gamma_{1,n,\rho}+\Gamma_{2,n,\rho}) + b_4\sum_{l=0}^{j-1}\sum_{\eta=0}^{i-1}\alpha_{n,m}^{\eta,l}\end{aligned} \tag{3.17}$$

where $b_1 = b_{1,1} + b_{2,1}, b_2 = b_{1,2} + b_{2,2}, b_3 = b_{1,3} + b_{2,3}, b_4 = b_{1,4} + b_{2,4}$

Using the notations of Lemma 1.7.3, we put for each fixed $n, m \in \mathbb{N}$

$\gamma(i, j) = a_{d,n,m}^{i,j}, \quad \mu(i, j) = b_4$ and

$\lambda(i, j) = c_1 + b_1\Delta x\Delta y\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}(\Gamma_{1,\xi,\rho}+\Gamma_{2,\xi,\rho}) + b_2\Delta x\sum_{\xi=0}^{n-1}(\Gamma_{1,\xi,m}+\Gamma_{2,\xi,m}) + b_3\Delta y\sum_{\rho=0}^{m-1}(\Gamma_{1,n,\rho}+\Gamma_{2,n,\rho})$

It is clear that the sequence $\lambda(i, j)$ is nondecreasing in each variable i and j.

Then, by Lemma 1.7.3, we obtain from (3.17)

$$\begin{aligned}\alpha_{n,m}^{i,j} \leq & a(i,j)\prod_{s=0}^{i-1}\left[1+\sum_{t=0}^{j-1}b_4\right]\\ \leq & \underbrace{c_1\left[1+pb_4\right]^p}_{b_5} + \Delta x\Delta y\underbrace{b_1\left[1+pb_4\right]^p}_{b_6}\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}(\Gamma_{1,\xi,\rho}+\Gamma_{2,\xi,\rho})\\ & + \Delta x\underbrace{b_2\left[1+pb_4\right]^p}_{b_7}\sum_{\xi=0}^{n-1}(\Gamma_{1,\xi,m}+\Gamma_{2,\xi,m}) + \Delta y\underbrace{b_3\left[1+pb_4\right]^p}_{b_8}\sum_{\rho=0}^{m-1}(\Gamma_{1,n,\rho}+\Gamma_{2,n,\rho}),\end{aligned}$$

it follows that, for all $n = 0, 1, \ldots, N-1; m = 0, \ldots, M-1$,

$$\begin{aligned}\Gamma_{d,n,m} \leq & b_5 + \Delta x\Delta y b_6\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}(\Gamma_{1,\xi,\rho}+\Gamma_{2,\xi,\rho}) + \Delta x b_7\sum_{\xi=0}^{n-1}(\Gamma_{1,\xi,m}+\Gamma_{2,\xi,m})\\ & + \Delta y b_8\sum_{\rho=0}^{m-1}(\Gamma_{1,n,\rho}+\Gamma_{2,n,\rho}), \qquad d = 1, 2\end{aligned} \tag{3.18}$$

Let $\Gamma_{n,m} = \Gamma_{1,n,m} + \Gamma_{2,n,m}$, for all $n = 0, \ldots, N-1; m = 0, \ldots, M-1$

Then, by Lemma 1.7.4, we obtain from (3.18) for fixed $d = 1, 2$

$$\begin{aligned}\Gamma_{d,n,m} &\leq b_5 + \Delta x \Delta y b_6 \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \Delta x b_7 \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + \Delta y b_8 \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} \\ &\leq b_5 \exp(\gamma_1 (N\Delta x + M\Delta y))\end{aligned} \tag{3.19}$$

where $\gamma_1 = \frac{1}{2}(b_7 + b_8 + \sqrt{(b_7 + b_8)^2 + 4b_6})$
Then

$$\Gamma_{d,n,m} \leq b_5 \exp(\gamma_1(\alpha + \beta)), \qquad d = 1, 2 \tag{3.20}$$

Hence, from (3.9), (3.15) and (3.20), by setting

$$\Lambda(p) = \max\{\Lambda_1(p), c_8 \exp(p\alpha c_9), b_5 \exp(\gamma_1(\alpha + \beta))\}$$

The proof of Lemma 3.3.1 is completed. ■

Main Convergence Result

We consider the space $L^\infty(\Omega, \mathbb{R}^2)$ with the norm

$$||\varphi||_{L^\infty(\Omega,\mathbb{R}^2)} = \max\{||\varphi_1||_{L^\infty(\Omega)}, ||\varphi_2||_{L^\infty(\Omega)}\}.$$

The following theorem gives the convergence of the presented method.

Theorem 3.3.1 *Let f_d and $H_{d,\delta}$ ($d, \delta = 1, 2$) be p times continuously differentiable on their respective domains. Then equations* (3.2), (3.3), (3.6) *define a unique approximation $w \in (S^{(-1)}_{p-1,p-1}(\mathcal{G}_{N,M}))^2$, and the resulting error function $E(x, y) = \psi(x, y) - w(x, y)$ satisfies:*

$$||E||_{L^\infty(\Omega,\mathbb{R}^2)} \leq C(\Delta x + \Delta y)^p,$$

where C is a finite constant independent of Δx and Δy.

Proof. The proof is split into three steps.

Claim 1. Define the error $E(x,y)$ on $R_{0,0}$ by $E_{d,0,0}(x,y) = \psi_d(x,y) - w_{d,0,0}(x,y)$, for $d = 1,2$. There exists a constant C_1 independent of Δx and Δy such that,

$$||E||_{L^\infty(R_{0,0},\mathbb{R}^2)} \leq C_1(\Delta x + \Delta y)^p.$$

Let $(x,y) \in R_{0,0}$, by using Taylor's theorem, we obtain from (3.2)

$$|E_{d,0,0}(x,y)| \leq \sum_{i+j=p} \frac{1}{i!j!} \left\| \frac{\partial^{i+j}\psi_d}{\partial x^i \partial y^j} \right\|_{L^\infty(R_{0,0})} (\Delta x)^i (\Delta y)^j, \qquad d = 1,2$$

Hence, by Lemma 3.3.1, we have

$$|E_{d,0,0}(x,y)| \leq \Lambda(p) \sum_{i+j=p} \frac{1}{i!j!} (\Delta x)^i (\Delta y)^j = \underbrace{\frac{\Lambda(p)}{p!}}_{C_1} (\Delta x + \Delta y)^p. \qquad d = 1,2 \tag{3.21}$$

Claim 2. Define the error $E(x,y)$ on $R_{n,0}$ by $E_{d,n,0}(x,y) = \psi_d(x,y) - w_{d,n,0}(x,y)$ for all $n \in \{0,\ldots,N-1\}$ and $d = 1,2$.

There exists a constant C_2 independent of Δx and Δy such that,

$$||E||_{L^\infty(R_{n,0},\mathbb{R}^2)} \leq C_2(\Delta x + \Delta y)^p,$$

for all $n = 1,\ldots,N-1$. Let's pose

$$||E||_{L^\infty(R_{n,0},\mathbb{R}^2)} = ||E_n||.$$

Let $(x, y) \in R_{n,0}$, we have from (3.4)

$$\psi_d(x,y) - \tilde{w}_{d,n,0}(x,y) = \sum_{\xi=0}^{n-1} \int_{x_\xi}^{x_{\xi+1}} \int_0^y (H_{d,1}(x,y,t,s)(\psi_1(t,s) - w_{1,\xi,0}(t,s)) + H_{d,2}(x,y,t,s)(\psi_2(t,s) - w_{2,\xi,0}(t,s)))dsdt + \int_{x_n}^{x} \int_0^y (H_{d,1}(x,y,t,s)(\psi_1(t,s) - \tilde{w}_{1,\xi,0}(t,s)) + H_{d,2}(x,y,t,s)(\psi_2(t,s) - \tilde{w}_{2,\xi,0}(t,s)))dsdt.$$

Hence,

$$|\psi_d(x,y) - \tilde{w}_{d,n,0}(x,y)| \leq \sum_{\xi=0}^{n-1} \Delta x \Delta y \overline{H}_d \|E\|_{L^\infty(R_{\xi,0},\mathbb{R}^2)} + \overline{H}_d \int_{x_n}^{x} \int_0^y |\psi_1(t,s) - \tilde{w}_{1,n,0}(t,s)| + |\psi_2(t,s) - \tilde{w}_{2,n,0}(t,s)| dsdt,$$

where, $\overline{H}_d = \|H_{d,1}\|_{L^\infty(\Omega)} + \|H_{d,2}\|_{L^\infty(\Omega)}$, $d = 1, 2$.
By adding the two inequalities, we have

$$|\psi_1(x,y) - \tilde{w}_{1,n,0}(x,y)| + |\psi_2(x,y) - \tilde{w}_{2,n,0}(x,y)| \leq \sum_{\xi=0}^{n-1} \Delta x \Delta y \overline{H} \|E_\xi\| + \overline{H} \int_{x_n}^{x} \int_0^y |\psi_1(t,s) - \tilde{w}_{1,n,0}(t,s)| + |\psi_2(t,s) - \tilde{w}_{2,n,0}(t,s)| dsdt,$$

where $\overline{H} = \overline{H}_1 + \overline{H}_2$
Then by Lemma 1.7.5,

$$|\psi_d(x,y) - \tilde{w}_{d,n,0}(x,y)| \leq \sum_{\xi=0}^{n-1} \Delta x \Delta y \overline{H} \|E_\xi\| \exp\left(\overline{H}\alpha\beta\right) \leq \sum_{\xi=0}^{n-1} \Delta x \underbrace{\beta \overline{H} \exp\left(\overline{H}\alpha\beta\right)}_{\lambda_1} \|E_\xi\|, \qquad d = 1, 2$$

Which implies, by using Taylor's theorem, that

$$\begin{aligned}|E_{d,n,0}(x,y)| &\leq |\psi_d(x,y)-\tilde{w}_{d,n,0}(x,y)|+|\tilde{w}_{d,n,0}(x,y)-w_{d,n,0}(x,y)|\\ &\leq \sum_{\xi=0}^{n-1}\Delta x\lambda_1\|E_\xi\|+\sum_{i+j=p}\frac{1}{i!j!}\left\|\frac{\partial^{i+j}\tilde{w}_{d,n,0}}{\partial x^i\partial y^j}\right\|_{L^\infty(R_{\xi,0})}(\Delta x)^i(\Delta y)^j.\end{aligned}$$

Hence, by Lemma 3.3.1, we obtain

$$|E_{d,n,0}(x,y)|\leq \sum_{\xi=0}^{n-1}\Delta x\lambda_1\|E_\xi\|+\frac{\Lambda(p)}{p!}(\Delta x+\Delta y)^p,\qquad d=1,2$$

Then, by Lemma 1.7.1, we have

$$\|E_n\|\leq \frac{\Lambda(p)}{p!}(\Delta x+\Delta y)^p\exp(\alpha\lambda_1).$$

Thus, we take $C_2=\frac{\Lambda(p)}{p!}\exp(\alpha\lambda_1)$.

Claim 3. Define the error $E(x,y)$ on $R_{n,m}$ by $E_{d,n,m}(x,y)=\psi_d(x,y)-w_{d,n,m}(x,y)$ for all $n\in\{0,\dots,N-1\}$, $m\in\{0,\dots,M-1\}$ and $d=1,2$.

There exists a constant C_3 independent of Δx and Δy such that,

$$\|E\|_{L^\infty(R_{n,m},\mathbb{R}^2)}\leq C_3(\Delta x+\Delta y)^p,$$

for all $n=0,\dots,N-1$ and $m=1,\dots,M-1$. Let's pose

$$\|E\|_{L^\infty(R_{n,m},\mathbb{R}^2)}=\|E_{n,m}\|.$$

Let $(x,y)\in R_{n,m}$, we have from (3.7)

$$\begin{aligned}|\psi_d(x,y)-\tilde{w}_{d,n,m}(x,y)| &\leq \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\Delta x\Delta y\overline{H}_d\|E_{\xi,\rho}\|+\sum_{\xi=0}^{n-1}\Delta x\Delta y\overline{H}_d\|E_{\xi,m}\|+\sum_{\rho=0}^{m-1}\Delta x\Delta y\overline{H}_d\|E_{n,\rho}\|\\ &+\overline{H}_d\int_{x_n}^{x}\int_{y_m}^{y}|\psi_1(t,s)-\tilde{w}_{1,n,m}(t,s)|+|\psi_2(t,s)-\tilde{w}_{2,n,m}(t,s)|dsdt,\qquad d=1,2\end{aligned}$$

By adding the two inequalities, we have

$$|\psi_1(t,s) - \tilde{w}_{1,n,m}(t,s)| + |\psi_2(t,s) - \tilde{w}_{2,n,m}(t,s)| \leq \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Delta x \Delta y \overline{H} ||E_{\xi,\rho}|| + \sum_{\xi=0}^{n-1} \Delta x \Delta y \overline{H} ||E_{\xi,m}||$$
$$+ \sum_{\rho=0}^{m-1} \Delta x \Delta y \overline{H} ||E_{n,\rho}|| + \overline{H} \int_{x_n}^{x} \int_{y_m}^{y} |\psi_1(t,s) - \tilde{w}_{1,n,m}(t,s)| + |\psi_2(t,s) - \tilde{w}_{2,n,m}(t,s)| ds dt,$$

Then by Lemma 1.7.5,

$$|\psi_d(x,y) - \tilde{w}_{d,n,m}(x,y)| \leq \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Delta x \Delta y \underbrace{\overline{H} \exp\left(\overline{H}\alpha\beta\right)}_{\lambda_2} ||E_{\xi,\rho}||$$
$$+ \sum_{\xi=0}^{n-1} \Delta x \underbrace{\beta\overline{H} \exp\left(\overline{H}\alpha\beta\right)}_{\lambda_3} ||E_{\xi,m}|| + \sum_{\rho=0}^{m-1} \Delta y \underbrace{\alpha\overline{H} \exp\left(\overline{H}\alpha\beta\right)}_{\lambda_4} ||E_{n,\rho}||, \qquad d = 1,2$$

which implies, by using Taylor's theorem, that

$$\begin{aligned} |E_{d,n,m}(x,y)| \leq& |\psi_d(x,y) - \tilde{w}_{d,n,m}(x,y)| + |\tilde{w}_{d,n,m}(x,y) - w_{d,n,m}(x,y)| \\ \leq& \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Delta x \Delta y \lambda_2 ||E_{\xi,\rho}|| + \sum_{\xi=0}^{n-1} \Delta x \lambda_3 ||E_{\xi,m}|| + \sum_{\rho=0}^{m-1} \Delta y \lambda_4 ||E_{n,\rho}|| \\ &+ \sum_{i+j=p} \frac{1}{i!j!} \left\| \frac{\partial^{i+j}\tilde{w}_{d,n,m}}{\partial x^i \partial y^j} \right\|_{L^\infty(R_{n,m})} (\Delta x)^i (\Delta y)^j, \qquad d = 1,2 \end{aligned}$$

Hence, by Lemma 3.3.1, we obtain

$$||E_{n,m}|| \leq \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Delta x \Delta y \lambda_2 ||E_{\xi,\rho}|| + \sum_{\xi=0}^{n-1} \Delta x \lambda_3 ||E_{\xi,m}|| + \sum_{\rho=0}^{m-1} \Delta y \lambda_4 ||E_{n,\rho}|| + \frac{\Lambda(p)}{p!}(\Delta x + \Delta y)^p, \qquad d = 1,2 \tag{3.22}$$

Then, by Lemma 1.7.4, we obtain

$$||E_{n,m}|| \leq \frac{\Lambda(p)}{p!}(\Delta x + \Delta y)^p \exp(\gamma_2(\alpha + \beta)) \tag{3.23}$$

where $\gamma_2 = \frac{1}{2}(\lambda_3 + \lambda_4 + \sqrt{(\lambda_3 + \lambda_4)^2 + 4\lambda_2})$

Thus, we take $C_3 = \frac{\Lambda(p)}{p!} \exp(\gamma_2(\alpha + \beta))$.
Taking $C = \max\{C_1, C_2, C_3\}$ completes the proof. ■

3.4 Numerical Implementation and Examples

Example 3.4.1 *Consider the following system of 2-DVIEs as defined in* (3.1)*:*

$$\begin{cases} \psi_1(x,y) = f_1(x,y) + \displaystyle\int_0^x \int_0^y \left[\cos(s)\,\psi_1(t,s) + \sin(s)\,\psi_2(t,s)\right] dsdt, \\ \psi_2(x,y) = f_2(x,y) + \displaystyle\int_0^x \int_0^y \left[xt^2\,\psi_1(t,s) + (x+t)\,\psi_2(t,s)\right] dsdt, \end{cases}$$

where the functions $f_1(x,y)$ and $f_2(x,y)$ are given by:

$$f_1(x,y) = x\cos(y) - \frac{1}{2}yx^2,$$

$$f_2(x,y) = x\sin(y) - \frac{1}{4}x^5\sin(y) + \frac{5}{6}x^3\cos(y) - \frac{5}{6}x^3.$$

The exact solution of this system is $\psi_1(x,y) = x\cos(y)$ and $\psi_2(x,y) = x\sin(y)$.

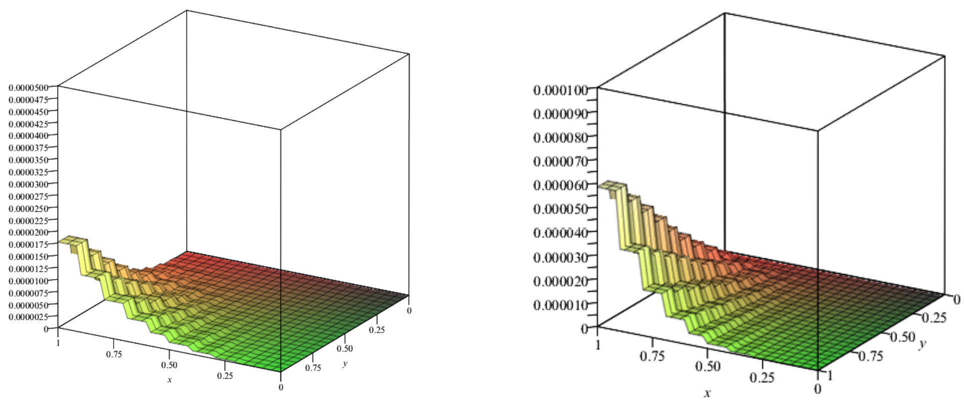
The absolute error of the proposed method for $N = M = 10$, $p = 3, 5$ is shown in Table 3.1. Also Figure 3.1 shows the comparison between exact and approximate solutions for $N = M = 10$ and $p = 5$.

Example 3.4.2 *Consider the following system of 2-DVIEs as defined in* (3.1)*:*

$$\begin{cases} \psi_1(x,y) = f_1(x,y) + \displaystyle\int_0^x \int_0^y \left[e^{-t}\,\psi_1(t,s) + xt\,\psi_2(t,s)\right] dsdt, \\ \psi_2(x,y) = f_2(x,y) + \displaystyle\int_0^x \int_0^y \left[ys\,\psi_1(t,s) - e^{-s}\,\psi_2(t,s)\right] dsdt, \end{cases}$$

Table 3.1: Absolute errors for Example 3.4.1 for $N = M = 10$

(x, y)	$p = 3$		$p = 5$	
	$\lvert\psi_1 - \tilde{\psi}_1\rvert$	$\lvert\psi_2 - \tilde{\psi}_2\rvert$	$\lvert\psi_1 - \tilde{\psi}_1\rvert$	$\lvert\psi_2 - \tilde{\psi}_2\rvert$
$(0, 0)$	0	0	0	0
$(0.1, 0.1)$	8.32×10^{-7}	3.88×10^{-9}	4.30×10^{-10}	6.00×10^{-12}
$(0.2, 0.2)$	3.34×10^{-6}	1.19×10^{-7}	2.00×10^{-9}	9.63×10^{-9}
$(0.3, 0.3)$	7.61×10^{-6}	7.06×10^{-7}	3.60×10^{-8}	1.08×10^{-7}
$(0.4, 0.4)$	1.37×10^{-5}	2.42×10^{-6}	1.91×10^{-7}	5.67×10^{-7}
$(0.5, 0.5)$	2.19×10^{-5}	6.29×10^{-6}	6.70×10^{-7}	2.01×10^{-6}
$(0.6, 0.6)$	3.23×10^{-5}	1.37×10^{-5}	1.84×10^{-6}	5.68×10^{-6}
$(0.7, 0.7)$	4.54×10^{-5}	2.68×10^{-5}	4.35×10^{-6}	1.36×10^{-5}
$(0.8, 0.8)$	6.17×10^{-5}	4.38×10^{-5}	9.15×10^{-6}	2.94×10^{-5}
$(0.9, 0.9)$	8.19×10^{-5}	8.24×10^{-5}	1.77×10^{-5}	5.83×10^{-5}



(a) The error $\lvert\psi_1 - \tilde{\psi}_1\rvert$ for $N = M = 10$. (b) The error $\lvert\psi_2 - \tilde{\psi}_2\rvert$ for $N = M = 10$.

Figure 3.1: Comparison between exact and approximate solutions for Example 3.4.1

where the functions $f_1(x, y)$ *and* $f_2(x, y)$ *are given by:*

$$f_1(x, y) = -x - \frac{1}{2}y + x(y+1)e^{-y} + (x^2 - x + 1)e^{-x} - x(xy + x + y + 1)e^{-x-y} + \frac{1}{2}ye^{-2x},$$

$$f_2(x, y) = \frac{1}{4} - \frac{1}{2}y^3 + ye^{-x-y} - \frac{1}{2}\left(y + \frac{1}{2}\right)e^{-y} + \frac{1}{2}\left(y^3 - \frac{1}{2}\right)e^{-x} + \frac{1}{2}\left(y + \frac{1}{2}\right)e^{-x-2y}.$$

The exact solution of this system is $\psi_1(x, y) = e^{-x}$ *and* $\psi_2(x, y) = ye^{-x-y}$.

The absolute error of the proposed method for $N = M = 15$, $p = 3$ is shown in Table 3.2.

Table 3.2: Absolute errors for Example 3.4.2 for $N = M = 15$

$(x, y) = (\frac{i}{15}, \frac{i}{15})$	$\lvert\psi_1 - \tilde{\psi}_1\rvert$	$\lvert\psi_2 - \tilde{\psi}_2\rvert$
$(0, 0)$	0	0
$i = 1$	4.99×10^{-8}	4.56×10^{-7}
$i = 2$	1.76×10^{-7}	1.61×10^{-6}
$i = 3$	3.33×10^{-7}	3.23×10^{-6}
$i = 5$	5.61×10^{-7}	7.11×10^{-6}
$i = 7$	4.23×10^{-7}	1.11×10^{-5}
$i = 10$	7.37×10^{-7}	1.65×10^{-5}
$i = 12$	1.95×10^{-6}	1.95×10^{-5}
$i = 14$	3.02×10^{-6}	2.20×10^{-5}
$i = 15$	1.63×10^{-5}	9.96×10^{-6}

Example 3.4.3 *Consider the following system of 2-DVIEs as defined in* (3.1)*:*

$$\begin{cases} \psi_1(x, y) = f_1(x, y) + \displaystyle\int_0^x \int_0^y \frac{x - y + t - s}{4}\,\psi_2(t, s)\, dsdt, \\ \psi_2(x, y) = f_2(x, y) + \displaystyle\int_0^x \int_0^y \frac{x - y + t - s}{4}\,\psi_1(t, s)\, dsdt, \end{cases}$$

where the functions $f_1(x, y)$ *and* $f_2(x, y)$ *are given by:*

$$f_1(x, y) = xe^{1-y} - \frac{e}{24}y^2\left(3 + 3x - 5y + e^{-x}(-3 - 6x + 5y)\right),$$

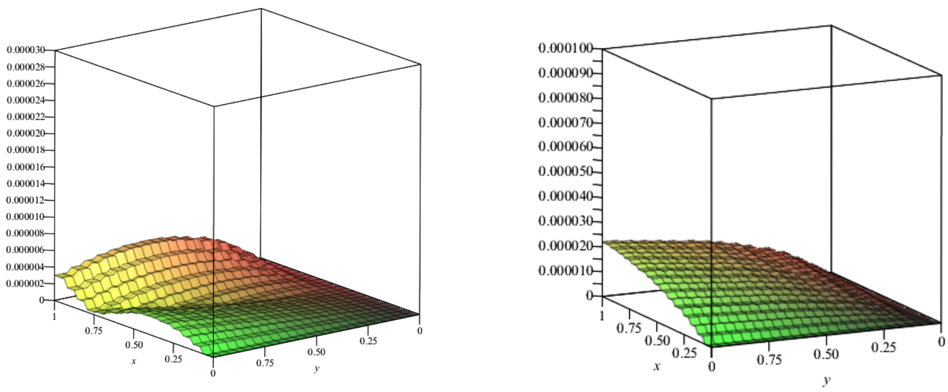
$$f_2(x, y) = ye^{1-x} + \frac{e}{24}x^2\left(3 - 5x + 3y + e^{-y}(-3 + 5x - 6y)\right).$$

The exact solution of this system is $\psi_1(x,y) = xe^{1-y}$ *and* $\psi_2(x,y) = ye^{1-x}$.

Comparison of the absolute errors between the hybrid functions method (HFM) [32] and the proposed method (TCM) for $N = M = 10$, $p = 4$ is shown in Table 3.3. Also Figure 3.2 shows the comparison between exact and approximate solutions for $N = M = 10$ and $p = 4$.

Table 3.3: Absolute errors for Example 3.4.3

(x,y)	HFM [32]		TCM	
	$\lvert\psi_1-\tilde{\psi}_1\rvert$	$\lvert\psi_2-\tilde{\psi}_2\rvert$	$\lvert\psi_1-\tilde{\psi}_1\rvert$	$\lvert\psi_2-\tilde{\psi}_2\rvert$
$(0,0)$	1.11×10^{-7}	5.02×10^{-11}	0	0
$(0.1,0.1)$	4.90×10^{-8}	3.25×10^{-8}	1.00×10^{-10}	2.00×10^{-10}
$(0.2,0.2)$	1.41×10^{-7}	1.34×10^{-7}	2.60×10^{-9}	1.90×10^{-9}
$(0.3,0.3)$	2.10×10^{-7}	2.03×10^{-7}	1.07×10^{-8}	8.50×10^{-9}
$(0.4,0.4)$	1.53×10^{-7}	1.37×10^{-7}	3.12×10^{-8}	2.71×10^{-8}
$(0.5,0.5)$	6.80×10^{-7}	6.80×10^{-7}	7.60×10^{-8}	6.45×10^{-8}
$(0.6,0.6)$	1.18×10^{-7}	1.18×10^{-7}	1.54×10^{-7}	1.35×10^{-7}
$(0.7,0.7)$	2.84×10^{-7}	2.84×10^{-7}	2.83×10^{-7}	2.51×10^{-7}
$(0.8,0.8)$	3.29×10^{-7}	3.28×10^{-7}	4.80×10^{-7}	4.26×10^{-7}
$(0.9,0.9)$	1.86×10^{-7}	1.87×10^{-7}	7.68×10^{-7}	6.76×10^{-7}



(a) The error $\lvert\psi_1-\tilde{\psi}_1\rvert$. (b) The error $\lvert\psi_2-\tilde{\psi}_2\rvert$.

Figure 3.2: Comparison between exact and approximate solutions for Example 3.4.3

3.5 Conclusion

This chapter presented the Taylor Collocation Method for solving systems of two-dimensional Volterra integral equations. The method employs Taylor polynomial expansions within a piecewise polynomial spline framework to directly approximate the solution without converting the integral system to algebraic equations.

Key contributions include:

1. A complete algorithmic description for systems of 2D-VIEs
2. Rigorous convergence analysis proving $(\Delta x + \Delta y)^p$ error bounds
3. Numerical verification of theoretical predictions

The method demonstrates high accuracy and is particularly effective for problems with smooth solutions. Future work will focus on extension to nonlinear systems.

CHAPTER 4

SOLVING SYSTEMS OF TWO-DIMENSIONAL VOLTERRA INTEGRAL EQUATIONS WITH PROPORTIONAL DELAYS VIA TAYLOR COLLOCATION

4.1 Introduction

Volterra integral equations (VIEs) are fundamental mathematical tools for modeling phenomena characterized by memory effects and hereditary properties, with applications spanning physics, biology, and engineering [7]. The inclusion of delays, particularly proportional delays, extends their applicability to areas such as population dynamics, control theory, and viscoelasticity, where system states depend on both current and past configurations [42, 18].

A variety of numerical methods have been developed for systems of one-dimensional integral and integro-differential equations. These include approaches based on Haar functions [30], hybrid Legendre functions [31], Chebyshev polynomials [2], homotopy perturbation techniques [51, 23], Taylor series expansions [44], B-spline collocation [38], and hybrid block-pulse functions [39]. For systems with delay arguments, contributions include the work of Yalçnba and Erdem [50], Shakourifar and Dehghan [40], Taghizadeh and Matinfar [45], and Yüzba and Ismailov [52], who employed differential transformation methods.

Systems of two-dimensional Volterra integral equations (2D-VIEs) extend these concepts to model problems with both spatial and temporal dependencies. Existing numerical strategies for such systems include operational matrix methods using hybrid functions [32] and Legendre wavelet approaches [25]. Modified Newton methods and adaptive iterative regularization schemes have also been explored for nonlinear and two-dimensional systems [16, 17].

The presence of delays introduces additional complexity, motivating the development of efficient numerical schemes. Collocation methods are particularly attractive due to their high accuracy and adaptability [8]. The Taylor collocation method, which combines the high-order approximation properties of Taylor series with the pointwise satisfaction of collocation techniques, has proven effective for various integral and integro-differential equations [24, 28]. However, its application to systems of two-dimensional Volterra integral equations with proportional delays remains an open area of investigation.

This chapter addresses this gap by developing a Taylor collocation method for such systems. We present a detailed derivation of the numerical scheme, provide a thorough convergence analysis, and demonstrate its efficacy through numerical experiments. The proposed method offers a computationally efficient and accurate alternative, suitable for practical applications requiring high precision.

4.2 Method Description

We consider a linear system of two-dimensional Volterra integral equations (2D-VIEs) with proportional delays of the form:

$$\begin{aligned}\sum_{\beta=1}^{r} \mathcal{A}_{\alpha,\beta}(\xi,\eta)\mathcal{U}_{\beta}(\xi,\eta) = \mathcal{G}_{\alpha}(\xi,\eta) &+ \sum_{\beta=1}^{r} \int_0^{\xi}\int_0^{\eta} \mathcal{K}_{\alpha,\beta}(\xi,\eta,\tau,\sigma)\mathcal{U}_{\beta}(\tau,\sigma)\, d\sigma\, d\tau \\ &+ \sum_{\beta=1}^{r} \int_0^{p_1\xi}\int_0^{p_2\eta} \mathcal{H}_{\alpha,\beta}(\xi,\eta,\tau,\sigma)\mathcal{U}_{\beta}(\tau,\sigma)\, d\sigma\, d\tau, \qquad (4.1)\end{aligned}$$

where $(\xi,\eta) \in \Omega = [0,\mathfrak{a}]\times[0,\mathfrak{b}] \subset \mathbb{R}^2$, and the delay parameters satisfy $0 \le p_1, p_2 < 1$. The functions $\mathcal{G}_{\alpha}$ and $\mathcal{A}_{\alpha,\beta}$, for $\alpha,\beta = 1,2,\ldots,r$, are continuous on Ω. The kernel functions $\mathcal{K}_{\alpha,\beta}$ and $\mathcal{H}_{\alpha,\beta}$ are continuous on their respective domains:

$$\Sigma_1 = \{(\xi,\eta,\tau,\sigma) : 0 \le \tau \le \xi \le \mathfrak{a},\ 0 \le \sigma \le \eta \le \mathfrak{b}\},$$
$$\Sigma_2 = \{(\xi,\eta,\tau,\sigma) : 0 \le \tau \le p_1\xi \le \mathfrak{a},\ 0 \le \sigma \le p_2\eta \le \mathfrak{b}\}.$$

The existence and uniqueness of solutions for such systems are discussed in [11, 41]. The system can be written in matrix form as:

$$\mathbf{A}(\xi,\eta)\mathbf{u}(\xi,\eta) = \mathbf{g}(\xi,\eta)+\int_0^{\xi}\int_0^{\eta}\mathbf{K}(\xi,\eta,\tau,\sigma)\mathbf{u}(\tau,\sigma)\,d\sigma\,d\tau+\int_0^{p_1\xi}\int_0^{p_2\eta}\mathbf{H}(\xi,\eta,\tau,\sigma)\mathbf{u}(\tau,\sigma)\,d\sigma\,d\tau, \tag{4.2}$$

where

$$\begin{aligned}
\mathbf{u}(\xi,\eta) &= [\mathcal{U}_1(\xi,\eta),\dots,\mathcal{U}_r(\xi,\eta)]^T,\\
\mathbf{g}(\xi,\eta) &= [\mathcal{G}_1(\xi,\eta),\dots,\mathcal{G}_r(\xi,\eta)]^T,\\
\mathbf{K}(\xi,\eta,\tau,\sigma) &= \{\mathcal{K}_{\alpha,\beta}(\xi,\eta,\tau,\sigma)\}_{\alpha,\beta=1}^r,\\
\mathbf{H}(\xi,\eta,\tau,\sigma) &= \{\mathcal{H}_{\alpha,\beta}(\xi,\eta,\tau,\sigma)\}_{\alpha,\beta=1}^r,\\
\mathbf{A}(\xi,\eta) &= \{\mathcal{A}_{\alpha,\beta}(\xi,\eta)\}_{\alpha,\beta=1}^r.
\end{aligned}$$

Assuming $\mathbf{A}(\xi,\eta)$ is invertible on Ω, the solution is given by:

$$\begin{aligned}
\mathbf{u}(\xi,\eta) = \mathbf{A}^{-1}(\xi,\eta)\mathbf{g}(\xi,\eta) &+ \int_0^{\xi}\int_0^{\eta}\mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,\sigma)\mathbf{u}(\tau,\sigma)\,d\sigma\,d\tau\\
&+\int_0^{p_1\xi}\int_0^{p_2\eta}\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,\sigma)\mathbf{u}(\tau,\sigma)\,d\sigma\,d\tau.
\end{aligned} \tag{4.3}$$

4.2.1 Discretization and Spline Space

To construct a numerical approximation, we introduce uniform partitions of the intervals $[0,\mathfrak{a}]$ and $[0,\mathfrak{b}]$. Let

$$\Pi_N = \{\xi_i = i\mathfrak{h} : i = 0,1,\dots,N\},\quad \mathfrak{h} = \frac{\mathfrak{a}}{N},$$

$$\Pi_M = \{\eta_j = j\mathfrak{k} : j = 0,1,\dots,M\},\quad \mathfrak{k} = \frac{\mathfrak{b}}{M},$$

and define the grid on Ω as $\Pi_{N,M} = \Pi_N \times \Pi_M = \{(\xi_n,\eta_m) : 0 \le n \le N, 0 \le m \le M\}$.

The subintervals and sub-rectangles are defined as:

$$
\begin{aligned}
\sigma_n &= [\xi_n, \xi_{n+1}), \quad n = 0, 1, \ldots, N-2; \quad \sigma_{N-1} = [\xi_{N-1}, \xi_N], \\
\delta_m &= [\eta_m, \eta_{m+1}), \quad m = 0, 1, \ldots, M-2; \quad \delta_{M-1} = [\eta_{M-1}, \eta_M], \\
\Omega_{n,m} &:= \sigma_n \times \delta_m, \quad n = 0, \ldots, N-1; \; m = 0, \ldots, M-1.
\end{aligned}
$$

Let $\pi_{\mathfrak{p}-1,\mathfrak{p}-1}$ denote the space of real polynomials of degree at most $\mathfrak{p}-1$ in both ξ and η. The bivariate polynomial spline space of degree $\mathfrak{p}-1$ is:

$$
\mathcal{S}^{(-1)}_{\mathfrak{p}-1,\mathfrak{p}-1}(\Pi_{N,M}) = \{\mathcal{V} : \mathcal{V}|_{\Omega_{n,m}} \in \pi_{\mathfrak{p}-1,\mathfrak{p}-1}, \; n = 0, \ldots, N-1; \; m = 0, \ldots, M-1\}.
$$

Our aim is to find an approximation $\mathcal{V} \in (\mathcal{S}^{(-1)}_{\mathfrak{p}-1,\mathfrak{p}-1}(\Pi_{N,M}))^r$ to the exact solution $\mathbf{u}$ of (4.1). The degrees of freedom are determined by imposing collocation conditions derived from Taylor polynomial expansions on each sub-rectangle $\Omega_{n,m}$.

The solution $\mathbf{u}$ is known along part of the boundary of Ω:

$$
\begin{aligned}
\mathbf{u}(\xi, 0) &= \mathbf{A}^{-1}(\xi, 0)\mathbf{g}(\xi, 0), \quad 0 \le \xi \le \mathfrak{a}, \\
\mathbf{u}(0, \eta) &= \mathbf{A}^{-1}(0, \eta)\mathbf{g}(0, \eta), \quad 0 \le \eta \le \mathfrak{b}.
\end{aligned}
$$

4.2.2 Construction of the Approximation

Approximation on $\Omega_{0,0}$

On the initial rectangle $\Omega_{0,0}$, we approximate $\mathbf{u}$ by its Taylor polynomial about $(0, 0)$:

$$
\mathcal{V}_{0,0}(\xi, \eta) = \sum_{\iota+\jmath=0}^{\mathfrak{p}-1} \frac{1}{\iota! \, \jmath!} \frac{\partial^{\iota+\jmath}\mathbf{u}(0,0)}{\partial \xi^{\iota} \partial \eta^{\jmath}} \xi^{\iota} \eta^{\jmath}, \quad (\xi, \eta) \in \Omega_{0,0}, \tag{4.4}
$$

where $\frac{\partial^{\iota+\jmath}\mathbf{u}(0,0)}{\partial\xi^{\iota}\partial\eta^{\jmath}}$ denotes the exact partial derivative at the origin. These derivatives can be obtained by successively differentiating the integral equation (4.3). For instance, differentiating $\jmath$ times with respect to η and then ι times with respect to ξ yields:

$$\begin{aligned}
\frac{\partial^{\iota+\jmath}\mathbf{u}(0,0)}{\partial\xi^{\iota}\partial\eta^{\jmath}} &= \partial_1^{(\iota)}\partial_2^{(\jmath)}\left(\mathbf{A}^{-1}(0,0)\mathbf{g}(0,0)\right) \\
&\quad + \sum_{\varrho=0}^{\jmath-1}\sum_{\kappa=0}^{\varrho}\sum_{\ell=0}^{\iota-1}\sum_{\mu=0}^{\ell}\binom{\varrho}{\kappa}\binom{\ell}{\mu}\times \\
&\quad \frac{\partial^{\ell-\mu}}{\partial\xi^{\ell-\mu}}\left[\left.\frac{\partial^{\iota-1-\ell}}{\partial\xi^{\iota-1-\ell}}\right|_{\tau=\xi}\left(\frac{\partial^{\varrho-\kappa}}{\partial\eta^{\varrho-\kappa}}\left[\partial_2^{(\jmath-1-\varrho)}\left(\mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,\eta)\right)\right]\right)\right]_{\xi=\eta=0}\frac{\partial^{\kappa+\mu}\mathbf{u}(0,0)}{\partial\xi^{\mu}\partial\eta^{\kappa}} \\
&\quad + p_1p_2\sum_{\varrho=0}^{\jmath-1}\sum_{\kappa=0}^{\varrho}\sum_{\ell=0}^{\iota-1}\sum_{\mu=0}^{\ell}\binom{\varrho}{\kappa}\binom{\ell}{\mu}\times \\
&\quad \frac{\partial^{\ell-\mu}}{\partial\xi^{\ell-\mu}}\left[\left.\frac{\partial^{\iota-1-\ell}}{\partial\xi^{\iota-1-\ell}}\right|_{\tau=p_1\xi}\left(\frac{\partial^{\varrho-\kappa}}{\partial\eta^{\varrho-\kappa}}\left[\partial_2^{(\jmath-1-\varrho)}\left(\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,p_2\eta)\right)\right]\right)\right]_{\xi=\eta=0}\frac{\partial^{\kappa+\mu}\mathbf{u}(0,0)}{\partial\xi^{\mu}\partial\eta^{\kappa}}.
\end{aligned}$$

Approximation on $\Omega_{n,0}$

For rectangles on the first row where $n \geq 1$, we define:

$$\mathcal{V}_{n,0}(\xi,\eta) = \sum_{\iota+\jmath=0}^{\mathrm{p}-1}\frac{1}{\iota!\,\jmath!}\frac{\partial^{\iota+\jmath}\widehat{\mathcal{V}}_{n,0}(\xi_n,0)}{\partial\xi^{\iota}\partial\eta^{\jmath}}(\xi-\xi_n)^{\iota}\eta^{\jmath}, \quad (\xi,\eta)\in\Omega_{n,0}, \tag{4.5}$$

where $\widehat{\mathcal{V}}_{n,0}$ satisfies the auxiliary integral equation:

$$\begin{aligned}
\widehat{\mathcal{V}}_{n,0}(\xi,\eta) = \mathbf{A}^{-1}(\xi,\eta)\mathbf{g}(\xi,\eta) &+ \sum_{\sigma=0}^{n-1}\int_{\xi_\sigma}^{\xi_{\sigma+1}}\int_0^{\eta}\mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,s)\mathcal{V}_{\sigma,0}(\tau,s)\,ds\,d\tau \\
&+ \int_{\xi_n}^{\xi}\int_0^{\eta}\mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,s)\widehat{\mathcal{V}}_{n,0}(\tau,s)\,ds\,d\tau \\
&+ \sum_{\sigma=0}^{n_0-1}\int_{\xi_\sigma}^{\xi_{\sigma+1}}\int_0^{p_2\eta}\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,s)\mathcal{V}_{\sigma,0}(\tau,s)\,ds\,d\tau \\
&+ \int_{\xi_{n_0}}^{p_1\xi}\int_0^{p_2\eta}\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,s)\mathcal{V}_{n_0,0}(\tau,s)\,ds\,d\tau,
\end{aligned} \tag{4.6}$$

with $p_1\xi \in [\xi_{n_0}, \xi_{n_0+1})$. To find $\dfrac{\partial^{\imath+\jmath}\widehat{\mathcal{V}}_{n,0}(\xi,\eta)}{\partial\xi^{\imath}\partial\eta^{\jmath}}$, we differentiate (4.6) $\jmath$-times with respect to η then ι-times with respect to ξ, to obtain:

$$
\begin{aligned}
&\frac{\partial^{\iota+\jmath}\widehat{\mathcal{V}}_{n,0}(\xi,\eta)}{\partial\xi^{\iota}\partial\eta^{\jmath}} = \partial_1^{(\iota)}\partial_2^{(\jmath)}\left(\mathbf{A}^{-1}(\xi,\eta)\mathbf{g}(\xi,\eta)\right) \\
&+ \sum_{\sigma=0}^{n-1}\sum_{\varrho=0}^{\jmath-1}\sum_{\kappa=0}^{\varrho}\binom{\varrho}{\kappa}\int_{\xi_\sigma}^{\xi_{\sigma+1}}\frac{\partial^{\iota}}{\partial\xi^{\iota}}\left[\frac{\partial^{\varrho-\kappa}}{\partial\eta^{\varrho-\kappa}}\left[\partial_2^{(\jmath-1-\varrho)}\left(\mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,\eta)\right)\right]\right]\frac{\partial^{\kappa}\mathcal{V}_{\sigma,0}(\tau,\eta)}{\partial\eta^{\kappa}}\,d\tau \\
&+ \sum_{\sigma=0}^{n-1}\int_{\xi_\sigma}^{\xi_{\sigma+1}}\int_0^{\eta}\partial_1^{(\iota)}\partial_2^{(\jmath)}\left(\mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,s)\right)\mathcal{V}_{\sigma,0}(\tau,s)\,ds\,d\tau \\
&+ \sum_{\varrho=0}^{\jmath-1}\sum_{\kappa=0}^{\varrho}\sum_{\ell=0}^{\iota-1}\sum_{\mu=0}^{\ell}\binom{\varrho}{\kappa}\binom{\ell}{\mu} \\
&\quad\times\frac{\partial^{\ell-\mu}}{\partial\xi^{\ell-\mu}}\left[\left.\frac{\partial^{\iota-1-\ell}}{\partial\xi^{\iota-1-\ell}}\right|_{\tau=\xi}\left(\frac{\partial^{\varrho-\kappa}}{\partial\eta^{\varrho-\kappa}}\left[\partial_2^{(\jmath-1-\varrho)}\left(\mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,\eta)\right)\right]\right)\right]\frac{\partial^{\kappa+\mu}\widehat{\mathcal{V}}_{n,0}(\xi,\eta)}{\partial\xi^{\mu}\partial\eta^{\kappa}} \\
&+ \sum_{\varrho=0}^{\jmath-1}\sum_{\kappa=0}^{\varrho}\binom{\varrho}{\kappa}\int_{\xi_n}^{\xi}\frac{\partial^{\iota}}{\partial\xi^{\iota}}\left[\frac{\partial^{\varrho-\kappa}}{\partial\eta^{\varrho-\kappa}}\left[\partial_2^{(\jmath-1-\varrho)}\left(\mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,\eta)\right)\right]\right]\frac{\partial^{\kappa}\widehat{\mathcal{V}}_{n,0}(\tau,\eta)}{\partial\eta^{\kappa}}\,d\tau \\
&+ \sum_{\ell=0}^{\iota-1}\sum_{\mu=0}^{\ell}\binom{\ell}{\mu}\int_0^{\eta}\frac{\partial^{\ell-\mu}}{\partial\xi^{\ell-\mu}}\left[\partial_1^{(\iota-1-\ell)}\partial_2^{(\jmath)}\left(\mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\xi,s)\right)\right]\frac{\partial^{\mu}\widehat{\mathcal{V}}_{n,0}(\xi,s)}{\partial\xi^{\mu}}\,ds \\
&+ \int_{\xi_n}^{\xi}\int_0^{\eta}\partial_1^{(\iota)}\partial_2^{(\jmath)}\left(\mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,s)\right)\widehat{\mathcal{V}}_{n,0}(\tau,s)\,ds\,d\tau \\
&+ p_2\sum_{\sigma=0}^{n_0-1}\sum_{\varrho=0}^{\jmath-1}\sum_{\kappa=0}^{\varrho}\binom{\varrho}{\kappa}p_2^{\kappa}\int_{\xi_\sigma}^{\xi_{\sigma+1}}\frac{\partial^{\iota}}{\partial\xi^{\iota}}\left[\frac{\partial^{\varrho-\kappa}}{\partial\eta^{\varrho-\kappa}}\left[\partial_2^{(\jmath-1-\varrho)}\left(\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,p_2\eta)\right)\right]\right]\frac{\partial^{\kappa}\mathcal{V}_{\sigma,0}(\tau,p_2\eta)}{\partial\eta^{\kappa}}\,d\tau \\
&+ \sum_{\sigma=0}^{n_0-1}\int_{\xi_\sigma}^{\xi_{\sigma+1}}\int_0^{p_2\eta}\partial_1^{(\iota)}\partial_2^{(\jmath)}\left(\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,s)\right)\mathcal{V}_{\sigma,0}(\tau,s)\,ds\,d\tau \\
&+ p_1p_2\sum_{\varrho=0}^{\jmath-1}\sum_{\kappa=0}^{\varrho}\sum_{\ell=0}^{\iota-1}\sum_{\mu=0}^{\ell}\binom{\varrho}{\kappa}\binom{\ell}{\mu}p_1^{\mu}p_2^{\kappa} \\
&\quad\times\frac{\partial^{\ell-\mu}}{\partial\xi^{\ell-\mu}}\left[\left.\frac{\partial^{\iota-1-\ell}}{\partial\xi^{\iota-1-\ell}}\right|_{\tau=p_1\xi}\left(\frac{\partial^{\varrho-\kappa}}{\partial\eta^{\varrho-\kappa}}\left[\partial_2^{(\jmath-1-\varrho)}\left(\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,p_2\eta)\right)\right]\right)\right]\frac{\partial^{\kappa+\mu}\mathcal{V}_{n_0,0}(p_1\xi,p_2\eta)}{\partial\xi^{\mu}\partial\eta^{\kappa}} \\
&+ p_2\sum_{\varrho=0}^{\jmath-1}\sum_{\kappa=0}^{\varrho}\binom{\varrho}{\kappa}p_2^{\kappa}\int_{\xi_{n_0}}^{p_1\xi}\frac{\partial^{\iota}}{\partial\xi^{\iota}}\left[\frac{\partial^{\varrho-\kappa}}{\partial\eta^{\varrho-\kappa}}\left[\partial_2^{(\jmath-1-\varrho)}\left(\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,p_2\eta)\right)\right]\right]\frac{\partial^{\kappa}\mathcal{V}_{n_0,0}(\tau,p_2\eta)}{\partial\eta^{\kappa}}\,d\tau \\
&+ p_1\sum_{\ell=0}^{\iota-1}\sum_{\mu=0}^{\ell}\binom{\ell}{\mu}p_1^{\mu}\int_0^{p_2\eta}\frac{\partial^{\ell-\mu}}{\partial\xi^{\ell-\mu}}\left[\partial_1^{(\iota-1-\ell)}\partial_2^{(\jmath)}\left(\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,p_1\xi,s)\right)\right]\frac{\partial^{\mu}\mathcal{V}_{n_0,0}(p_1\xi,s)}{\partial\xi^{\mu}}\,ds \\
&+ \int_{\xi_{n_0}}^{p_1\xi}\int_0^{p_2\eta}\partial_1^{(\iota)}\partial_2^{(\jmath)}\left(\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,s)\right)\mathcal{V}_{n_0,0}(\tau,s)\,ds\,d\tau.
\end{aligned}
\tag{4.7}
$$

Hence, evaluating at $(\xi_n, 0)$:

$$
\begin{aligned}
\frac{\partial^{\iota+\jmath}\widehat{\mathcal{V}}_{n,0}(\xi_n,0)}{\partial\xi^\iota\partial\eta^\jmath} &= \partial_1^{(\iota)}\partial_2^{(\jmath)}\left(\mathbf{A}^{-1}(\xi_n,0)\mathbf{g}(\xi_n,0)\right) \\
&+ \sum_{\sigma=0}^{n-1}\sum_{\varrho=0}^{\jmath-1}\sum_{\kappa=0}^{\varrho}\binom{\varrho}{\kappa}\int_{\xi_\sigma}^{\xi_{\sigma+1}}\frac{\partial^\iota}{\partial\xi^\iota}\left[\frac{\partial^{\varrho-\kappa}}{\partial\eta^{\varrho-\kappa}}\left[\partial_2^{(\jmath-1-\varrho)}\left(\mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,\eta)\right)\right]\right]\Bigg|_{\xi=\xi_n,\eta=0}\frac{\partial^\kappa\mathcal{V}_{\sigma,0}(\tau,0)}{\partial\eta^\kappa}\,d\tau \\
&+ \sum_{\varrho=0}^{\jmath-1}\sum_{\kappa=0}^{\varrho}\sum_{\ell=0}^{\iota-1}\sum_{\mu=0}^{\ell}\binom{\varrho}{\kappa}\binom{\ell}{\mu} \\
&\quad\times\frac{\partial^{\ell-\mu}}{\partial\xi^{\ell-\mu}}\left[\frac{\partial^{\iota-1-\ell}}{\partial\xi^{\iota-1-\ell}}\bigg|_{\tau=\xi}\left(\frac{\partial^{\varrho-\kappa}}{\partial\eta^{\varrho-\kappa}}\left[\partial_2^{(\jmath-1-\varrho)}\left(\mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,\eta)\right)\right]\right)\right]\Bigg|_{\xi=\xi_n,\eta=0}\frac{\partial^{\kappa+\mu}\widehat{\mathcal{V}}_{n,0}(\xi_n,0)}{\partial\xi^\mu\partial\eta^\kappa} \\
&+ p_2\sum_{\sigma=0}^{n_0-1}\sum_{\varrho=0}^{\jmath-1}\sum_{\kappa=0}^{\varrho}\binom{\varrho}{\kappa}p_2^\kappa\int_{\xi_\sigma}^{\xi_{\sigma+1}}\frac{\partial^\iota}{\partial\xi^\iota}\left[\frac{\partial^{\varrho-\kappa}}{\partial\eta^{\varrho-\kappa}}\left[\partial_2^{(\jmath-1-\varrho)}\left(\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,p_2\eta)\right)\right]\right]\Bigg|_{\xi=\xi_n,\eta=0}\frac{\partial^\kappa\mathcal{V}_{\sigma,0}(\tau,0)}{\partial\eta^\kappa}\,d\tau \\
&+ p_1p_2\sum_{\varrho=0}^{\jmath-1}\sum_{\kappa=0}^{\varrho}\sum_{\ell=0}^{\iota-1}\sum_{\mu=0}^{\ell}\binom{\varrho}{\kappa}\binom{\ell}{\mu}p_1^\mu p_2^\kappa \\
&\quad\times\frac{\partial^{\ell-\mu}}{\partial\xi^{\ell-\mu}}\left[\frac{\partial^{\iota-1-\ell}}{\partial\xi^{\iota-1-\ell}}\bigg|_{\tau=p_1\xi}\left(\frac{\partial^{\varrho-\kappa}}{\partial\eta^{\varrho-\kappa}}\left[\partial_2^{(\jmath-1-\varrho)}\left(\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,p_2\eta)\right)\right]\right)\right]\Bigg|_{\xi=\xi_n,\eta=0}\frac{\partial^{\kappa+\mu}\mathcal{V}_{n_0,0}(p_1\xi_n,0)}{\partial\xi^\mu\partial\eta^\kappa} \\
&+ p_2\sum_{\varrho=0}^{\jmath-1}\sum_{\kappa=0}^{\varrho}\binom{\varrho}{\kappa}p_2^\kappa\int_{\xi_{n_0}}^{p_1\xi_n}\frac{\partial^\iota}{\partial\xi^\iota}\left[\frac{\partial^{\varrho-\kappa}}{\partial\eta^{\varrho-\kappa}}\left[\partial_2^{(\jmath-1-\varrho)}\left(\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,p_2\eta)\right)\right]\right]_{\xi=\xi_n,\eta=0}\frac{\partial^\kappa\mathcal{V}_{n_0,0}(\tau,0)}{\partial\eta^\kappa}\,d\tau.
\end{aligned}
\tag{4.8}
$$

Approximation on $\Omega_{n,m}$

For interior rectangles with $n \geq 0$ and $m \geq 1$, we define:

$$
\mathcal{V}_{n,m}(\xi,\eta) = \sum_{\iota+\jmath=0}^{\mathfrak{p}-1}\frac{1}{\iota!\,\jmath!}\frac{\partial^{\iota+\jmath}\widehat{\mathcal{V}}_{n,m}(\xi_n,\eta_m)}{\partial\xi^\iota\partial\eta^\jmath}(\xi-\xi_n)^\iota(\eta-\eta_m)^\jmath, \quad (\xi,\eta)\in\Omega_{n,m}, \tag{4.9}
$$

where $\widehat{\mathcal{V}}_{n,m}$ satisfies:

$$
\begin{aligned}
\widehat{\mathcal{V}}_{n,m}(\xi,\eta) = \mathbf{A}^{-1}(\xi,\eta)\mathbf{g}(\xi,\eta) &+ \sum_{\sigma=0}^{n-1}\sum_{\lambda=0}^{m-1}\int_{\xi_\sigma}^{\xi_{\sigma+1}}\int_{\eta_\lambda}^{\eta_{\lambda+1}} \mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,s)\mathcal{V}_{\sigma,\lambda}(\tau,s)\,ds\,d\tau \\
&+ \sum_{\sigma=0}^{n-1}\int_{\xi_\sigma}^{\xi_{\sigma+1}}\int_{\eta_m}^{\eta} \mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,s)\mathcal{V}_{\sigma,m}(\tau,s)\,ds\,d\tau \\
&+ \sum_{\lambda=0}^{m-1}\int_{\xi_n}^{\xi}\int_{\eta_\lambda}^{\eta_{\lambda+1}} \mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,s)\mathcal{V}_{n,\lambda}(\tau,s)\,ds\,d\tau \\
&+ \int_{\xi_n}^{\xi}\int_{\eta_m}^{\eta} \mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,s)\widehat{\mathcal{V}}_{n,m}(\tau,s)\,ds\,d\tau \\
&+ \sum_{\sigma=0}^{n_0-1}\sum_{\lambda=0}^{m_0-1}\int_{\xi_\sigma}^{\xi_{\sigma+1}}\int_{\eta_\lambda}^{\eta_{\lambda+1}} \mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,s)\mathcal{V}_{\sigma,\lambda}(\tau,s)\,ds\,d\tau \\
&+ \sum_{\sigma=0}^{n_0-1}\int_{\xi_\sigma}^{\xi_{\sigma+1}}\int_{\eta_{m_0}}^{p_2\eta} \mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,s)\mathcal{V}_{\sigma,m_0}(\tau,s)\,ds\,d\tau \\
&+ \sum_{\lambda=0}^{m_0-1}\int_{\xi_{n_0}}^{p_1\xi}\int_{\eta_\lambda}^{\eta_{\lambda+1}} \mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,s)\mathcal{V}_{n_0,\lambda}(\tau,s)\,ds\,d\tau \\
&+ \int_{\xi_{n_0}}^{p_1\xi}\int_{\eta_{m_0}}^{p_2\eta} \mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,s)\mathcal{V}_{n_0,m_0}(\tau,s)\,ds\,d\tau,
\end{aligned}
\tag{4.10}
$$

with $p_1\xi \in [\xi_{n_0}, \xi_{n_0+1})$ and $p_2\eta \in [\eta_{m_0}, \eta_{m_0+1})$. Differentiation yields the partial derivatives at (ξ_n, η_m).

$$
\begin{aligned}
\frac{\partial^{\iota+\jmath}\widehat{\mathcal{V}}_{n,m}(\xi,\eta)}{\partial\xi^\iota\partial\eta^\jmath} &= \partial_1^{(\iota)}\partial_2^{(\jmath)}\Big(\mathbf{A}^{-1}(\xi,\eta)\mathbf{g}(\xi,\eta)\Big) \\
&+ \sum_{\sigma=0}^{n-1}\sum_{\lambda=0}^{m-1}\int_{\xi_\sigma}^{\xi_{\sigma+1}}\int_{\eta_\lambda}^{\eta_{\lambda+1}} \partial_1^{(\iota)}\partial_2^{(\jmath)}\Big(\mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,s)\Big)\mathcal{V}_{\sigma,\lambda}(\tau,s)\,ds\,d\tau \\
&+ \sum_{\sigma=0}^{n-1}\sum_{\varrho=0}^{\jmath-1}\sum_{\kappa=0}^{\varrho}\binom{\varrho}{\kappa}\int_{\xi_\sigma}^{\xi_{\sigma+1}} \frac{\partial^\iota}{\partial\xi^\iota}\left[\frac{\partial^{\varrho-\kappa}}{\partial\eta^{\varrho-\kappa}}\Big[\partial_2^{(\jmath-1-\varrho)}(\mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,\eta))\Big]\right]\frac{\partial^\kappa\mathcal{V}_{\sigma,m}(\tau,\eta)}{\partial\eta^\kappa}\,d\tau \\
&+ \sum_{\sigma=0}^{n-1}\int_{\xi_\sigma}^{\xi_{\sigma+1}}\int_{\eta_m}^{\eta} \partial_1^{(\iota)}\partial_2^{(\jmath)}\Big(\mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,s)\Big)\mathcal{V}_{\sigma,m}(\tau,s)\,ds\,d\tau \\
&+ \sum_{\lambda=0}^{m-1}\sum_{\ell=0}^{\iota-1}\sum_{\mu=0}^{\ell}\binom{\ell}{\mu}\int_{\eta_\lambda}^{\eta_{\lambda+1}} \frac{\partial^{\ell-\mu}}{\partial\xi^{\ell-\mu}}\left[\partial_1^{(\iota-1-\ell)}\partial_2^{(\jmath)}\Big(\mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\xi,s)\Big)\right]\frac{\partial^\mu\mathcal{V}_{n,\lambda}(\xi,s)}{\partial\xi^\mu}\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\lambda=0}^{m-1}\int_{\xi_n}^{\xi}\int_{\eta_\lambda}^{\eta_{\lambda+1}}\partial_1^{(\iota)}\partial_2^{(\jmath)}\Big(\mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,s)\Big)\mathcal{V}_{n,\lambda}(\tau,s)\,ds\,d\tau\\
&+\sum_{\varrho=0}^{\jmath-1}\sum_{\kappa=0}^{\varrho}\sum_{\ell=0}^{\iota-1}\sum_{\mu=0}^{\ell}\binom{\varrho}{\kappa}\binom{\ell}{\mu}\\
&\quad\times\frac{\partial^{\ell-\mu}}{\partial\xi^{\ell-\mu}}\left[\frac{\partial^{\iota-1-\ell}}{\partial\xi^{\iota-1-\ell}}\bigg|_{\tau=\xi}\left(\frac{\partial^{\varrho-\kappa}}{\partial\eta^{\varrho-\kappa}}\Big[\partial_2^{(\jmath-1-\varrho)}(\mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,\eta))\Big]\right)\right]\frac{\partial^{\kappa+\mu}\widehat{\mathcal{V}}_{n,m}(\xi,\eta)}{\partial\xi^{\mu}\partial\eta^{\kappa}}\\
&+\sum_{\varrho=0}^{\jmath-1}\sum_{\kappa=0}^{\varrho}\binom{\varrho}{\kappa}\int_{\xi_n}^{\xi}\frac{\partial^{\iota}}{\partial\xi^{\iota}}\left[\frac{\partial^{\varrho-\kappa}}{\partial\eta^{\varrho-\kappa}}\Big[\partial_2^{(\jmath-1-\varrho)}(\mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,\eta))\Big]\right]\frac{\partial^{\kappa}\widehat{\mathcal{V}}_{n,m}(\tau,\eta)}{\partial\eta^{\kappa}}\,d\tau\\
&+\sum_{\ell=0}^{\iota-1}\sum_{\mu=0}^{\ell}\binom{\ell}{\mu}\int_{\eta_m}^{\eta}\frac{\partial^{\ell-\mu}}{\partial\xi^{\ell-\mu}}\left[\partial_1^{(\iota-1-\ell)}\partial_2^{(\jmath)}\Big(\mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\xi,s)\Big)\right]\frac{\partial^{\mu}\widehat{\mathcal{V}}_{n,m}(\xi,s)}{\partial\xi^{\mu}}\,ds\\
&+\int_{\xi_n}^{\xi}\int_{\eta_m}^{\eta}\partial_1^{(\iota)}\partial_2^{(\jmath)}\Big(\mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,s)\Big)\widehat{\mathcal{V}}_{n,m}(\tau,s)\,ds\,d\tau\\
&+\sum_{\sigma=0}^{n_0-1}\sum_{\lambda=0}^{m_0-1}\int_{\xi_\sigma}^{\xi_{\sigma+1}}\int_{\eta_\lambda}^{\eta_{\lambda+1}}\partial_1^{(\iota)}\partial_2^{(\jmath)}\Big(\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,s)\Big)\mathcal{V}_{\sigma,\lambda}(\tau,s)\,ds\,d\tau\\
&+p_2\sum_{\sigma=0}^{n_0-1}\sum_{\varrho=0}^{\jmath-1}\sum_{\kappa=0}^{\varrho}\binom{\varrho}{\kappa}p_2^{\kappa}\int_{\xi_\sigma}^{\xi_{\sigma+1}}\frac{\partial^{\iota}}{\partial\xi^{\iota}}\left[\frac{\partial^{\varrho-\kappa}}{\partial\eta^{\varrho-\kappa}}\Big[\partial_2^{(\jmath-1-\varrho)}(\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,p_2\eta))\Big]\right]\frac{\partial^{\kappa}\mathcal{V}_{\sigma,m_0}(\tau,p_2\eta)}{\partial\eta^{\kappa}}\,d\tau\\
&+\sum_{\sigma=0}^{n_0-1}\int_{\xi_\sigma}^{\xi_{\sigma+1}}\int_{\eta_{m_0}}^{p_2\eta}\partial_1^{(\iota)}\partial_2^{(\jmath)}\Big(\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,s)\Big)\mathcal{V}_{\sigma,m_0}(\tau,s)\,ds\,d\tau\\
&+p_1\sum_{\lambda=0}^{m_0-1}\sum_{\ell=0}^{\iota-1}\sum_{\mu=0}^{\ell}\binom{\ell}{\mu}p_1^{\mu}\int_{\eta_\lambda}^{\eta_{\lambda+1}}\frac{\partial^{\ell-\mu}}{\partial\xi^{\ell-\mu}}\left[\partial_1^{(\iota-1-\ell)}\partial_2^{(\jmath)}\Big(\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,p_1\xi,s)\Big)\right]\frac{\partial^{\mu}\mathcal{V}_{n_0,\lambda}(p_1\xi,s)}{\partial\xi^{\mu}}\,ds\\
&+\sum_{\lambda=0}^{m_0-1}\int_{\xi_{n_0}}^{p_1\xi}\int_{\eta_\lambda}^{\eta_{\lambda+1}}\partial_1^{(\iota)}\partial_2^{(\jmath)}\Big(\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,s)\Big)\mathcal{V}_{n_0,\lambda}(\tau,s)\,ds\,d\tau\\
&+p_1p_2\sum_{\varrho=0}^{\jmath-1}\sum_{\kappa=0}^{\varrho}\sum_{\ell=0}^{\iota-1}\sum_{\mu=0}^{\ell}\binom{\varrho}{\kappa}\binom{\ell}{\mu}p_1^{\mu}p_2^{\kappa}\\
&\quad\times\frac{\partial^{\ell-\mu}}{\partial\xi^{\ell-\mu}}\left[\frac{\partial^{\iota-1-\ell}}{\partial\xi^{\iota-1-\ell}}\bigg|_{\tau=p_1\xi}\left(\frac{\partial^{\varrho-\kappa}}{\partial\eta^{\varrho-\kappa}}\Big[\partial_2^{(\jmath-1-\varrho)}(\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,p_2\eta))\Big]\right)\right]\frac{\partial^{\kappa+\mu}\mathcal{V}_{n_0,m_0}(p_1\xi,p_2\eta)}{\partial\xi^{\mu}\partial\eta^{\kappa}}\\
&+p_2\sum_{\varrho=0}^{\jmath-1}\sum_{\kappa=0}^{\varrho}\binom{\varrho}{\kappa}p_2^{\kappa}\int_{\xi_{n_0}}^{p_1\xi}\frac{\partial^{\iota}}{\partial\xi^{\iota}}\left[\frac{\partial^{\varrho-\kappa}}{\partial\eta^{\varrho-\kappa}}\Big[\partial_2^{(\jmath-1-\varrho)}(\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,p_2\eta))\Big]\right]\frac{\partial^{\kappa}\mathcal{V}_{n_0,m_0}(\tau,p_2\eta)}{\partial\eta^{\kappa}}\,d\tau\\
&+p_1\sum_{\ell=0}^{\iota-1}\sum_{\mu=0}^{\ell}\binom{\ell}{\mu}p_1^{\mu}\int_{\eta_{m_0}}^{p_2\eta}\frac{\partial^{\ell-\mu}}{\partial\xi^{\ell-\mu}}\left[\partial_1^{(\iota-1-\ell)}\partial_2^{(\jmath)}\Big(\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,p_1\xi,s)\Big)\right]\frac{\partial^{\mu}\mathcal{V}_{n_0,m_0}(p_1\xi,s)}{\partial\xi^{\mu}}\,ds
\end{aligned}
$$

$$+ \int_{\xi_{n_0}}^{p_1\xi} \int_{\eta_{m_0}}^{p_2\eta} \partial_1^{(\iota)} \partial_2^{(\jmath)} \Big(\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,s)\Big) \mathcal{V}_{n_0,m_0}(\tau,s)\, ds\, d\tau. \tag{4.11}$$

And here's the evaluated version at the grid point

$$\begin{aligned}
&\frac{\partial^{\iota+\jmath}\widehat{\mathcal{V}}_{n,m}(\xi_n,\eta_m)}{\partial\xi^\iota\partial\eta^\jmath} = \partial_1^{(\iota)}\partial_2^{(\jmath)}(\mathbf{A}^{-1}(\xi_n,\eta_m)\mathbf{g}(\xi_n,\eta_m)) \\
&+ \sum_{\sigma=0}^{n-1}\sum_{\lambda=0}^{m-1} \int_{\xi_\sigma}^{\xi_{\sigma+1}} \int_{\eta_\lambda}^{\eta_{\lambda+1}} \partial_1^{(\iota)}\partial_2^{(\jmath)}(\mathbf{A}^{-1}(\xi_n,\eta_m)\mathbf{K}(\xi_n,\eta_m,\tau,s))\mathcal{V}_{\sigma,\lambda}(\tau,s)\, ds\, d\tau \\
&+ \sum_{\sigma=0}^{n-1}\sum_{\varrho=0}^{\jmath-1}\sum_{\kappa=0}^{\varrho} \binom{\varrho}{\kappa} \int_{\xi_\sigma}^{\xi_{\sigma+1}} \frac{\partial^\iota}{\partial\xi^\iota}\left[\frac{\partial^{\varrho-\kappa}}{\partial\eta^{\varrho-\kappa}}[\partial_2^{(\jmath-1-\varrho)}(\mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,\eta))]\right]_{\substack{\xi=\xi_n\\ \eta=\eta_m}} \frac{\partial^\kappa\mathcal{V}_{\sigma,m}(\tau,\eta_m)}{\partial\eta^\kappa}\, d\tau \\
&+ \sum_{\lambda=0}^{m-1}\sum_{\ell=0}^{\iota-1}\sum_{\mu=0}^{\ell} \binom{\ell}{\mu} \int_{\eta_\lambda}^{\eta_{\lambda+1}} \frac{\partial^{\ell-\mu}}{\partial\xi^{\ell-\mu}}\left[\partial_1^{(\iota-1-\ell)}\partial_2^{(\jmath)}(\mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\xi,s))\right]_{\substack{\xi=\xi_n\\ \eta=\eta_m}} \frac{\partial^\mu\mathcal{V}_{n,\lambda}(\xi_n,s)}{\partial\xi^\mu}\, ds \\
&+ \sum_{\varrho=0}^{\jmath-1}\sum_{\kappa=0}^{\varrho}\sum_{\ell=0}^{\iota-1}\sum_{\mu=0}^{\ell} \binom{\varrho}{\kappa}\binom{\ell}{\mu}\times \\
&\quad \frac{\partial^{\ell-\mu}}{\partial\xi^{\ell-\mu}}\left[\frac{\partial^{\iota-1-\ell}}{\partial\xi^{\iota-1-\ell}}\bigg|_{\tau=\xi}\left(\frac{\partial^{\varrho-\kappa}}{\partial\eta^{\varrho-\kappa}}\left[\partial_2^{(\jmath-1-\varrho)}(\mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,\eta))\right]\right)\right]_{\substack{\xi=\xi_n\\ \eta=\eta_m}} \frac{\partial^{\kappa+\mu}\widehat{\mathcal{V}}_{n,m}(\xi_n,\eta_m)}{\partial\xi^\mu\partial\eta^\kappa} \\
&+ \sum_{\sigma=0}^{n_0-1}\sum_{\lambda=0}^{m_0-1} \int_{\xi_\sigma}^{\xi_{\sigma+1}} \int_{\eta_\lambda}^{\eta_{\lambda+1}} \partial_1^{(\iota)}\partial_2^{(\jmath)}(\mathbf{A}^{-1}(\xi_n,\eta_m)\mathbf{H}(\xi_n,\eta_m,\tau,s))\mathcal{V}_{\sigma,\lambda}(\tau,s)\, ds\, d\tau \\
&+ p_2\sum_{\sigma=0}^{n_0-1}\sum_{\varrho=0}^{\jmath-1}\sum_{\kappa=0}^{\varrho} \binom{\varrho}{\kappa} p_2^\kappa \int_{\xi_\sigma}^{\xi_{\sigma+1}} \frac{\partial^\iota}{\partial\xi^\iota}\left[\frac{\partial^{\varrho-\kappa}}{\partial\eta^{\varrho-\kappa}}[\partial_2^{(\jmath-1-\varrho)}(\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,p_2\eta))]\right]_{\substack{\xi=\xi_n\\ \eta=\eta_m}} \frac{\partial^\kappa\mathcal{V}_{\sigma,m_0}(\tau,p_2\eta_m)}{\partial\eta^\kappa}\, d\tau \\
&+ \sum_{\sigma=0}^{n_0-1} \int_{\xi_\sigma}^{\xi_{\sigma+1}} \int_{\eta_{m_0}}^{p_2\eta_m} \partial_1^{(\iota)}\partial_2^{(\jmath)}\Big(\mathbf{A}^{-1}(\xi_n,\eta_m)\mathbf{H}(\xi_n,\eta_m,\tau,s)\Big)\mathcal{V}_{\sigma,m_0}(\tau,s)\, ds\, d\tau \\
&+ p_1\sum_{\lambda=0}^{m_0-1}\sum_{\ell=0}^{\iota-1}\sum_{\mu=0}^{\ell} \binom{\ell}{\mu} p_1^\mu \int_{\eta_\lambda}^{\eta_{\lambda+1}} \frac{\partial^{\ell-\mu}}{\partial\xi^{\ell-\mu}}\left[\partial_1^{(\iota-1-\ell)}\partial_2^{(\jmath)}(\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,p_1\xi,s))\right]_{\substack{\xi=\xi_n\\ \eta=\eta_m}} \frac{\partial^\mu\mathcal{V}_{n_0,\lambda}(p_1\xi_n,s)}{\partial\xi^\mu}\, ds \\
&+ \sum_{\lambda=0}^{m_0-1} \int_{\xi_{n_0}}^{p_1\xi_n} \int_{\eta_\lambda}^{\eta_{\lambda+1}} \partial_1^{(\iota)}\partial_2^{(\jmath)}\Big(\mathbf{A}^{-1}(\xi_n,\eta_m)\mathbf{H}(\xi_n,\eta_m,\tau,s)\Big)\mathcal{V}_{n_0,\lambda}(\tau,s)\, ds\, d\tau \\
&+ p_1p_2\sum_{\varrho=0}^{\jmath-1}\sum_{\kappa=0}^{\varrho}\sum_{\ell=0}^{\iota-1}\sum_{\mu=0}^{\ell} \binom{\varrho}{\kappa}\binom{\ell}{\mu}\times p_1^\mu p_2^\kappa \\
&\quad \frac{\partial^{\ell-\mu}}{\partial\xi^{\ell-\mu}}\left[\frac{\partial^{\iota-1-\ell}}{\partial\xi^{\iota-1-\ell}}\bigg|_{\tau=p_1\xi}\left(\frac{\partial^{\varrho-\kappa}}{\partial\eta^{\varrho-\kappa}}\left[\partial_2^{(\jmath-1-\varrho)}(\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,p_2\eta))\right]\right)\right]_{\substack{\xi=\xi_n\\ \eta=\eta_m}} \frac{\partial^{\kappa+\mu}\mathcal{V}_{n_0,m_0}(p_1\xi_n,p_2\eta_m)}{\partial\xi^\mu\partial\eta^\kappa}
\end{aligned}$$

$$+ p_2 \sum_{\varrho=0}^{\jmath-1} \sum_{\kappa=0}^{\varrho} \binom{\varrho}{\kappa} p_2^{\kappa} \int_{\xi_{n_0}}^{p_1\xi_n} \frac{\partial^{\iota}}{\partial \xi^{\iota}} \left[\frac{\partial^{\varrho-\kappa}}{\partial \eta^{\varrho-\kappa}} \left[\partial_2^{(\jmath-1-\varrho)} (\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,p_2\eta)) \right] \right]_{\substack{\xi=\xi_n \\ \eta=\eta_m}} \frac{\partial^{\kappa} \mathcal{V}_{n_0,m_0}(\tau, p_2\eta_m)}{\partial \eta^{\kappa}} \, d\tau$$
$$+ p_1 \sum_{\ell=0}^{\iota-1} \sum_{\mu=0}^{\ell} \binom{\ell}{\mu} p_1^{\mu} \int_{\eta_{m_0}}^{p_2\eta_m} \frac{\partial^{\ell-\mu}}{\partial \xi^{\ell-\mu}} \left[\partial_1^{(\iota-1-\ell)} \partial_2^{(\jmath)} \Big(\mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,p_1\xi,s) \Big) \right]_{\substack{\xi=\xi_n \\ \eta=\eta_m}} \frac{\partial^{\mu} \mathcal{V}_{n_0,m_0}(p_1\xi_n, s)}{\partial \xi^{\mu}} \, ds$$
$$+ \int_{\xi_{n_0}}^{p_1\xi_n} \int_{\eta_{m_0}}^{p_2\eta_m} \partial_1^{(\iota)} \partial_2^{(\jmath)} \Big(\mathbf{A}^{-1}(\xi_n,\eta_m)\mathbf{H}(\xi_n,\eta_m,\tau,s) \Big) \mathcal{V}_{n_0,m_0}(\tau,s) \, ds \, d\tau.$$

4.3 Error Analysis

We begin with a lemma establishing the boundedness of the derivatives of the auxiliary functions.

Lemma 4.3.1 *Let* $\mathbf{g}$*,* $\mathbf{K}$ *and* $\mathbf{H}$ *be* $\mathfrak{p}$ *times continuously differentiable on their respective domains. Then, there exists a positive number* $\vartheta(\mathfrak{p})$ *such that for all* $n = 0, \ldots, N-1$*,* $m = 0, \ldots, M-1$*, and* $\iota + \jmath = 0, 1, \ldots, \mathfrak{p}$*, we have:*

$$\left\| \frac{\partial^{\iota+\jmath} \widehat{\mathcal{V}}_{n,m}}{\partial \xi^{\iota} \partial \eta^{\jmath}} \right\|_{L^{\infty}(\Omega_{n,m}, \mathbb{R}^r)} \leq \vartheta(\mathfrak{p}),$$

where $\widehat{\mathcal{V}}_{0,0}(\xi,\eta) = \mathbf{u}(\xi,\eta)$ *for* $(\xi,\eta) \in \Omega_{0,0}$*.*

Proof. Let $a_{n,m}^{\iota,\jmath} = \left\| \frac{\partial^{\iota+\jmath} \widehat{\mathcal{V}}_{n,m}}{\partial \xi^{\iota} \partial \eta^{\jmath}} \right\|_{L^{\infty}(\Omega_{n,m}, \mathbb{R}^r)}$ for all $\iota + \jmath = 0, 1, \ldots, \mathfrak{p}$. We proceed by induction on n and m.

Step 1: Initial rectangle $\Omega_{0,0}$. For the initial rectangle, we have the exact solution $\widehat{\mathcal{V}}_{0,0}(\xi,\eta) = \mathbf{u}(\xi,\eta)$. Therefore,

$$a_{0,0}^{\iota,\jmath} \leq \max \left\{ \left\| \frac{\partial^{\iota+\jmath} \mathbf{u}}{\partial \xi^{\iota} \partial \eta^{\jmath}} \right\|_{L^{\infty}(\Omega_{0,0}, \mathbb{R}^r)} : \iota + \jmath = 0, 1, \ldots, \mathfrak{p} \right\} = \vartheta_1(\mathfrak{p}). \tag{4.12}$$

Step 2: Boundary rectangles $\Omega_{n,0}$ for $n \geq 1$. Since $0 \leq p_1, p_2 < 1$, for $(\xi,\eta) \in \Omega_{n,0}$, we have $p_1\xi \in [\xi_{n_0}, \xi_{n_0+1})$ and $p_2\eta \in [\eta_0, \eta_1)$, where $n_0 \le n$. Thus, the largest possible value for n_0 is n, and the largest possible value for ξ_{n_0} is ξ_n.

Define $\Gamma_n = \max\{a^{\iota,\jmath}_{n,0} : \iota + \jmath = 0, 1, \ldots, \mathfrak{p}\}$ for $n = 0, 1, \ldots, N-1$.

From the differentiated form of the auxiliary integral equation for $\Omega_{n,0}$ (corresponding to (4.7) in the method description), we obtain for all $n = 1, \ldots, N-1$ and $\iota + \jmath = 0, 1, \ldots, \mathfrak{p}$:

$$
\begin{aligned}
a^{\iota,\jmath}_{n,0} \le C_1 &+ (C_2 + p_2C_2) \sum_{\kappa=0}^{n-1} \sum_{\varrho=0}^{\jmath-1} \sum_{\lambda=0}^{\varrho} \int_{\xi_\kappa}^{\xi_{\kappa+1}} \Gamma_\kappa \, d\tau + 2C_3 \sum_{\kappa=0}^{n-1} \int_{\xi_\kappa}^{\xi_{\kappa+1}} \int_0^\eta \Gamma_\kappa \, d\sigma d\tau \\
&+ (C_4 + p_1p_2C_4) \sum_{\varrho=0}^{\jmath-1} \sum_{\lambda=0}^{\varrho} \sum_{\ell=0}^{\iota-1} \sum_{\mu=0}^{\ell} a^{\mu,\lambda}_{n,0} + (C_2 + p_2C_2) \sum_{\varrho=0}^{\jmath-1} \sum_{\lambda=0}^{\varrho} \int_{\xi_n}^{\xi} a^{0,\lambda}_{n,0} \, d\tau \\
&+ (C_5 + p_1C_5) \int_0^\eta \sum_{\ell=0}^{\iota-1} \sum_{\mu=0}^{\ell} a^{\mu,0}_{n,0} \, d\sigma + 2C_3 \int_{\xi_n}^{\xi} \int_0^\eta a^{0,0}_{n,0} \, d\sigma d\tau,
\end{aligned}
$$

where the constants C_i ($i = 1, \ldots, 5$) are positive and independent of N and M, defined as:

$$
C_1 = \max\left\{ \left\| \partial_1^{(\iota)} \partial_2^{(\jmath)} (\mathbf{A}^{-1}\mathbf{g}) \right\|_{L^\infty(\Omega)} : \iota + \jmath = 0, 1, \ldots, \mathfrak{p} \right\},
$$

$$
\begin{aligned}
C_2 = \max\Bigg\{ &\binom{\varrho}{\lambda} \left\| \frac{\partial^\iota}{\partial \xi^\iota} \left[\frac{\partial^{\varrho-\lambda}}{\partial \eta^{\varrho-\lambda}} \left[\partial_2^{(\jmath-1-\varrho)} (\mathbf{A}^{-1}\mathbf{K}) \right] \right] \right\|_{L^\infty(\Omega)}, \\
&\binom{\varrho}{\lambda} \left\| \frac{\partial^\iota}{\partial \xi^\iota} \left[\frac{\partial^{\varrho-\lambda}}{\partial \eta^{\varrho-\lambda}} \left[\partial_2^{(\jmath-1-\varrho)} (\mathbf{A}^{-1}\mathbf{H}) \right] \right] \right\|_{L^\infty(\Omega)} : \varrho = 0, \ldots, \jmath-1; \lambda = 0, \ldots, \varrho \Bigg\},
\end{aligned}
$$

$$
C_3 = \max\left\{ \left\| \partial_1^{(\iota)} \partial_2^{(\jmath)} (\mathbf{A}^{-1}\mathbf{K}) \right\|_{L^\infty(\Omega)}, \left\| \partial_1^{(\iota)} \partial_2^{(\jmath)} (\mathbf{A}^{-1}\mathbf{H}) \right\|_{L^\infty(\Omega)} \right\},
$$

$$
\begin{aligned}
C_4 = \max\Bigg\{ &\binom{\varrho}{\lambda}\binom{\ell}{\mu} \left\| \frac{\partial^{\ell-\mu}}{\partial \xi^{\ell-\mu}} \left[\frac{\partial^{\iota-1-\ell}}{\partial \xi^{\iota-1-\ell}} \bigg|_{\tau=\xi} \left(\frac{\partial^{\varrho-\lambda}}{\partial \eta^{\varrho-\lambda}} \left[\partial_2^{(\jmath-1-\varrho)} (\mathbf{A}^{-1}\mathbf{K}) \right] \right) \right] \right\|_{L^\infty(\Omega)}, \\
&\binom{\varrho}{\lambda}\binom{\ell}{\mu} \left\| \frac{\partial^{\ell-\mu}}{\partial \xi^{\ell-\mu}} \left[\frac{\partial^{\iota-1-\ell}}{\partial \xi^{\iota-1-\ell}} \bigg|_{\tau=\xi} \left(\frac{\partial^{\varrho-\lambda}}{\partial \eta^{\varrho-\lambda}} \left[\partial_2^{(\jmath-1-\varrho)} (\mathbf{A}^{-1}\mathbf{H}) \right] \right) \right] \right\|_{L^\infty(\Omega)} : \\
&\varrho = 0, \ldots, \jmath-1; \lambda = 0, \ldots, \varrho; \ell = 0, \ldots, \iota-1; \mu = 0, \ldots, \ell \Bigg\},
\end{aligned}
$$

$$C_5 = \max\left\{\binom{\ell}{\mu}\left\|\frac{\partial^{\ell-\mu}}{\partial\xi^{\ell-\mu}}\left[\partial_1^{(\iota-1-\ell)}\partial_2^{(\jmath)}(\mathbf{A}^{-1}\mathbf{K})\right]\right\|_{L^\infty(\Omega)},\right.$$
$$\left.\binom{\ell}{\mu}\left\|\frac{\partial^{\ell-\mu}}{\partial\xi^{\ell-\mu}}\left[\partial_1^{(\iota-1-\ell)}\partial_2^{(\jmath)}(\mathbf{A}^{-1}\mathbf{H})\right]\right\|_{L^\infty(\Omega)} : \ell = 0,\dots,\iota-1; \mu = 0,\dots,\ell\right\}.$$

Noting that the sums over the indices introduce factors of at most $\mathfrak{p}$, we obtain:

$$\begin{aligned}
a_{n,0}^{\iota,\jmath} &\le C_1 + 2C_2\mathfrak{p}^2\mathfrak{h}\sum_{\kappa=0}^{n-1}\Gamma_\kappa + 2C_3\mathfrak{h}\mathfrak{k}\sum_{\kappa=0}^{n-1}\Gamma_\kappa + 2C_4\mathfrak{p}^2\sum_{\lambda=0}^{\jmath-1}\sum_{\mu=0}^{\iota-1}a_{n,0}^{\mu,\lambda} \\
&\quad + 2C_2\mathfrak{p}\mathfrak{h}\sum_{\lambda=0}^{\jmath-1}a_{n,0}^{0,\lambda} + 2C_5\mathfrak{p}\mathfrak{k}\sum_{\mu=0}^{\iota-1}a_{n,0}^{\mu,0} + 2C_3\mathfrak{h}\mathfrak{k}a_{n,0}^{0,0} \\
&\le C_1 + C_6\mathfrak{h}\sum_{\kappa=0}^{n-1}\Gamma_\kappa + C_7\sum_{\lambda=0}^{\jmath-1}\sum_{\mu=0}^{\iota-1}a_{n,0}^{\mu,\lambda},
\end{aligned} \tag{4.13}$$

where $C_6 = 2(C_2\mathfrak{p}^2 + C_3\mathfrak{b})$ and $C_7 = 2(C_4\mathfrak{p}^2 + C_2\mathfrak{p}\mathfrak{a} + C_5\mathfrak{p}\mathfrak{b} + C_3\mathfrak{a}\mathfrak{b})$, with a and b denoting the lengths of the intervals in the ξ and η directions, respectively.

Now, for each fixed n, we apply Lemma 1.7.3 (Pachpatte's inequality) with:

$$\varepsilon(\iota,\jmath) = a_{n,0}^{\iota,\jmath}, \quad a(\iota,\jmath) = C_1 + C_6\mathfrak{h}\sum_{\kappa=0}^{n-1}\Gamma_\kappa, \quad b(\iota,\jmath) = C_7.$$

Clearly, $a(\iota,\jmath)$ is nondecreasing in both variables. Applying the lemma yields:

$$\begin{aligned}
a_{n,0}^{\iota,\jmath} &\le \left(C_1 + C_6\mathfrak{h}\sum_{\kappa=0}^{n-1}\Gamma_\kappa\right)\prod_{s=0}^{\iota-1}\left[1 + \sum_{t=0}^{\jmath-1}C_7\right] \\
&\le \underbrace{C_1\left[1+\mathfrak{p}C_7\right]^{\mathfrak{p}}}_{C_8} + \underbrace{C_6\left[1+\mathfrak{p}C_7\right]^{\mathfrak{p}}}_{C_9}\mathfrak{h}\sum_{\kappa=0}^{n-1}\Gamma_\kappa.
\end{aligned} \tag{4.14}$$

Since $\Gamma_n = \max\{a_{n,0}^{\iota,\jmath}\}$, we obtain from (4.14):

$$\Gamma_n \leq C_8 + C_9\mathfrak{h}\sum_{\kappa=0}^{n-1}\Gamma_\kappa. \tag{4.15}$$

Applying Lemma 1.7.1 (the discrete Gronwall inequality) gives for all $n = 0, 1, \ldots, N-1$:

$$\Gamma_n \leq C_8 \exp(C_9\mathfrak{a}). \tag{4.16}$$

Step 3: Interior rectangles $\Omega_{n,m}$ for $m \geq 1$. Now consider the sequence $\Gamma_{n,m} = \max\{a_{n,m}^{\iota,\jmath} : \iota + \jmath = 0, 1, \ldots, \mathfrak{p}\}$ for $n = 0, 1, \ldots, N-1$ and $m = 0, 1, \ldots, M-1$.

Since $0 \leq p_1, p_2 < 1$, for $(\xi, \eta) \in \Omega_{n,m}$, we have $p_1\xi \in [\xi_{n_0}, \xi_{n_0+1})$ and $p_2\eta \in [\eta_{m_0}, \eta_{m_0+1})$, where $n_0 \leq n$ and $m_0 \leq m$. Thus, the largest possible values for (n_0, m_0) are (n, m), and the largest possible values for (ξ_{n_0}, η_{m_0}) are (ξ_n, η_m).

From the differentiated form of the auxiliary integral equation for $\Omega_{n,m}$ (corresponding to (3.8) in the method description), we obtain for all $n = 0, \ldots, N-1$, $m = 1, \ldots, M-1$, and $\iota + \jmath = 0, 1, \ldots, \mathfrak{p}$:

$$\begin{aligned}
a_{n,m}^{\iota,\jmath} \leq C_1 &+ 2C_3\mathfrak{h}\mathfrak{k}\sum_{\kappa=0}^{n-1}\sum_{\lambda=0}^{m-1}\Gamma_{\kappa,\lambda} + (C_2 + p_2C_2)\mathfrak{p}^2\mathfrak{h}\sum_{\kappa=0}^{n-1}\Gamma_{\kappa,m} \\
&+ 2C_3\mathfrak{h}\mathfrak{k}\sum_{\kappa=0}^{n-1}\Gamma_{\kappa,m} + (C_5 + p_1C_5)\mathfrak{p}^2\mathfrak{k}\sum_{\lambda=0}^{m-1}\Gamma_{n,\lambda} \\
&+ 2C_3\mathfrak{h}\mathfrak{k}\sum_{\lambda=0}^{m-1}\Gamma_{n,\lambda} + (C_4 + p_1p_2C_4)\mathfrak{p}^2\sum_{\lambda=0}^{\jmath-1}\sum_{\mu=0}^{\iota-1}a_{n,m}^{\mu,\lambda} \\
&+ (C_2 + p_2C_2)\mathfrak{p}\mathfrak{h}\sum_{\lambda=0}^{\jmath-1}a_{n,m}^{0,\lambda} + (C_5 + p_1C_5)\mathfrak{p}\mathfrak{k}\sum_{\mu=0}^{\iota-1}a_{n,m}^{\mu,0} \\
&+ 2C_3\mathfrak{h}\mathfrak{k}a_{n,m}^{0,0}.
\end{aligned}$$

Simplifying the constants, we obtain:

$$a_{n,m}^{\iota,\jmath} \leq C_1 + \mathcal{B}_1 \mathfrak{h}\mathfrak{k} \sum_{\kappa=0}^{n-1} \sum_{\lambda=0}^{m-1} \Gamma_{\kappa,\lambda} + \mathcal{B}_2 \mathfrak{h} \sum_{\kappa=0}^{n-1} \Gamma_{\kappa,m} + \mathcal{B}_3 \mathfrak{k} \sum_{\lambda=0}^{m-1} \Gamma_{n,\lambda}$$
$$+ \mathcal{B}_4 \sum_{\lambda=0}^{\jmath-1} \sum_{\mu=0}^{\iota-1} a_{n,m}^{\mu,\lambda}, \tag{4.17}$$

where

$$\begin{aligned}
\mathcal{B}_1 &= 2C_3,\\
\mathcal{B}_2 &= 2(C_2 \mathfrak{p}^2 + C_3 \mathfrak{b}),\\
\mathcal{B}_3 &= 2(C_5 \mathfrak{p}^2 + \mathfrak{a} C_3),\\
\mathcal{B}_4 &= 2(C_4 \mathfrak{p}^2 + C_2 \mathfrak{p}\mathfrak{a} + C_5 \mathfrak{p}\mathfrak{b} + C_3 \mathfrak{a}\mathfrak{b}).
\end{aligned}$$

For each fixed n and m, we again apply Lemma 1.7.3 with:

$$\varepsilon(\iota,\jmath) = a_{n,m}^{\iota,\jmath}, \quad a(\iota,\jmath) = C_1 + \mathcal{B}_1 \mathfrak{h}\mathfrak{k} \sum_{\kappa=0}^{n-1} \sum_{\lambda=0}^{m-1} \Gamma_{\kappa,\lambda} + \mathcal{B}_2 \mathfrak{h} \sum_{\kappa=0}^{n-1} \Gamma_{\kappa,m} + \mathcal{B}_3 \mathfrak{k} \sum_{\lambda=0}^{m-1} \Gamma_{n,\lambda}, \quad b(\iota,\jmath) = \mathcal{B}_4.$$

Applying Pachpatte's inequality yields:

$$\begin{aligned}
a_{n,m}^{\iota,\jmath} &\leq \left(C_1 + \mathcal{B}_1 \mathfrak{h}\mathfrak{k} \sum_{\kappa=0}^{n-1} \sum_{\lambda=0}^{m-1} \Gamma_{\kappa,\lambda} + \mathcal{B}_2 \mathfrak{h} \sum_{\kappa=0}^{n-1} \Gamma_{\kappa,m} + \mathcal{B}_3 \mathfrak{k} \sum_{\lambda=0}^{m-1} \Gamma_{n,\lambda} \right)\\
&\quad \times \prod_{s=0}^{\iota-1} \left[1 + \sum_{t=0}^{\jmath-1} \mathcal{B}_4 \right]\\
&\leq \underbrace{C_1 \left[1 + \mathfrak{p}\mathcal{B}_4\right]^{\mathfrak{p}}}_{\mathcal{B}_5} + \mathfrak{h}\mathfrak{k} \underbrace{\mathcal{B}_1 \left[1 + \mathfrak{p}\mathcal{B}_4\right]^{\mathfrak{p}}}_{\mathcal{B}_6} \sum_{\kappa=0}^{n-1} \sum_{\lambda=0}^{m-1} \Gamma_{\kappa,\lambda}\\
&\quad + \mathfrak{h} \underbrace{\mathcal{B}_2 \left[1 + \mathfrak{p}\mathcal{B}_4\right]^{\mathfrak{p}}}_{\mathcal{B}_7} \sum_{\kappa=0}^{n-1} \Gamma_{\kappa,m} + \mathfrak{k} \underbrace{\mathcal{B}_3 \left[1 + \mathfrak{p}\mathcal{B}_4\right]^{\mathfrak{p}}}_{\mathcal{B}_8} \sum_{\lambda=0}^{m-1} \Gamma_{n,\lambda}.
\end{aligned}$$

Since $\Gamma_{n,m} = \max\{a_{n,m}^{\iota,\jmath}\}$, we obtain:

$$\Gamma_{n,m} \leq \mathcal{B}_5 + \mathcal{B}_6\mathfrak{h}\mathfrak{k}\sum_{\kappa=0}^{n-1}\sum_{\lambda=0}^{m-1}\Gamma_{\kappa,\lambda} + \mathcal{B}_7\mathfrak{h}\sum_{\kappa=0}^{n-1}\Gamma_{\kappa,m} + \mathcal{B}_8\mathfrak{k}\sum_{\lambda=0}^{m-1}\Gamma_{n,\lambda}. \tag{4.18}$$

Now, applying Lemma 1.7.4 (the two-dimensional discrete Gronwall inequality) to (4.18) gives for all $n = 0, 1, \ldots, N-1$ and $m = 0, 1, \ldots, M-1$:

$$\Gamma_{n,m} \leq \mathcal{B}_5 \exp\left(\gamma_1(\mathfrak{a} + \mathfrak{b})\right), \tag{4.19}$$

where $\gamma_1 = \frac{1}{2}\left(\mathcal{B}_7 + \mathcal{B}_8 + \sqrt{(\mathcal{B}_7 + \mathcal{B}_8)^2 + 4\mathcal{B}_6}\right)$.

Step 4: Combining the bounds. From (4.12), (4.16), and (4.19), we define:

$$\vartheta(\mathfrak{p}) = \max\left\{\vartheta_1(\mathfrak{p}),\ C_8\exp(\mathfrak{a}C_9),\ \mathcal{B}_5\exp\left(\gamma_1(\mathfrak{a} + \mathfrak{b})\right)\right\}.$$

Thus, for all $n = 0, \ldots, N-1$, $m = 0, \ldots, M-1$, and $\iota + \jmath = 0, 1, \ldots, \mathfrak{p}$, we have:

$$\left\|\frac{\partial^{\iota+\jmath}\widehat{\mathcal{V}}_{n,m}}{\partial\xi^{\iota}\partial\eta^{\jmath}}\right\|_{L^\infty(\Omega_{n,m},\mathbb{R}^r)} \leq \vartheta(\mathfrak{p}),$$

which completes the proof of Lemma 4.3.1. ■

The main convergence theorem is stated below.

Theorem 4.3.1 *Let* **g**, **K** *and* **H** *be* $\mathfrak{p}$ *times continuously differentiable on their respective domains. Then equations* (4.4), (4.5), (4.9) *define a unique approximation* $\mathcal{V} \in (\mathcal{S}_{\mathfrak{p}-1,\mathfrak{p}-1}^{(-1)}(\Pi_{N,M}))^r$, *and the resulting error function* $\mathbf{e}(\xi, \eta) = \mathbf{u}(\xi, \eta) - \mathcal{V}(\xi, \eta)$ *satisfies:*

$$\|\mathbf{e}\|_{L^\infty(\Omega,\mathbb{R}^r)} \leq \mathfrak{C}(\mathfrak{h} + \mathfrak{k})^{\mathfrak{p}},$$

where $\mathfrak{C}$ *is a finite constant independent of* $\mathfrak{h}$ *and* $\mathfrak{k}$.

Proof. The proof is conducted in three stages, corresponding to the different types of subdomains.

Claim 1: Error on the initial rectangle $\Omega_{0,0}$. Define the error on $\Omega_{0,0}$ by $\mathbf{e}_{0,0}(\xi,\eta) = \mathbf{u}(\xi,\eta) - \mathcal{V}_{0,0}(\xi,\eta)$. We show that there exists a constant $\mathfrak{C}_1$ independent of $\mathfrak{h}$ and $\mathfrak{k}$ such that:

$$\|\mathbf{e}\|_{L^\infty(\Omega_{0,0},\mathbb{R}^r)} \leq \mathfrak{C}_1(\mathfrak{h}+\mathfrak{k})^{\mathfrak{p}}.$$

For $(\xi,\eta) \in \Omega_{0,0}$, applying Taylor's theorem to (4.4) yields:

$$\|\mathbf{e}\|_{L^\infty(\Omega_{0,0},\mathbb{R}^r)} \leq \sum_{\iota+\jmath=\mathfrak{p}} \frac{1}{\iota!\,\jmath!} \left\| \frac{\partial^{\iota+\jmath}\mathbf{u}}{\partial\xi^\iota \partial\eta^\jmath} \right\|_{L^\infty(\Omega_{0,0},\mathbb{R}^r)} \mathfrak{h}^\iota \mathfrak{k}^\jmath.$$

Using Lemma 4.3.1 to bound the derivatives of $\mathbf{u}$, we obtain:

$$\|\mathbf{e}\|_{L^\infty(\Omega_{0,0},\mathbb{R}^r)} \leq \vartheta(\mathfrak{p}) \sum_{\iota+\jmath=\mathfrak{p}} \frac{1}{\iota!\,\jmath!} \mathfrak{h}^\iota \mathfrak{k}^\jmath = \underbrace{\frac{\vartheta(\mathfrak{p})}{\mathfrak{p}!}}_{\mathfrak{C}_1} (\mathfrak{h}+\mathfrak{k})^{\mathfrak{p}}. \tag{4.20}$$

Claim 2: Error on boundary rectangles $\Omega_{n,0}$ for $n \geq 1$. Define the error on $\Omega_{n,0}$ by $\mathbf{e}_{n,0}(\xi,\eta) = \mathbf{u}(\xi,\eta) - \mathcal{V}_{n,0}(\xi,\eta)$ for $n = 1,\ldots,N-1$. Let $\|\mathbf{e}_n\| = \|\mathbf{e}\|_{L^\infty(\Omega_{n,0},\mathbb{R}^r)}$. We show that there exists a constant $\mathfrak{C}_2$ independent of $\mathfrak{h}$ and $\mathfrak{k}$ such that:

$$\|\mathbf{e}_n\| \leq \mathfrak{C}_2(\mathfrak{h}+\mathfrak{k})^{\mathfrak{p}}, \quad \forall n = 1,\ldots,N-1.$$

For $(\xi, \eta) \in \Omega_{n,0}$, from the auxiliary integral equation (4.6) we have:

$$\begin{aligned}
\mathbf{u}(\xi,\eta) - \widehat{\mathcal{V}}_{n,0}(\xi,\eta) &= \sum_{\kappa=0}^{n-1} \int_{\xi_\kappa}^{\xi_{\kappa+1}} \int_0^{\eta} \mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,\sigma)\mathbf{e}_{\kappa,0}(\tau,\sigma)\, d\sigma d\tau \\
&+ \int_{\xi_n}^{\xi} \int_0^{\eta} \mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,\sigma)(\mathbf{u}(\tau,\sigma) - \widehat{\mathcal{V}}_{n,0}(\tau,\sigma))\, d\sigma d\tau \\
&+ \sum_{\kappa=0}^{n_0-1} \int_{\xi_\kappa}^{\xi_{\kappa+1}} \int_0^{p_2\eta} \mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,\sigma)\mathbf{e}_{\kappa,0}(\tau,\sigma)\, d\sigma d\tau \\
&+ \int_{\xi_{n_0}}^{p_1\xi} \int_0^{p_2\eta} \mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,\sigma)\mathbf{e}_{n_0,0}(\tau,\sigma)\, d\sigma d\tau.
\end{aligned}$$

Since $n_0 \le n$, $p_1\xi \le \xi$, and $p_2\eta \le \eta$, we obtain:

$$\|\mathbf{u} - \widehat{\mathcal{V}}_{n,0}\| \le \sum_{\kappa=0}^{n-1} \mathfrak{h}\mathfrak{k}\overline{\mathcal{K}}\|\mathbf{e}_\kappa\| + \overline{\mathcal{K}} \int_{\xi_n}^{\xi} \int_0^{\eta} \|\mathbf{u} - \widehat{\mathcal{V}}_{n,0}\|\, d\sigma d\tau,$$

where $\overline{\mathcal{K}} = 2\max\{r\|\mathbf{A}^{-1}\mathbf{K}\|_{L^\infty(\Omega)}, r\|\mathbf{A}^{-1}\mathbf{H}\|_{L^\infty(\Omega)}\}$.

Applying Lemma 1.7.5 (Wendroff's inequality) yields:

$$\|\mathbf{u} - \widehat{\mathcal{V}}_{n,0}\| \le \sum_{\kappa=0}^{n-1} \mathfrak{h}\mathfrak{k}\overline{\mathcal{K}}\|\mathbf{e}_\kappa\| \exp\left(\overline{\mathcal{K}}ab\right) \le \sum_{\kappa=0}^{n-1} \mathfrak{h}\, \underbrace{b\overline{\mathcal{K}} \exp\left(\overline{\mathcal{K}}ab\right)}_{\lambda_1} \|\mathbf{e}_\kappa\|.$$

Now, using Taylor's theorem and Lemma 4.3.1:

$$\begin{aligned}
\|\mathbf{e}_n\| &\le \|\mathbf{u} - \widehat{\mathcal{V}}_{n,0}\| + \|\widehat{\mathcal{V}}_{n,0} - \mathcal{V}_{n,0}\| \\
&\le \sum_{\kappa=0}^{n-1} \mathfrak{h}\lambda_1\|\mathbf{e}_\kappa\| + \sum_{\iota+\jmath=\mathfrak{p}} \frac{1}{\iota!\,\jmath!} \left\| \frac{\partial^{\iota+\jmath}\widehat{\mathcal{V}}_{n,0}}{\partial\xi^\iota \partial\eta^\jmath} \right\| \mathfrak{h}^\iota \mathfrak{k}^\jmath \\
&\le \sum_{\kappa=0}^{n-1} \mathfrak{h}\lambda_1\|\mathbf{e}_\kappa\| + \frac{\vartheta(\mathfrak{p})}{\mathfrak{p}!}(\mathfrak{h}+\mathfrak{k})^{\mathfrak{p}}.
\end{aligned}$$

Applying Lemma 1.7.1 (the discrete Gronwall inequality) gives:

$$\|\mathbf{e}_n\| \leq \frac{\vartheta(\mathfrak{p})}{\mathfrak{p}!}(\mathfrak{h}+\mathfrak{k})^{\mathfrak{p}} \exp\left(a\lambda_1\right).$$

Thus, we set $\mathfrak{C}_2 = \dfrac{\vartheta(\mathfrak{p})}{\mathfrak{p}!}\exp(a\lambda_1)$.

Claim 3: Error on interior rectangles $\Omega_{n,m}$ for $m \geq 1$. Define the error on $\Omega_{n,m}$ by $\mathbf{e}_{n,m}(\xi,\eta) = \mathbf{u}(\xi,\eta) - \mathcal{V}_{n,m}(\xi,\eta)$ for $n = 0,\ldots,N-1$ and $m = 1,\ldots,M-1$. Let $\|\mathbf{e}_{n,m}\| = \|\mathbf{e}\|_{L^\infty(\Omega_{n,m},\mathbb{R}^r)}$. We show that there exists a constant $\mathfrak{C}_3$ independent of $\mathfrak{h}$ and $\mathfrak{k}$ such that:

$$\|\mathbf{e}_{n,m}\| \leq \mathfrak{C}_3(\mathfrak{h}+\mathfrak{k})^{\mathfrak{p}}, \quad \forall n = 0,\ldots,N-1,\ m = 1,\ldots,M-1.$$

For $(\xi,\eta) \in \Omega_{n,m}$, from the auxiliary integral equation (4.10) we have:

$$\begin{aligned}
\mathbf{u}(\xi,\eta) - \widehat{\mathcal{V}}_{n,m}(\xi,\eta) &= \sum_{\kappa=0}^{n-1}\sum_{\lambda=0}^{m-1}\int_{\xi_\kappa}^{\xi_{\kappa+1}}\int_{\eta_\lambda}^{\eta_{\lambda+1}} \mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,\sigma)\mathbf{e}_{\kappa,\lambda}(\tau,\sigma)\,d\sigma d\tau \\
&+ \sum_{\kappa=0}^{n-1}\int_{\xi_\kappa}^{\xi_{\kappa+1}}\int_{\eta_m}^{\eta} \mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,\sigma)\mathbf{e}_{\kappa,m}(\tau,\sigma)\,d\sigma d\tau \\
&+ \sum_{\lambda=0}^{m-1}\int_{\xi_n}^{\xi}\int_{\eta_\lambda}^{\eta_{\lambda+1}} \mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,\sigma)\mathbf{e}_{n,\lambda}(\tau,\sigma)\,d\sigma d\tau \\
&+ \int_{\xi_n}^{\xi}\int_{\eta_m}^{\eta} \mathbf{A}^{-1}(\xi,\eta)\mathbf{K}(\xi,\eta,\tau,\sigma)(\mathbf{u}(\tau,\sigma) - \widehat{\mathcal{V}}_{n,m}(\tau,\sigma))\,d\sigma d\tau \\
&+ \sum_{\kappa=0}^{n_0-1}\sum_{\lambda=0}^{m_0-1}\int_{\xi_\kappa}^{\xi_{\kappa+1}}\int_{\eta_\lambda}^{\eta_{\lambda+1}} \mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,\sigma)\mathbf{e}_{\kappa,\lambda}(\tau,\sigma)\,d\sigma d\tau \\
&+ \sum_{\kappa=0}^{n_0-1}\int_{\xi_\kappa}^{\xi_{\kappa+1}}\int_{\eta_{m_0}}^{p_2\eta} \mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,\sigma)\mathbf{e}_{\kappa,m_0}(\tau,\sigma)\,d\sigma d\tau \\
&+ \sum_{\lambda=0}^{m_0-1}\int_{\xi_{n_0}}^{p_1\xi}\int_{\eta_\lambda}^{\eta_{\lambda+1}} \mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,\sigma)\mathbf{e}_{n_0,\lambda}(\tau,\sigma)\,d\sigma d\tau \\
&+ \int_{\xi_{n_0}}^{p_1\xi}\int_{\eta_{m_0}}^{p_2\eta} \mathbf{A}^{-1}(\xi,\eta)\mathbf{H}(\xi,\eta,\tau,\sigma)\mathbf{e}_{n_0,m_0}(\tau,\sigma)\,d\sigma d\tau.
\end{aligned}$$

Since $n_0 \leq n$, $m_0 \leq m$, $p_1\xi \leq \xi$, and $p_2\eta \leq \eta$, we obtain:

$$\|\mathbf{u}-\widehat{\mathcal{V}}_{n,m}\| \leq \sum_{\kappa=0}^{n-1}\sum_{\lambda=0}^{m-1} \mathfrak{h}\mathfrak{k}\overline{\mathcal{K}}\|\mathbf{e}_{\kappa,\lambda}\| + \sum_{\kappa=0}^{n-1} \mathfrak{h}\mathfrak{k}\overline{\mathcal{K}}\|\mathbf{e}_{\kappa,m}\| + \sum_{\lambda=0}^{m-1} \mathfrak{h}\mathfrak{k}\overline{\mathcal{K}}\|\mathbf{e}_{n,\lambda}\| + \overline{\mathcal{K}} \int_{\xi_n}^{\xi}\int_{\eta_m}^{\eta} \|\mathbf{u}-\widehat{\mathcal{V}}_{n,m}\|\, d\sigma d\tau.$$

Applying Lemma 1.7.5 yields:

$$\begin{aligned}
\|\mathbf{u}-\widehat{\mathcal{V}}_{n,m}\| \leq{}& \sum_{\kappa=0}^{n-1}\sum_{\lambda=0}^{m-1} \mathfrak{h}\mathfrak{k}\underbrace{\overline{\mathcal{K}}\exp\left(\overline{\mathcal{K}}ab\right)}_{\lambda_2}\|\mathbf{e}_{\kappa,\lambda}\| \\
&+ \sum_{\kappa=0}^{n-1} \mathfrak{h}\underbrace{b\overline{\mathcal{K}}\exp\left(\overline{\mathcal{K}}ab\right)}_{\lambda_3}\|\mathbf{e}_{\kappa,m}\| \\
&+ \sum_{\lambda=0}^{m-1} \mathfrak{k}\underbrace{a\overline{\mathcal{K}}\exp\left(\overline{\mathcal{K}}ab\right)}_{\lambda_4}\|\mathbf{e}_{n,\lambda}\|.
\end{aligned}$$

Now, using Taylor's theorem and Lemma 4.3.1:

$$\begin{aligned}
\|\mathbf{e}_{n,m}\| &\leq \|\mathbf{u}-\widehat{\mathcal{V}}_{n,m}\| + \|\widehat{\mathcal{V}}_{n,m}-\mathcal{V}_{n,m}\| \\
&\leq \sum_{\kappa=0}^{n-1}\sum_{\lambda=0}^{m-1} \mathfrak{h}\mathfrak{k}\lambda_2\|\mathbf{e}_{\kappa,\lambda}\| + \sum_{\kappa=0}^{n-1}\mathfrak{h}\lambda_3\|\mathbf{e}_{\kappa,m}\| + \sum_{\lambda=0}^{m-1}\mathfrak{k}\lambda_4\|\mathbf{e}_{n,\lambda}\| \\
&\quad + \sum_{\iota+\jmath=\mathfrak{p}} \frac{1}{\iota!\,\jmath!}\left\|\frac{\partial^{\iota+\jmath}\widehat{\mathcal{V}}_{n,m}}{\partial\xi^{\iota}\partial\eta^{\jmath}}\right\|\mathfrak{h}^{\iota}\mathfrak{k}^{\jmath} \\
&\leq \sum_{\kappa=0}^{n-1}\sum_{\lambda=0}^{m-1} \mathfrak{h}\mathfrak{k}\lambda_2\|\mathbf{e}_{\kappa,\lambda}\| + \sum_{\kappa=0}^{n-1}\mathfrak{h}\lambda_3\|\mathbf{e}_{\kappa,m}\| + \sum_{\lambda=0}^{m-1}\mathfrak{k}\lambda_4\|\mathbf{e}_{n,\lambda}\| \\
&\quad + \frac{\vartheta(\mathfrak{p})}{\mathfrak{p}!}(\mathfrak{h}+\mathfrak{k})^{\mathfrak{p}}.
\end{aligned}$$

Applying Lemma 1.7.4 to this recursive inequality gives:

$$\|\mathbf{e}_{n,m}\| \leq \frac{\vartheta(\mathfrak{p})}{\mathfrak{p}!}(\mathfrak{h}+\mathfrak{k})^{\mathfrak{p}}\exp\left(\gamma_2(a+b)\right),$$

where $\gamma_2 = \frac{1}{2}\left(\lambda_3 + \lambda_4 + \sqrt{(\lambda_3+\lambda_4)^2 + 4\lambda_2}\right)$.

Thus, we set $\mathfrak{C}_3 = \dfrac{\vartheta(\mathfrak{p})}{\mathfrak{p}!}\exp(\gamma_2(a+b))$.

Conclusion. Combining the three claims, we obtain:

$$\|\mathbf{e}\|_{L^\infty(\Omega,\mathbb{R}^r)} \leq \mathfrak{C}(\mathfrak{h}+\mathfrak{k})^{\mathfrak{p}},$$

where $\mathfrak{C} = \max\{\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3\}$ is independent of $\mathfrak{h}$ and $\mathfrak{k}$. This completes the proof of Theorem 4.3.1. ■

4.4 Numerical Examples

To illustrate the theoretical results, we present numerical experiments for selected test problems. For each example, we compute the approximate solution $\mathcal{V}$ using the proposed Taylor collocation method with various values of $\mathfrak{p}$, N, and M. The maximum error $E_{N,M} = \max_{(\xi,\eta)\in\Omega} \|\mathbf{u}(\xi,\eta) - \mathcal{V}(\xi,\eta)\|_\infty$ is estimated at a set of test points. The numerical order of convergence is calculated as:

$$\text{Order} \approx \frac{\log(E_{N,M}/E_{2N,2M})}{\log(2)}.$$

In this section, several numerical examples are given to show the efficiency of our proposed method for approximating the solution of system of 2D-VIEs.

Example 4.4.1 *Consider the following system of 2D-VIEs as defined in* (4.1)*:*

$$\sum_{\beta=1}^{3} \mathcal{A}_{\alpha,\beta}(x,y)\mathcal{U}_\beta(x,y) = \mathcal{G}_\alpha(x,y) + \sum_{\beta=1}^{3}\int_0^x\int_0^y \mathcal{K}_{\alpha,\beta}(x,y,t,s)\mathcal{U}_\beta(t,s)\,ds\,dt$$
$$+ \sum_{\beta=1}^{3}\int_0^{\frac{x}{4}}\int_0^{\frac{y}{3}} \mathcal{H}_{\alpha,\beta}(x,y,t,s)\mathcal{U}_\beta(t,s)\,ds\,dt,$$

for $\alpha = 1, 2, 3$, where the coefficient matrices are given by:

$$\{\mathcal{A}_{\alpha,\beta}(x,y)\}_{\alpha,\beta=1}^{3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\{\mathcal{K}_{\alpha,\beta}(x,y,t,s)\}_{\alpha,\beta=1}^{3} = \begin{pmatrix} 0 & 2x & e^{-s} \\ t^2 s & 0 & sy \\ x & s-t & 0 \end{pmatrix},$$

$$\{\mathcal{H}_{\alpha,\beta}(x,y,t,s)\}_{\alpha,\beta=1}^{3} = \begin{pmatrix} \cos(y) & e^{-s} & 0 \\ sy & 0 & 0 \\ e^{-t} & 0 & x \end{pmatrix}.$$

The functions $\mathcal{G}_\alpha(x,y)$, $\alpha = 1, 2, 3$, are chosen so that the exact solution of this system is $\mathcal{U}_1(x,y) = xe^y$, $\mathcal{U}_2(x,y) = xy^3$ and $\mathcal{U}_3(x,y) = \cos(y)$.

The absolute error of the proposed method for $N = M = 15$, $p = 4$ is shown in Table 4.1. Also Figure 4.1 shows the comparison between exact and approximate solutions for $N = M = 15$ and $p = 4$. Also table 4.2 presents a convergence study demonstrating the expected $O((\mathfrak{h} + \mathfrak{k}))$ rate.

Example 4.4.2 *Consider the following system of 2D-VIEs as defined in* (4.1) *with $r = 3$, $\mathcal{A}_{\alpha,\beta}(x,y) = \delta_{\alpha,\beta}$ (Kronecker delta), and the kernel functions given by:*

$$\mathcal{K}_{1,1}(x,y,t,s) = 0, \quad \mathcal{K}_{1,2}(x,y,t,s) = 2x, \quad \mathcal{K}_{1,3}(x,y,t,s) = e^{-s},$$

$$\mathcal{K}_{2,1}(x,y,t,s) = st^2, \quad \mathcal{K}_{2,2}(x,y,t,s) = 0, \quad \mathcal{K}_{2,3}(x,y,t,s) = sy,$$

$$\mathcal{K}_{3,1}(x,y,t,s) = x, \quad \mathcal{K}_{3,2}(x,y,t,s) = s-t, \quad \mathcal{K}_{3,3}(x,y,t,s) = 0,$$

and $\mathcal{H}_{\alpha,\beta}(x,y,t,s) = 0$ for all $\alpha, \beta = 1, 2, 3$. The functions $\mathcal{G}_\alpha(x,y)$, $\alpha = 1, 2, 3$, are given by:

$$\mathcal{G}_1(x,y) = xe^y - \frac{1}{2}x - \frac{1}{4}x^3y^4 + \frac{1}{2e^y}x\cos(y) - \frac{1}{2e^y}x\sin(y),$$

Table 4.1: Absolute errors for Example 4.4.1 for $N = M = 15$

$(x, y) = (\frac{i}{15}, \frac{i}{15})$	$\lvert\mathcal{U}_1 - \mathcal{V}_1\rvert$	$\lvert\mathcal{U}_2 - \mathcal{V}_2\rvert$	$\lvert\mathcal{U}_3 - \mathcal{V}_3\rvert$
$(0, 0)$	0	0	0
$i = 1$	2.20×10^{-9}	2.23×10^{-12}	1.00×10^{-9}
$i = 2$	1.48×10^{-8}	4.40×10^{-11}	2.40×10^{-9}
$i = 3$	5.20×10^{-8}	2.20×10^{-10}	7.10×10^{-9}
$i = 4$	1.14×10^{-7}	7.44×10^{-10}	1.42×10^{-8}
$i = 5$	2.01×10^{-7}	1.89×10^{-9}	2.41×10^{-8}
$i = 6$	3.20×10^{-7}	5.25×10^{-9}	4.83×10^{-8}
$i = 7$	4.31×10^{-7}	1.28×10^{-8}	8.34×10^{-8}
$i = 8$	4.93×10^{-7}	2.84×10^{-8}	1.32×10^{-7}
$i = 9$	4.12×10^{-7}	6.18×10^{-8}	2.11×10^{-7}
$i = 10$	1.70×10^{-7}	1.24×10^{-7}	3.12×10^{-7}
$i = 11$	9.23×10^{-7}	2.43×10^{-7}	4.56×10^{-7}
$i = 12$	2.76×10^{-6}	4.53×10^{-7}	6.34×10^{-7}
$i = 13$	6.10×10^{-6}	7.97×10^{-7}	8.24×10^{-7}
$i = 14$	1.16×10^{-5}	1.33×10^{-6}	9.82×10^{-7}
$i = 15$	9.07×10^{-6}	5.91×10^{-4}	1.14×10^{-4}
CPU time/s		875.62	

Table 4.2: Convergence study for Example 4.4.1 with $p = 3$

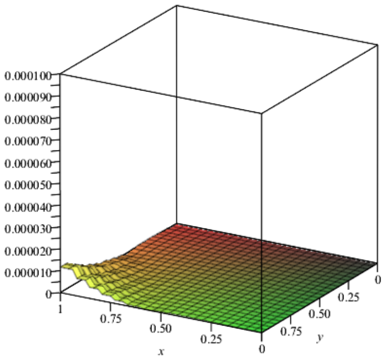
$N = M$	$\lVert e_1\rVert_\infty$	EOC	$\lVert e_2\rVert_\infty$	EOC	$\lVert e_3\rVert_\infty$	EOC	CPU (s)
4	2.83×10^{-2}	—	5.15×10^{-2}	—	3.88×10^{-3}	—	5.62
8	4.27×10^{-3}	2.73	7.24×10^{-3}	2.83	6.57×10^{-4}	2.56	35.03
16	5.85×10^{-4}	2.87	9.55×10^{-4}	2.92	9.45×10^{-5}	2.80	720.60

$$\mathcal{G}_2(x, y) = xy^3 - \frac{1}{4}x^4e^y y + \frac{1}{4}x^4e^y - \frac{1}{4}x^4 - \sin(y)xy^2 - xy\cos(y) + xy,$$

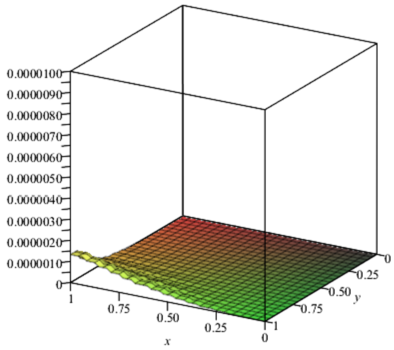
$$\mathcal{G}_3(x, y) = \cos(y) + \frac{1}{12}x^3y^4 + \frac{1}{2}x^3 - \frac{1}{2}x^3e^y - \frac{1}{10}x^2y^5.$$

The exact solution of this system is $\mathcal{U}_1(x, y) = xe^y$, $\mathcal{U}_2(x, y) = xy^3$ *and* $\mathcal{U}_3(x, y) = \cos(y)$.

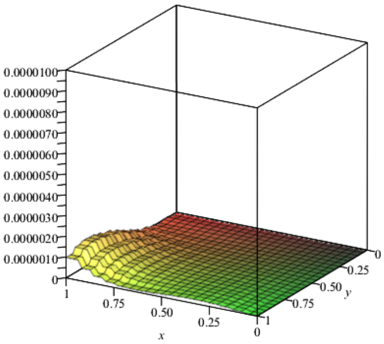
The absolute error of the proposed method for $N = M = 10$, $p = 3$ *is shown in Table 4.3. Also Figure 4.2 shows the comparison between exact and approximate solutions for* $N = M = 10$ *and* $p = 3$.



(a) The error $|\mathcal{U}_1 - \mathcal{V}_1|$ for $N = M = 15$.



(b) The error $|\mathcal{U}_2 - \mathcal{V}_2|$ for $N = M = 15$.



(c) The error $|\mathcal{U}_3 - \mathcal{V}_3|$ for $N = M = 15$.

Figure 4.1: Comparison between exact and approximate solutions for Example 4.4.1

Example 4.4.3 *Consider the following system of 2D-VIEs as defined in* (4.1) *with* $r = 2$, $\mathcal{A}_{\alpha,\beta}(x,y) = \delta_{\alpha,\beta}$, *and the kernel functions given by:*

$$\mathcal{K}_{1,1}(x,y,t,s) = \cos(s), \quad \mathcal{K}_{1,2}(x,y,t,s) = \sin(s),$$

$$\mathcal{K}_{2,1}(x,y,t,s) = xt^2, \quad \mathcal{K}_{2,2}(x,y,t,s) = x + t,$$

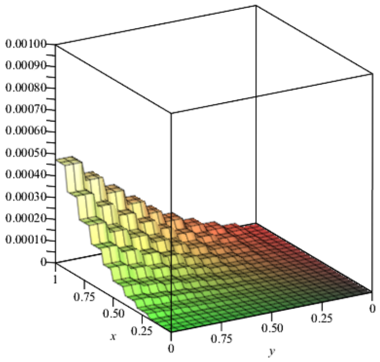
$$\mathcal{H}_{1,1}(x,y,t,s) = 1, \quad \mathcal{H}_{1,2}(x,y,t,s) = 0, \quad \mathcal{H}_{2,1}(x,y,t,s) = 0, \quad \mathcal{H}_{2,2}(x,y,t,s) = 1,$$

with $p_1 = \frac{1}{4}$ *and* $p_2 = \frac{1}{3}$. *The functions* $\mathcal{G}_1(x,y)$ *and* $\mathcal{G}_2(x,y)$ *are given by:*

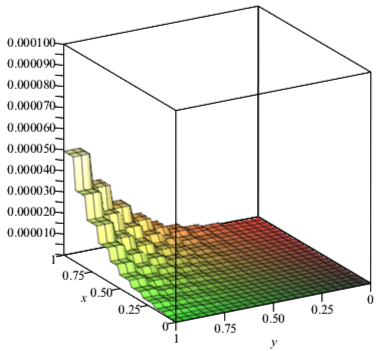
$$\mathcal{G}_1(x,y) = x\cos(y) - \frac{1}{2}x^2 y - 0.03125x^2 \sin\left(\frac{y}{3}\right),$$

Table 4.3: Absolute errors for Example 4.4.2 for $N = M = 10$

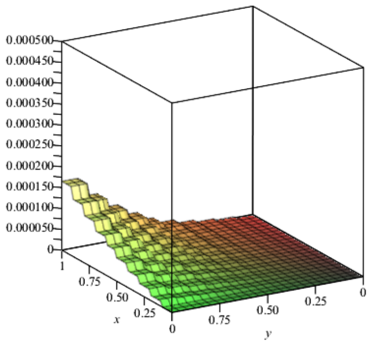
(x, y)	$\lvert\mathcal{U}_1 - \mathcal{V}_1\rvert$	$\lvert\mathcal{U}_2 - \mathcal{V}_2\rvert$	$\lvert\mathcal{U}_3 - \mathcal{V}_3\rvert$
$(0, 0)$	0	0	0
$(0.1, 0.1)$	3.26×10^{-8}	3.91×10^{-10}	8.81×10^{-8}
$(0.2, 0.2)$	8.99×10^{-7}	1.19×10^{-8}	7.93×10^{-7}
$(0.3, 0.3)$	5.09×10^{-6}	9.16×10^{-8}	2.96×10^{-6}
$(0.4, 0.4)$	1.69×10^{-5}	4.08×10^{-7}	7.75×10^{-6}
$(0.5, 0.5)$	4.24×10^{-5}	1.36×10^{-6}	1.68×10^{-5}
$(0.6, 0.6)$	8.98×10^{-5}	3.82×10^{-6}	3.24×10^{-5}
$(0.7, 0.7)$	1.69×10^{-4}	9.59×10^{-6}	5.82×10^{-5}
$(0.8, 0.8)$	2.92×10^{-4}	2.23×10^{-5}	9.96×10^{-5}
$(0.9, 0.9)$	4.73×10^{-4}	4.92×10^{-5}	1.65×10^{-4}



(a) The error $\lvert\mathcal{U}_1 - \mathcal{V}_1\rvert$ for $N = M = 10$.



(b) The error $\lvert\mathcal{U}_2 - \mathcal{V}_2\rvert$ for $N = M = 10$.



(c) The error $\lvert\mathcal{U}_3 - \mathcal{V}_3\rvert$ for $N = M = 10$.

Figure 4.2: Comparison between exact and approximate solutions for Example 4.4.2

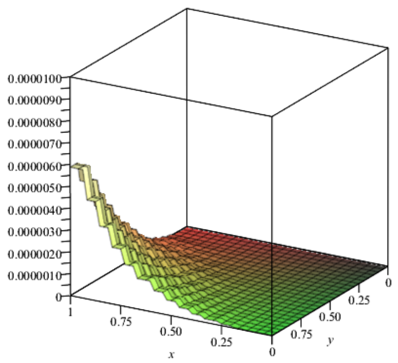
$$\mathcal{G}_2(x, y) = x\sin(y) - \frac{1}{4}\sin(y)x^5 + \frac{5}{6}x^3 - \frac{5}{6}x^3 + 0.03125x^2\cos\left(\frac{y}{3}\right) - 0.03125x^2.$$

The exact solution of this system is $\mathcal{U}_1(x, y) = x\cos(y)$ *and* $\mathcal{U}_2(x, y) = x\sin(y)$.

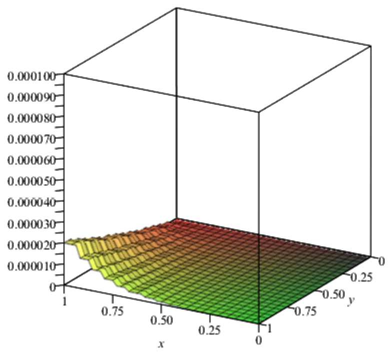
The absolute error of the proposed method for $N = M = 15$, $p = 4$ is shown in Table 4.4. Also Figure 4.3 shows the comparison between exact and approximate solutions for $N = M = 15$ and $p = 4$.

Table 4.4: Absolute errors for Example 4.4.3 for $N = M = 15$

$(x, y) = (\frac{i}{15}, \frac{i}{15})$	$\lvert\mathcal{U}_1 - \mathcal{V}_1\rvert$	$\lvert\mathcal{U}_2 - \mathcal{V}_2\rvert$
$(0, 0)$	0	0
$i = 1$	7.00×10^{-11}	2.07×10^{-10}
$i = 2$	3.00×10^{-10}	1.76×10^{-9}
$i = 3$	1.10×10^{-9}	8.36×10^{-9}
$i = 4$	4.60×10^{-9}	2.66×10^{-8}
$i = 5$	1.68×10^{-8}	7.37×10^{-8}
$i = 6$	4.75×10^{-8}	1.84×10^{-7}
$i = 7$	1.14×10^{-7}	4.09×10^{-7}
$i = 8$	2.44×10^{-7}	8.41×10^{-7}
$i = 9$	4.77×10^{-7}	1.61×10^{-6}
$i = 10$	8.65×10^{-7}	2.89×10^{-6}
$i = 11$	1.49×10^{-6}	4.94×10^{-6}
$i = 12$	2.43×10^{-6}	8.14×10^{-6}
$i = 13$	3.84×10^{-6}	1.29×10^{-5}
$i = 14$	5.87×10^{-6}	1.99×10^{-5}
$i = 15$	5.43×10^{-6}	2.85×10^{-4}
CPU time/s	312.57	



(a) The error $\lvert\mathcal{U}_1 - \mathcal{V}_1\rvert$ for $N = M = 15$.



(b) The error $\lvert\mathcal{U}_2 - \mathcal{V}_2\rvert$ for $N = M = 15$.

Figure 4.3: Comparison between exact and approximate solutions for Example 4.4.3

Example 4.4.4 *Consider the following system of 2D-VIEs as defined in* (4.1) *with $r = 2$ and* *the coefficient matrix:*

$$\{\mathcal{A}_{\alpha,\beta}(x,y)\}_{\alpha,\beta=1}^{2} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix},$$

and the kernel functions given by:

$$\mathcal{K}_{1,1}(x,y,t,s) = 2x, \quad \mathcal{K}_{1,2}(x,y,t,s) = e^{-s},$$

$$\mathcal{K}_{2,1}(x,y,t,s) = t^2 s^3, \quad \mathcal{K}_{2,2}(x,y,t,s) = sy,$$

with $\mathcal{H}_{\alpha,\beta}(x,y,t,s) = 0$ for all $\alpha, \beta = 1,2$. The functions $\mathcal{G}_1(x,y)$ and $\mathcal{G}_2(x,y)$ are given by:

$$\mathcal{G}_1(x,y) = \frac{1}{12} x \left(4x^2 e^{-y}(y^3 + 3y^2 + 6y + 6) - 3(x^2-1)e^{y} - 3x(7x + 4y^3)\right),$$

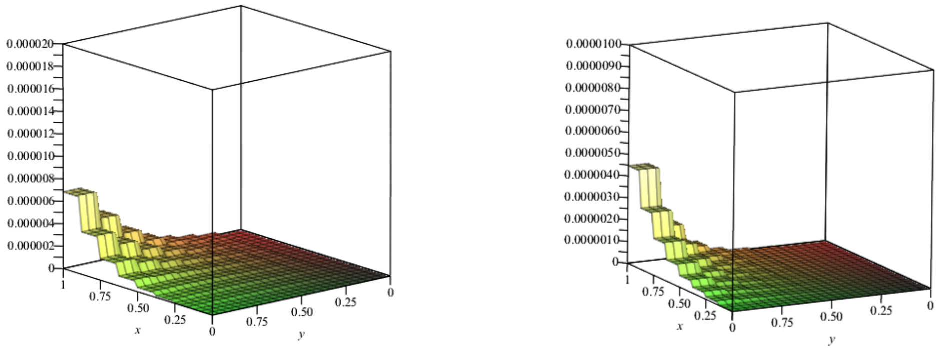
$$\mathcal{G}_2(x,y) = -\frac{1}{16} x^4 \left(e^{y}(y((y-3)y+6)-6)+6\right) - \frac{1}{15} x^3 y^6 - x^2 y^3.$$

The exact solution of this system is $\mathcal{U}_1(x,y) = \frac{1}{4} x e^{y}$ and $\mathcal{U}_2(x,y) = x^2 y^3$.

Comparison of the absolute errors between the Legendre wavelets functions method (LWFM) [25] and the proposed method (TCM) for $N = M = 8$, $p = 5$ is shown in Table 4.5. Also Figure 4.4 shows the comparison between exact and approximate solutions for $N = M = 8$ and $p = 5$.

Table 4.5: Absolute errors for Example 4.4.4

(x,y)	LWFM [25]		TCM	
	$\lvert\mathcal{U}_1 - \mathcal{V}_1\rvert$	$\lvert\mathcal{U}_2 - \mathcal{V}_2\rvert$	$\lvert\mathcal{U}_1 - \mathcal{V}_1\rvert$	$\lvert\mathcal{U}_2 - \mathcal{V}_2\rvert$
$(\frac{1}{8},\frac{1}{8})$	2.44×10^{-4}	2.44×10^{-4}	3.56×10^{-8}	4.97×10^{-10}
$(\frac{3}{8},\frac{3}{8})$	6.63×10^{-3}	6.45×10^{-3}	2.72×10^{-7}	3.07×10^{-8}
$(\frac{5}{8},\frac{5}{8})$	3.24×10^{-2}	2.81×10^{-2}	1.01×10^{-6}	3.25×10^{-7}
$(\frac{7}{8},\frac{7}{8})$	9.32×10^{-2}	6.15×10^{-2}	6.83×10^{-6}	4.47×10^{-6}
$(\frac{1}{8},\frac{7}{8})$	1.73×10^{-3}	1.67×10^{-3}	6.48×10^{-8}	1.15×10^{-7}
$(\frac{7}{8},\frac{1}{8})$	1.22×10^{-2}	1.17×10^{-2}	2.60×10^{-7}	3.55×10^{-9}
$(\frac{1}{8},\frac{3}{8})$	7.32×10^{-4}	7.28×10^{-4}	8.62×10^{-8}	1.00×10^{-8}
$(\frac{5}{8},\frac{1}{8})$	6.13×10^{-3}	6.02×10^{-3}	1.82×10^{-7}	2.48×10^{-9}



(a) The error $|\mathcal{U}_1 - \mathcal{V}_1|$ for $N = M = 8$. (b) The error $|\mathcal{U}_2 - \mathcal{V}_2|$ for $N = M = 8$.

Figure 4.4: Comparison between exact and approximate solutions for Example 4.4.4

4.5 Conclusion

This chapter presented a Taylor collocation method for solving systems of two-dimensional Volterra integral equations with proportional delays. The method constructs piecewise polynomial approximations using Taylor series expansions on each sub-rectangle. A convergence analysis proved that the method achieves $(\mathfrak{h} + \mathfrak{k})^{\mathfrak{p}}$ accuracy. Numerical experiments confirmed the theoretical results. Future work includes extension to non-linear systems.

CONCLUSION AND PERSPECTIVES

This thesis has presented a comprehensive investigation of Taylor collocation methods for solving Volterra integral equations and their extensions. Three main contributions have been developed and analyzed.

First, a Taylor collocation method was formulated for Volterra integral equations with proportional delays. The approach constructs local Taylor polynomial approximations on a uniform mesh, determining coefficients through collocation. A rigorous convergence analysis established that the method achieves $O((\Delta t)^r)$ accuracy for sufficiently smooth solutions, and numerical experiments confirmed this high-order convergence across various delay parameters.

Second, the methodology was extended to systems of two-dimensional Volterra integral equations. The Taylor collocation technique was adapted to handle the increased dimensionality and coupling between equations. Convergence analysis demonstrated that the method retains its high-order accuracy in the two-dimensional setting, with numerical examples validating the theoretical results.

Third, the method was further generalized to systems of two-dimensional Volterra integral equations with proportional delays. This extension addresses the additional complexity introduced by delay arguments in a multi-dimensional, multi-equation framework. Theoretical convergence guarantees were established, and computational experiments confirmed the method's effectiveness and precision.

Across all three contributions, the Taylor collocation method has proven to be a powerful and versatile numerical tool. Its key advantages include high-order accuracy for smooth solutions, conceptual simplicity, computational efficiency through local approximation, and flexibility in handling various equation types including delay arguments and coupled systems. The rigorous convergence analyses and numerical validations presented throughout this work establish the method as a reliable and competitive approach for solving a wide range of Volterra integral equation problems.

Future research directions may include extending the method to nonlinear problems, developing adaptive step-size strategies for solutions with sharp gradients, and applying the Taylor collocation approach to other classes of functional equations such as integro-differential equations and problems with state-dependent delays.

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