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Analysis 4 : Courses and Exercises with Solutions

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Contents

Preface

This booklet presents the official syllabus of the course Analysis 4, primarily intended for second-year undergraduate students in Mathematics. It is structured into three main chapters: Topology in $\mathbb{R}^n$, Functions of Several Variables, and Multiple Integrals. The first chapter provides an introductory overview of the fundamental concepts related to the Euclidean space $\mathbb{R}^n$, while the last two chapters conclude with solved exercises aimed at reinforcing understanding and facilitating the assimilation of the mathematical notions and results developed throughout this booklet.

Finally, I would like to express sincere gratitude to the reviewers for their careful evaluation and valuable remarks, which contributed significantly to the improvement of the scientific and pedagogical quality of this course.

Chapter 1

Topology in $\mathbb{R}^n$

In this chapter, we introduce fundamental concepts related to the Euclidean space $\mathbb{R}^n$, which serves as the foundation for advanced studies in analysis and topology. We begin by defining the structure of $\mathbb{R}^n$ as an n-dimensional real vector space equipped with standard algebraic operations. Next, we present the notion of a **norm**, which provides a way to measure the length or magnitude of vectors, and discuss the properties that characterize a **normed vector space**. We also introduce the **inner product**.

Furthermore, we explore the idea of **equivalent norms**, emphasizing that different norms on $\mathbb{R}^n$ can induce the same topology. The notions of **open and closed balls** are then defined as fundamental building blocks for understanding neighborhoods and open sets. We also discuss the **neighborhood** of a point and its role in defining continuity, convergence, and openness. Finally, we introduce the concepts of **interior**, **closure**, and **compact sets**, which are essential for characterizing boundedness and completeness in metric and topological spaces. Together, these notions provide the analytical tools needed for the rigorous study of functions and mappings in higher dimensions.

1.1 The space $\mathbb{R}^n$

Definition 1.1.1. *The space $\mathbb{R}^n$ is the set of ordered n-tuples $(x_1, x_2, ..., x_n)$ of real numbers.*
We equip $\mathbb{R}^n$ with the following two operations:

1. $\forall x, y \in \mathbb{R}^n$; $x + y = (x_1 + y_1, ..., x_n + y_n)$, .
2. $\forall x \in \mathbb{R}^n, \forall \lambda \in \mathbb{k}$ $(\mathbb{k} = \mathbb{R}$ *or* $\mathbb{k} = \mathbb{C})$; $\lambda x = (\lambda x_1, \lambda x_2, ..., \lambda x_n)$.

With these two operations, it is easy to verify that $\mathbb{R}^n$ is a vector space on $\mathbb{k}$ with dimension n.

1.2 Norm

Definition 1.2.1. *Let $E \subset \mathbb{R}^n$ be a vector space on $\mathbb{k}$ ($\mathbb{k} = \mathbb{R}$ or $\mathbb{k} = \mathbb{C}$). We call a norm on E, any application:*

$$\begin{array}{rccl} N: & E & \to & \mathbb{R} \\ & x & \mapsto & N(x) = \|x\| \end{array} \tag{1.1}$$

that satisfies the following properties for all vectors $x, y \in \mathbb{R}^n$ and for all $\lambda \in \mathbb{k}$:
a) $\|x\| = 0 \Leftrightarrow x = 0_E$.
b) $\|\lambda x\| = |\lambda| \|x\|$.
c) $\|x + y\| \leq \|x\| + \|y\|$ *(the triangle inequality).*

1.3 Normed vector space

Definition 1.3.1. *If the space E is equipped with a norm $\|.\|$, we then say that E is a normed vector space, and it is denoted by $(E, \|.\|)$.*

Example 1.3.1. *(Usual norm on $\mathbb{R}^p$) Let $x = (x_1, ..., x_p) \in \mathbb{R}^p$.*
1. *The application $x \mapsto N_\infty(x) = \|x\|_\infty = \sup\limits_{1 \leq i \leq p} \{|x_i|\}$, is a norm.*
2. *The application $x \mapsto N_1(x) = \|x\|_1 = \sum\limits_{i=1}^{p} |x_i|$, is a norm.*

3. *The application* $x \mapsto N_2(x) = \|x\|_2 = \sqrt{\sum_{i=1}^{p} x_i^2}$, *is a norm, called the Euclidean norm.*

1.4 Inner product in a real vector space

Definition 1.4.1. *In the real vector space* $\mathbb{R}^p$, *an inner product is any application* $\langle .,. \rangle : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ *that satisfies the following three properties:*

1. Linearity in the first argument

$$\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle, \text{ for } x, y, z \in \mathbb{R}^p \text{ and } \alpha \in \mathbb{k}.$$

2. Symmetry

$$\langle x, y \rangle = \langle y, x \rangle, \text{ for } x, y \in \mathbb{R}^p.$$

3. Positive definiteness

$$\langle x, x \rangle \geq 0 \text{ and } \langle x, x \rangle = 0 \Longleftrightarrow x = 0.$$

1.4.1 Euclidean case(standart inner product in $\mathbb{R}^p$)

The most common inner product in $\mathbb{R}^p$ is given by:

$$\langle x, y \rangle = \sum_{i=1}^{p} x_i y_i, \text{ for all } x, y \in \mathbb{R}^p.$$

We therefore have the following properties:

1. $\|x\| = \sqrt{\langle x, x \rangle}$.
2. $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$.
3. $\langle x, y \rangle = \langle y, x \rangle$.
4. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ (bilinearity).
5. If $\langle x, y \rangle = 0 \Rightarrow \|x + y\|^2 = \|x\|^2 + \|y\|^2$ (Euality of Pythagore).
6. $|\langle x, y \rangle| \leq \|x\| \|y\|$.(inequality of Cauchy-Schwarz).

1.5 Equivalent norms

Definition 1.5.1. *Two norms N_1 and N_2 defined on the same normed vector space E, are said to be equivalent, if and only if there exist two non negative real numbers α and β, such that:*

$$\forall x \in E : \alpha N_1(x) \leq N_2(x) \leq \beta N_1(x). \tag{1.2}$$

We then have the following theorem :

Theorem 1.5.1. *For all $p \in \mathbb{N}^*$, the norms N_∞, N_1 and N_2 on $E = \Bbbk^p$, are pairwise equivalent. Moreover, for all $x \in E$, we have:*

$$N_\infty(x) \leq N_1(x) \leq \sqrt{p} N_2(x) \leq p N_\infty(x). \tag{1.3}$$

Proof. The relation $N_\infty(x) \leq N_1(x)$ is obvious, because $N_1(x)$ is equal to the sum of $N_\infty(x)$.
Moreover, by using the inequality of Cauchy-Schwarz, we get:

$$N_1^2(x) \leq (\sum_{i=1}^{p} 1 \times |x_i|)^2 \leq \sum_{i=1}^{p} 1^2 \times \sum_{i=1}^{p} |x_i|^2 = pN_2^2(x).$$

Therefore $N_1(x) \leq \sqrt{p}N_2(x)$.
Finally, $N_\infty(x)$ is the greatest of the positive real numbers $|x_i|$, $i = 1, ..., p$. So, the sum $\sum\limits_{i=1}^{p} |x_i|^2$ consists of p terms, all less or equal to $N_\infty^2(x)$.
We deduce that:

$$N_2(x) = \sqrt{\sum_{i=1}^{p} |x_i|^2} \leq \sqrt{\sum_{i=1}^{p} N_\infty^2(x)} = \sqrt{pN_\infty^2(x)} = \sqrt{p}N_\infty(x).$$

that is: $N_\infty(x) \leq N_1(x) \leq \sqrt{p}N_2(x) \leq pN_\infty(x)$. □

1.6 Open ball and closed ball

Definition 1.6.1. *Let $(E, \|.\|)$ be a normed vector space, and let $a \in E$ and r be a non negative real number. An open ball (respectively a closed ball) of a normed vector* *space E, is a set $B(a, r)$ (respectively $\bar{B}(a, r)$) defined by:*

$$B(a, r) = \{x \in E \ / \ \|x - a\| < r.\} \quad (1.4)$$

(respectively

$$\bar{B}(a, r) = \{x \in E \ / \ \|x - a\| \leq r\}), \quad (1.5)$$

where a is the center of the ball and r is its radius.

1.7 Neighborhood of a point

Definition 1.7.1. *A subset V of a normed vector space E is called a neighborhood of a point $a \in E$, if it is contains an open ball centrered at a.*

1.8 Open set and closed set

Definition 1.8.1. ** A subset U of E is said to be open, if it is a neighborhood of each of its points.*
** A subset F of E is said to be closed if it complement in E is an open set.*
** A non-empty subset F of E is said to be closed, if it satisfies the following property: Every sequence of points in F that converges in E has its limit in F.*

1.9 Compact set

Definition 1.9.1. *A non-empty subset K of E is said to be compact if and only if one of the following properties is satisfied:*
1) K is closed and bounded.
2) Every bounded sequence of point in K has a convergent subsequence whose limit belongs to K (property of Bolzano-Weierstrass).

1.10 Interior point

Definition 1.10.1. *Let E be a normed vector space and A a non-empty subset in E. A point $a \in A$ is called an interior point of A if there exists an open set $U \subseteq E$, such that $a \in U$.*
The set of interior points of A is denoted by $Int(A)$ or $\overset{\circ}{A}$ and it is called interior of A. It is always the largest open subset contained in A.

Chapter 2

Functions of several real variables

In this chapter we want to go over some of the basic ideas about functions of more than one variable.

2.1 General definition

In mathematical analysis, a function of several real variables is a function with more than one argument, with all arguments being real variables. This concept extends the idea of a function of a real variable to several variables

Definition 2.1.1. *A function of n real variables is any application f from a subset D of $\mathbb{R}^n$ to $\mathbb{R}$. This part D is called the domain of definition of f.*

So, the domain of a function of n variables is the subset of $\mathbb{R}^n$ for which the function is defined. As usual, the domain of a function of several real variables is supposed to contain a nonempty open subset of $\mathbb{R}^n$.

Example 2.1.1. *A simple example of a function in two variables is: $f : D \to \mathbb{R}$ defined by*

$$f(x, y) = \frac{1}{3}xy, \tag{2.1}$$

where

$$D = (x, y) \in \mathbb{R}^2 \;\; x > 0 \text{ and } y > 0.$$

The function f represents the volume of a cone with base area x and height h measured perpendicularly from the base. The domain restricts all variables to be positive since lengths and areas must be positive. Furthermore, D is called domain of definition of f.

2.2 Limit at a point a

Definition 2.2.1. *Let f be a real function of n real variables, defined in a neighborhood of a point a of $\mathbb{R}^n$, except possibly at a itself, and let l be a given real number. So:*

$$\lim_{x \longrightarrow a} f(x) = l \iff \forall \varepsilon > 0, \exists \alpha > 0, \; \forall x \in B(a, \alpha), \text{ we have } \left| f(x) - l \right| < \varepsilon.$$

Example 2.2.1. *Consider the function*

$$(x, y) \mapsto f(x, y) = \frac{xy^3}{x^2 + y^2}, \; (x_0, y_0) = (0, 0).$$

1. The first method: By definition, we have:

$$|x| \leq \sqrt{x^2 + y^2} \text{ and } \left| y \right| \leq \sqrt{x^2 + y^2}.$$

So

$$\left| \frac{xy^3}{x^2 + y^2} \right| \leq \frac{(x^2 + y^2)^2}{x^2 + y^2} \leq (\sqrt{(x^2 + y^2)})^2 < \varepsilon.$$

Hence, for all $\varepsilon > 0, \exists \alpha = \sqrt{\varepsilon}$ such that:

$$\sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2} < \alpha \Longrightarrow \left| f(x, y) - 0 \right| < \varepsilon.$$

That is: $f(x, y) \to 0$, when $(x, y) \to (0, 0)$.

2.The second method: We set $\begin{cases} x = \rho \cos \theta \\ y = \rho \sin \theta \end{cases}$

We can write

$$f(x, y) = \frac{\rho^4 \sin 2\theta}{2\rho^2} = \frac{\rho^2}{2} \sin 2\theta \to 0, \text{ as } \rho \to 0.$$

Example 2.2.2. $f(x,y) = \dfrac{x-y}{x+y}$, *and* $(x_0, y_0) = (0,0)$.
1. The first method:

$$f(x, \lambda x) = \frac{x - \lambda x}{x + \lambda x} = \frac{1-\alpha}{1+\alpha}.$$

Since the limit at $(0,0)$ *of* $f(x, \lambda x)$ *depends on the value of* λ*, the function* f *has no limit at* $(0,0)$.
2. The second method, $f(x,0) = 1$ *and* $f(0,y) = -1$*. Therefore, the limit of* f *does not exist (because a limit, whenever it exists, it is unique).*

2.3 Continuous function

2.3.1 Continuity at a point

Definition 2.3.1. *Let* $f : N(a) \subset \mathbb{R}^n \to \mathbb{R}$ *be a function defined on the neighborhood of* $a \in \mathbb{R}^n$ *and in particular at the point* a.
We say that f *is continuous at* a*, if* $\lim\limits_{x \longrightarrow a} f(x) = f(a)$*; that is:*

$$\forall \varepsilon > 0, \exists \alpha > 0, \text{ such that } \forall x \in B(a, \alpha), \text{ we have } \left| f(x) - f(a) \right| < \varepsilon. \tag{2.2}$$

Proposition 2.3.1. *Let* f *be a continuous function. If a sequence of points* $\{x_n\} \subset \mathbb{R}^n$ *converges to the point* $a \in \mathbb{R}^n$*, then the sequence of the images* $\{f(x_n)\}$ *converges to* $f(a)$.

Remark 2.1. *The properties of a continuous function in* $a \in \mathbb{R}^n$ *are identical to that in the case of a function of a single variable.*

Definition 2.3.2. *Let* $f : E \longrightarrow F$ *(*E, F *two normed vector subspaces of* $\mathbb{R}^n$*)) be a function and let* $a = (a_1, ..., a_n) \in E$.
We say that f *is continuous at* a*, if* $\lim\limits_{X \longrightarrow a} f(X) = f(a)$*; that is:*

$$\forall \varepsilon > 0, \exists \alpha > 0, \text{ such that } \forall x \in B_E(a, \alpha), \text{ we have } \left\| f(x) - f(a) \right\| < \varepsilon$$

Proposition 2.3.2.

$$\lim_{x \to a} f(x) = f(a) \in \mathbb{R}^n \Longleftrightarrow \lim_{x \longrightarrow a} f_i(x) = f_i(a), \text{ for all } i=1,2,...,n.$$

Proof. $\Longrightarrow$) Suppose that: $\lim\limits_{x\longrightarrow a} f(x) = f(a)$. We then have:

$$\forall \varepsilon > 0, \exists \alpha > 0, \text{ such that } \forall x \in B_E(a,\alpha), \text{ we have } \left\|f(x) - f(a)\right\| < \varepsilon. \qquad (2.3)$$

On the other hand

$$\begin{aligned} \left|f_i(x) - f_i(a)\right| &\leq \sqrt{\sum_{i=1}^{n}(f_i(x) - f_i(a))^2}, \text{ for all } i = 1, ..., n. \\ &= \left\|f(x) - f(a)\right\|_F < \varepsilon. \end{aligned}$$

Which ensures $\lim\limits_{x\longrightarrow a} f_i(x) = f_i(a)$, for all $= 1, ..., n$.
$\underline{\Leftarrow}$) Now, suppose that: $\lim\limits_{x\longrightarrow a} f_i(x) = f_i(a)$, $i = 1, ..., n$. So:

$$\forall \varepsilon > 0, \exists \alpha > 0, \text{ such that } \forall x \in B_E(a,\alpha), \text{ we have } \left|f_i(x) - f_i(a)\right| < \frac{\varepsilon}{\sqrt{n}},$$

for all $i = 1, ..., n$. That is:

$$\forall \varepsilon > 0, \exists \alpha > 0, \text{ such that } \forall x \in B_E(a,\alpha) \text{ we have } \left|f_i(x) - f_i(a)\right|^2 < \frac{\varepsilon^2}{n},$$

for all $i = 1, ..., n$. It follows that, $\forall \varepsilon > 0, \exists \alpha > 0, \ \forall x \in B_E(a,\alpha)$, we have

$$\left\|f(x) - f(a)\right\|_F = \sqrt{\sum_{i=1}^{n}(f_i(x) - f_i(a))^2} < \varepsilon.$$

□

2.3.2 Continuity on a subset

Definition 2.3.3. *Let f be a function defined on a subset E of $\mathbb{R}^n$. We say that f is continuous on E if, for all point $a \in E$, we have:*

$$\forall \varepsilon > 0, \exists \alpha > 0, \textit{ such that } \forall x \in E \cap B_E(a,\alpha) \textit{ we have } \left\|f(x) - f(a)\right\| < \varepsilon.$$

Proposition 2.3.3. *Let $f : E \to \mathbb{R}^q$ ($E \subset \mathbb{R}^p$). There is an equivalence between the three following assertions:*

1. *The application f is continuous on E.*
2. *The inverse image under f of every open set of $\mathbb{R}^q$ is an open of $\mathbb{R}^p$.*

3. *The inverse image under f of every closed set of $\mathbb{R}^q$ is a closed of $\mathbb{R}^p$.*

Proof. 01) $\Longrightarrow$ 02) Suppose that f is continuous on E.
Let A be an open of $\mathbb{R}^q$ and let $a \in f^{-1}(A)$, so $f(a) \in A$.
A is an open of $\mathbb{R}^q \Longleftrightarrow \exists r > 0$, such that $\forall y \in A$, $\left\|y - f(a)\right\| < r$. That is: $B(f(a), r) \subset A$.
But f is continuous at a, we then have: $\forall \varepsilon > 0, \exists \alpha > 0$, such that:

$$\forall x \in E \cap B(a, \alpha) \Rightarrow \left\|f(x) - f(a)\right\| < \varepsilon = r.$$

It follows that $f(x) \in A$, i.e. $x \in f^{-1}(A)$. We then deduce that the open ball of center a and of radius α is included in $f^{-1}(A)$, which shows that $f^{-1}(A)$ is an open.
02) $\Longrightarrow$ 01) : Let $a \in E$ (arbitrary). It is sufficient to show that f is continuous at a.
Let $\varepsilon > 0$ be given, and let $f(a) \in A$, such that: $B(f(a), \varepsilon) \subset A$ (because A is open). So

$$f(x) \in B(f(a), \varepsilon) \subset A \Longleftrightarrow \left\|f(x) - f(a)\right\| < \varepsilon,$$

Let $x \in f^{-1}(A)$. Since $f^{-1}(A)$ is an open, then $\forall \varepsilon > 0, \exists \alpha > 0$, such that $\|x - a\| < \alpha \Longrightarrow \left\|f(x) - f(a)\right\| < \varepsilon$.This is indeed the definition of the continuity of f at the point a.
01) $\Longrightarrow$ 03) We just need to switch to the supplementary plan.
Let $f : E \subset \mathbb{R}^p \to \mathbb{R}^q$ and let B^c a closed of $\mathbb{R}^q$. We have $f^{-1}(B^c) = [f^{-1}(B)]^c$.
So B^c is closed $\Longleftrightarrow B$ is open$\Longleftrightarrow f^{-1}(B)$ is open (because f is continuous), therefore $[f^{-1}(B)]^c = f^{-1}(B^c)$ is closed. ◻

Proposition 2.3.4. *Let $f : E \subset \mathbb{R}^p \to \mathbb{R}^q$ be a continuous function. Then the image of any compact A of E is a compact of $\mathbb{R}^q$.*

Proof. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence of $f(A)$.
So $\exists (x_n)_{n\in\mathbb{N}} \subset A$, such that: $y_n = f(x_n), \forall n \in \mathbb{N}$.
Since A is a compact, then from any sequence (x_n) of A, we can extract a subsequencee (x'_n) converging to $a \in A$. So from the sequence $(y_n)_{n\in\mathbb{N}}$, we can extract a subsequence (y'_n) defined by $y'_n = f(x'_n) \in f(A)$ which converges to

$f(a) \in f(A)$ (because f is continuous).
Consequently $f(A)$ is a compact in $\mathbb{R}^q$. □

2.4 Differentiable functions

2.4.1 Higher-order partial derivatives

Definition 2.4.1. *Let $f : U \to \mathbb{R}$ (U be an open of $\mathbb{R}^p$). We say that f has at the point $a \in U$ a first partial derivative, with respect to x_i, if and only if:*

$$\lim_{h \longrightarrow 0} \frac{f(a + he_i) - f(a)}{h} \text{ exists.}$$

This limit, whenever exists, tt is denoted by $f'_{x_i}(a)$ or $\dfrac{\partial f}{\partial x_i}(a)$.

Definition 2.4.2. *Let $f : U \subset \mathbb{R}^P \to \mathbb{R}$. We say that f is of class C^1 on U, if all partial derivatives of order one of f exist and are continuous on U .*

The set of functions of class C^1 on U is denoted by $C^1(U)$.

Example 2.4.1. *Consider the function:*

$$(x, y) \mapsto f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, \text{ if } (x, y) \neq (0, 0) \\ 0, \text{ if } (x, y) = (0, 0), \end{cases}$$

** For $(x, y) \neq (0, 0)$, we have:*

$$\begin{cases} \dfrac{\partial f}{\partial x}(x, y) = \dfrac{y(-x^2 + y^2)}{x^2 + y^2} \\ \dfrac{\partial f}{\partial y}(x, y) = \dfrac{x(x^2 - y^2)}{(x^2 + y^2)^2}, \end{cases}$$

which are continuous on $\mathbb{R}^2 - \{(0, 0)\}$.
So f is of class C^1 on $\mathbb{R}^2 - \{(0, 0)\}$.
** At the point $(0, 0)$:*

$$\begin{cases} \dfrac{\partial f}{\partial x}(0, 0) = \lim_{h \to 0} \dfrac{f(h, 0) - f(0, 0)}{h} = 0 \\ \dfrac{\partial f}{\partial y}(0, 0) = \lim_{h \to 0} \dfrac{f(0, h) - f(0, 0)}{h} = 0. \end{cases}$$

So, the function f has the partial derivatives on $\mathbb{R}^2$ and it is of class C^1 on $\mathbb{R}^2 - \{(0,0)\}$.
It is clear that $\frac{\partial f}{\partial x}$ is not continuous at $(0,0)$.
For example: $\frac{\partial f}{\partial x}\left(\frac{2}{n}, \frac{1}{n}\right) = \frac{-3n}{25} \to -\infty$, *as* $n \to +\infty$.

Definition 2.4.3. *Let $f \in C^1(U)$ (U is an open of $\mathbb{R}^p$).*
We say that f has a second partial derivative at a point $a \in U$, with respect to x_j and x_i, $i, j = 1, 2, ..., p$, if:

$$\lim_{h \longrightarrow 0} \frac{\frac{\partial f}{\partial x_i}(a + he_j) - \frac{\partial f}{\partial x_i}(a)}{h} \text{ exists.}$$

If $i \neq j$ (case of mixed second partial derivative), it is denoted by:

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)(a) \quad or \quad \frac{\partial^2 f}{\partial x_i \partial x_j}(a) \tag{2.4}$$

If $i = j$ (case of pure second partial derivative), it is denoted by $\frac{\partial^2 f}{\partial x_i^2}(a)$.
If f admits second partial derivatives at every point $a \in U$, we say that it admits second partial derivatives on U. If these derivatives are continuous on U, We say that f is of class C^2 on U.
The set of functions of class C^2 on U is of course denoted by $C^2(U)$.

Definition 2.4.4. *Let f be a function defined on an open U of $\mathbb{R}^p$. If its partial derivatives of order 1 are still differentiable with respect to each variable, their partial derivatives are called second partial derivatives. By induction, we define the partial derivatives of order n as the partial derivatives of the derivatives of order $n - 1$.*

Remark 2.2. *A partial derivative of order n is therefore obtained by successively differentiating one of the variables n times with respect to a.*

For example, we obtain a derivative of order 4 of a function of three variables x, y, z by first differentiating with respect to x, then with respect to y, then again with respect to x, then with respect to z; or by differentiating with respect to y, then with respect to z, and then twice with respect to x.

Notation: The partial derivative of order n of a function of p variables $x_1, x_2, ..., x_p$ obtained by differentiating n_1 times with respect to x_1, n_2 times with respect to $x_2, ..., n_k$ times with respect to x_p, where $n_1, ..., n_k$ are positive integers, such that $n_1 + n_2 + ... + n_k = n$ is denoted by $\dfrac{\partial^n f}{\partial x_1^{n_1} ... \partial x_p^{n_k}}$.

2.4.2 Schwarz's theorem

Theorem 2.4.1. *Let $f \in C^1(U)$, (U is an open of $\mathbb{R}^p$) admitting second partial derivatives on U.*

1. *If $\dfrac{\partial^2 f}{\partial x_i \partial x_j}$ and $\dfrac{\partial^2 f}{\partial x_j \partial x_i}$ are continuous at a, for all $i, j = 1, ..., p$, then:*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(a) = \frac{\partial^2 f}{\partial x_j \partial x_i}(a). \tag{2.5}$$

2. *If f is of class C^2 on U, we then have:*

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}. \tag{2.6}$$

Proof. Without loss of generality, we will assume that $p = 2$.
Let $a = (\alpha, \beta) \in U$ and let $(h, k) \in \mathbb{R}^2$, such that: $(a + h, \beta + k) \in U$ (because U is open). We evaluate the following real in two different ways, we get:

$$u(h, k) = [f(\alpha + h, \beta + k) - f(\alpha + h, \beta)] - [f(\alpha, \beta + k) - f(\alpha, \beta)].$$

Let $F_1(x) = f(x, \beta + k) - f(\alpha, \beta)$. Using Mean Value Theorem, we get

$$u(h, k) = F_1(\alpha + h) - F_1(a) = hF_1'(\alpha + \theta_1 h); \text{ where } \theta_1 \in]0, 1[.$$

On the other hand

$$F_1'(x) = \frac{\partial f}{\partial x}(x, \beta + k) - \frac{\partial f}{\partial x}(x, \beta).$$

So

$$u(h, k) = h\left[\frac{\partial f}{\partial x}(\alpha + \theta_1 h, \beta + k) - \frac{\partial f}{\partial x}(\alpha + \theta_1 h, \beta)\right]$$

Now, let's $F_2(y) = \frac{\partial f}{\partial x}(\alpha + \theta_1 h, y)$. Using Mean Value Theorem, we get

$$u(h,k) = h(F_2(\beta + k) - F_2(\beta)] = hkF_2'(\beta + \theta_2 k); \text{ where } \theta_2 \in]0,1[$$

Since $F_2'(y) = \frac{\partial^2 f}{\partial y \partial x}(\alpha + \theta_1 h, y)$, we have

$$u(h,k) = hk\frac{\partial^2 f}{\partial y \partial x}(\alpha + \theta_1 h, \beta + \theta_2 k) \tag{2.7}$$

Let $G_1(y) = f(\alpha + \theta_1 h, y) - f(\alpha, y)$. So

$$u(h,k) = G_1(\beta + k) - G_1(\beta) = kG_1'(\beta + \theta_3 k); \text{ where } \theta_3 \in]0,1[\,.$$

Since $G_1' = \frac{\partial f}{\partial y}(\alpha + h, y) - \frac{\partial f}{\partial x}(\alpha, y)$, we have

$$u(h,k) = k\left[\frac{\partial f}{\partial y}(\alpha + h, \beta + \theta_3 k) - \frac{\partial f}{\partial y}(\alpha, \beta + \theta_3 k)\right]$$

Let $G_2(x) = \frac{\partial f}{\partial y}(x, \beta + \theta_3 k)$. Thus

$$u(h,k) = k\,[G_2(\alpha + h) - G_2(\alpha)] = khG_2'(\alpha + \theta_4 k); \text{ where } \theta_4 \in]0,1[\,.$$

Since $G_2'(x) = \frac{\partial^2 f}{\partial x \partial y}((x, \beta + \theta_3 k)$, we then have

$$u(h,k) = kh\frac{\partial^2 f}{\partial x \partial y}(\alpha + \theta_4 h, \beta + \theta_3 k) \tag{2.8}$$

From (2.7) and (2.8), we deduce the statement 02.
For the statement 01: If $(h,k) \longrightarrow (0,0)$, the second partial derivative functions are assuming continuous at $a = (\alpha, \beta)$, we therefore have:

$$\frac{\partial^2 f}{\partial x \partial y}(a) = \frac{\partial^2 f}{\partial y \partial x}(a).$$

□

2.4.3 Laplacian of a function

Definition 2.4.5. *Let $f \in C^1(U)$ (U is an open of $\mathbb{R}^p$) be a function whose p second partial derivatives $\frac{\partial^2 f}{(\partial x_i)^2} : U \to \mathbb{R}$, $i = 1, 2, ..., p$ exist. Then, the function*

$\Delta f : U \rightarrow \mathbb{R}$ defined by:

$$\Delta f(x_1, x_2, ..., x_p) = \sum_{i=1}^{p} \frac{\partial^2 f}{(\partial x_i)^2}(x_1, x_2, ..., x_p),$$

is called Laplacian of the function f.

2.4.4 Harmonic function

Definition 2.4.6. *A function $f : U \rightarrow \mathbb{R}$ is called harmonic if it is of class C^2, and $\Delta f(x_1, x_2, ..., x_p) = 0$, for all $x \in U$.*

2.4.5 Continuous linear applications

Definition 2.4.7. *Let $E \subset \mathbb{R}^p$ and $F \subset \mathbb{R}^q$ two normed vector subspaces defined on the same field $\mathbb{k}$, and let $L : E \longrightarrow F$ be an application. We say that L is linear if:*

$$\text{for all } x, y \in E, \text{ and for all } \alpha, \beta \in \mathbb{k} : L(\alpha x + \beta y) = \alpha L(x) + \beta L(y).$$

Theorem 2.4.2. *Let $L : E \longrightarrow F$ be a linear application. Therefore, the following three assertions are equivalent:*

1. L is continuous on E.
2. L is continuous at the point $x_0 = o_p$.
3. L is bounded on the unit ball $B(o_p, 1)$.

Proof.
$1 \Longrightarrow 2$ Obvious.
$2 \Longrightarrow 3$: Suppose that L is continuous at the point $x_0 = o_p$. Thus

$$\forall \varepsilon > 0, \exists \alpha > 0 \text{ such that } \forall x \in E; \|X - 0\|_E \leq \alpha \Longrightarrow \|L(X) - L(0)\|_F < \varepsilon.$$

$L(0) = 0$ (because L is linear). We take $\varepsilon = 1$, we then have:

$$\exists \alpha > 0 \text{ such that} : \forall x \in E : \|X\|_E \leq \alpha \Longrightarrow \|L(X)\|_F < 1.$$

Let $X = \alpha Y$. So:

$$\|X\|_E \leq \alpha \Longrightarrow \|Y\|_E \leq 1,$$

and

$$\|Y\|_E \leq 1 \Longrightarrow \|L(Y)\|_F < \frac{1}{\alpha} = M.$$

Therefore

$$\exists M > 0, \text{ such that } \forall Y \in B(o_p, 1); \text{ we have } \|L(Y)\| < M.$$

So L is bounded on the ball $B(o_p, 1)$.

03) $\Longrightarrow$ 01): Suppose that L is bounded on the ball $B(o_p, 1)$, and we show that L is bounded on E, that is:

$$\exists M > 0, \forall X \in E, \text{ we have } \|L(X)\|_F < M \|X\|_E.$$

Indeed, if $\|X\|_E = 0$, The previous relationship is true.
Suppose that: $\|X\|_E = r > 0$, and we take $y = \frac{1}{r}x$, $\forall x \in E$.
So $\|y\| = 1 \Longrightarrow y \in B(o_p, 1)$.
Since L is bounded on $B(o_p, 1)$, then $\|Ly\|_F \leq M$, $\forall y \in B(o_p, 1)$. So

$$\forall x \in E : \|L(x)\|_F = \|L(ry)\|_F = r\|L(y)\|_F = M\|x\|_E \leq rM.$$

Let us now show that L is continuous on E.
Let $a \in E$ (a arbitrary), and let $\varepsilon > 0$. then

$$\|L(X) - L(a)\|_F = \|L(X-a)\|_F \leq M\|X-a\|_E \leq \varepsilon \Longrightarrow \|X-a\| < \frac{\varepsilon}{M}.$$

Therefore, it suffices to take $\alpha = \frac{\varepsilon}{M}$. So:

$$\forall \varepsilon > 0, \exists \alpha = \frac{\varepsilon}{M} \text{ such that } \forall x \in E; \ \|X-a\| < \alpha, \text{ we have } \|L(X) - L(a)\|_F < \varepsilon.$$

Therefore L is continuous on E. □

Remark 2.3. *We denote by $L(E, F)$ the set of linear applications from E to F, and we equip $L(E, F)$ with the norm:*

$$\|L\| = \underset{\|X\| \leq 1}{Sup} \|L(x)\|. \tag{2.9}$$

2.4.6 Differentiable functions

Definition 2.4.8. *Let $f : U \to \mathbb{R}^q$ (U an open of $\mathbb{R}^p$). We say that f is differentiable at a point $a \in U$, if there exists a linear application L and a function ε from U to $\mathbb{R}^p$ such that:*

$$f(a+h) = f(a) + L(h) + \|h\|\,\varepsilon(h), \text{ where } \varepsilon(h) \underset{h\to 0_q}{\longrightarrow} 0_q \tag{2.10}$$

Theorem 2.4.3. *If f is differentiable at $a \in U$, then the linear application L of the previous definition is unique. It is called the differential of f at a and it is denoted by df_a.*

Proof. Suppose that we have two functions L_1 and L_2, satisfying the definition of differentiability, we then have:

$$L_1(h) - L_2(h) = \|h\| \left[\varepsilon_2(h) - \varepsilon_1(h\right]. \tag{2.11}$$

Which implies that

$$\lim_{h\to 0_p} \frac{L_2(h) - L_1(h)}{\|h\|} = 0, \forall h \neq 0_p$$

Now, let $h \in U^*$ and consider the function $g :]0, \infty[\to \mathbb{R}^q$ defined by:

$$g(t) = \frac{L_2(th) - L_1(th)}{\|th\|}$$

So $\lim_{t\to 0} g(t) = 0$. But for all $t \in]0, \infty[$;

$$\begin{aligned} g(t) &= \frac{L_2(th) - L_1(th)}{\|th\|} = \frac{L_2(h) - L_1(h)}{\|h\|} \\ &= \frac{L_2(1,h) - L_1(1,h)}{\|1.h\|} = g(1). \end{aligned}$$

Thus $0 = \lim_{t\to 0} g(t) = g(1), \forall h \neq 0_p$. Which implies that:

$$L_2(h) - L_1(h) = 0, \forall h \neq 0_p.$$

Therefore $L_2 = L_1$. □

2.4.7 Properties of differentiable functions

Let f and g be two differentiable functions at $a \in U$. Then we have:

1. f and g are continuous at a.
2. $f + g$ is differentiable at a, and furthermore: $d(f+g)_a = df_a + dg_a$
3. αf is differentiable at a, and furthermore: $d(\alpha f)_a = \alpha df_a$

2.4.8 Expressions of the differential

Let $f : U \to \mathbb{R}^q$ (U is an open of $\mathbb{R}^p$).

Theorem 2.4.4. *If $P = 1$ and $q = 1$. Then, f is differentiable at $a \in U$, if f is derivable at a, and moreover:*

$$\forall h \in \mathbb{R}, df_a(h) = f'(a).h. \tag{2.12}$$

Proof. It suffices to refer to the definition of the differentiability of a numerical function at a point $a \in \mathbb{R}$. □

Theorem 2.4.5. *If $p = 1$ and q arbitrary. Then, f is differentiable at the point $a \in U$ if and only if the q coordinate functions $f_1, f_2, ..., f_q$ of f are differentiable at a, and moreover*

$$\forall h \in \mathbb{R}, df_a(h) = (f_1'(a)h, ..., f_q'(a)h). \tag{2.13}$$

Proof. $\Leftarrow$) If each function coordinates f_i $(1 \leq i \leq q)$ is differentiable at $a \in U$, we then have :

$$\begin{aligned} f(a+h) &= (f_1(a+h), ..., f_q(a+h)) \\ &= \left[f_1(a) + f_1'(a)h + |h|\,\varepsilon_1(h), ..., f_q(a) + f_q'(a)h + |h|\,\varepsilon_q(h)\right] \\ &= f(a) + \left(f_1'(a), ..., f_q'(a)\right)h + |h|\,(\varepsilon_1(h), ..., \varepsilon_q(h)), \end{aligned}$$

where $\lim_{h\to 0}(\varepsilon_1(h), ..., \varepsilon_q(h)) = 0_q$.

$\Rightarrow$) Suppose that f is differentiable at a. The application df_a is linear, then there exist $l_1, ..., l_q$, such that

$$df_a = (l_1 h, ..., l_q h).$$

Let $\varepsilon = (\varepsilon_1, ..., \varepsilon_q) \in \mathbb{R}^q$, we then have

$$f_i(a+h) = f_i(a) + l_i h + |h|\, \varepsilon_i(h), \text{ for all } i = 1, 2, ...nq,$$

where $\lim_{h\to 0} \varepsilon_i(h) = 0$.
Consequently, each coordinate function f_i est derivable at a, and $f_i'(a) = l_i$.
that is :

$$df_a(h) = (f_1'(a)h, ..., f_q'(a)h).$$

□

Theorem 2.4.6. *If p arbitrary and $q = 1$.*

1. *If f is differentiable at the point a, then f admits p partial derivatives in a, and moreover*
$$\forall h \in \mathbb{R}^p, df_a(h) = \sum_{i=1}^{p} \frac{\partial f}{\partial x_i}(a) h_i. \quad (2.14)$$
2. *If f admits p continuous partial derivatives at a, then f is differentiable at a and moreover:*
$$\forall h \in \mathbb{R}^p, df_a(h) = \sum_{i=1}^{p} \frac{\partial f}{\partial x_i}(a) h_i \quad (2.15)$$

Proof. 1. The applicationn df_a is a linear on $\mathbb{R}^p$, then there exist p real $\alpha_1, ..., \alpha_p$, such that for all $h = (h_1, ..., h_{p)} \in \mathbb{R}^p$, we have

$$df_a(h) = \sum_{i=1}^{p} \alpha_i h_i.$$

So

$$df_a(0, ..., h_i, ..., 0) = \alpha_i h_i.$$

On the other hand

$$f(a+h) = f(a) + df_a(h) + \|h\|\, \varepsilon_q(h)$$

So

$$\begin{aligned} f(a_1, ..., a_{i-1}, a_i + h_i, a_{i+1}, ..., a_p) &= f(a_1, ..., a_p) + df_a(0, ..., h_i, ..., 0) + |h_i|\, \varepsilon(0, ..., h_i, ..., 0) \\ &= f(a_1, ..., a_p) + \alpha h_i + |h_i|\, \varepsilon_i(h_i), \end{aligned}$$

where $\lim\limits_{h_i \longrightarrow 0} \varepsilon_i(h_i) = 0$.
which means that the i-th partial function of f is differentiable at the point $a = (a_1, ..., a_i, ..., a_p)$, and $\frac{\partial f}{\partial x_i}(a) = \alpha_i$. Consequently

$$df_a(h) = \sum_{i=1}^{p} \alpha_i h_i = \sum_{i=1}^{p} \frac{\partial f}{\partial x_i}(a) h_i.$$

2. We have:

$$\begin{aligned} f(a+h) - f(a) &= f(a_1 + h_1, ..., a_p + h_p) - f(a_1, ..., a_p) = \\ &\left[f(a_1 + h_1, ..., a_p + h_p) - f(a_1, a_2 + h_2, ..., a_p + h_p) \right] + \\ &-[f(a_1, a_2 + h_2, ..., a_p + h_p) - f(a_1, a_2, a_3 + h_3, ..., a_p + h_p)] + \\ &. \\ &. \\ &. \\ &+[f(a_1, a_2, ..., a_{p-1}, a_p + h_p) - f(a_1, a_2, ..., a_{p-1}, a_p)]. \end{aligned}$$

Applying the Mean Value Theorem to each difference above, we get:

$$\begin{aligned} f(a+h) - f(a) &= h_1 \frac{\partial f}{\partial x_1}(a_1 + \theta_1 h_1, a_2 + h_2, ..., a_p + h_p) + \\ &... + h_p \frac{\partial f}{\partial x_p}(a_1, a_2, ..., a_{p-1}, a_p + \theta_p h_p), \end{aligned}$$

where $0 \leq \theta_i \leq 1$, for all $1 \leq i \leq p$.
Since each partial derivative above is continuous at a, we then have:

$$\begin{aligned} f(a+h) - f(a) &= h_1[\frac{\partial f}{\partial x_1}(a_1, a_2, ..., a_p) + \varepsilon_1(h)] + \\ &... + h_p[\frac{\partial f}{\partial x_p}(a_1, a_2, ..., a_p) + \varepsilon_p(h)], \end{aligned}$$

with $\lim\limits_{h \to 0_p} \varepsilon_i(h) = 0 \; \forall i = 1, ..., p$.
If $h \neq 0_p$, then $\|h\| = \sup\limits_{1 \leq i \leq p} \{|h_i|\} = \left|h_{i_0}\right|$, so

$$\frac{\left|\sum\limits_{i=1}^{p} h_i \varepsilon_i(h)\right|}{\|h\|} = \left|\sum_{i=1}^{p} \frac{h_i}{h_{i_0}} \varepsilon_i(h)\right| \leq \sum_{i=1}^{p} |\varepsilon_i(h)|.$$

Consequeently

$$\lim_{h\to 0_p}\frac{\left|f(a+h)-f(a)-\sum_{i=1}^{p}h_i\frac{\partial f}{\partial \lambda_i}(a)\right|}{\|h\|}=\lim_{h\to 0}\frac{\left|\sum_{i=1}^{p}h_i\varepsilon_i(h)\right|}{\|h\|}=0.$$

This ensures the differentiability of f in a, and moreover,

$$\forall h\in\mathbb{R}^p, df_a(h)=\sum_{i=1}^{p}\frac{\partial f}{\partial x_i}(a)h_i.$$

□

Theorem 2.4.7. *If p and q are arbitrary. Then f is differentiable at a if and only if the q coordinate functions of f are differentiable, and moreover:*

$$\forall h\in\mathbb{R}^p, df_a(h)=(d(f_1)_a(h),...,d(f_q)_a(h)). \tag{2.16}$$

Proof. $\Longrightarrow$) Suppose that f is differentiable at a. Then:

$$f(a+h)=f(a)+df_a(h)+\|h\|\,\varepsilon(h).$$

Which implies that

$$f_i(a+h)=f_i(a)+\varphi_i(h)+\|h\|\,\varepsilon_i(h),$$

where φ_i is a linear form on $\mathbb{R}^p$, and $\lim_{h\to 0_p}\varepsilon_i(h)=0$.
This proves that each coordinate function f_i is differentiable at a, and that:

$$(df_i)_a(h)=\varphi_i(h)=(df_a)_i(h).$$

$\Longleftarrow$) Suppose that each coordinate function f_i is differentiable at a, then if the linear transformation is defined by:

$$l(h)=(df_1)_a(h),...,(df_q)_ah),$$

we can easily verify that this application is suitable as a differential of f at the point a. □

2.4.9 Jacobian matrix

Definition 2.4.9. *Let f be a differentiable function at $a \in U$. The Jacobian matrix of f at a point a is a matrix denoted by $J_f(a)$, where:*

$$J_f(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & . & . & . & \frac{\partial f_1}{\partial x_p}(a) \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ \frac{\partial f_q}{\partial x_1}(a) & . & . & . & \frac{\partial f_q}{\partial x_p}(a) \end{pmatrix}$$

We then have, for $h = \left(h_1, ..., h_p\right)^T$, we have $df_a(h) = J_f(a).h$.
If $p = q$, the Jacobian of f at a is the determinant of the Jacobian matrix of f at a, i.e, $\mathcal{J}_f(a) = \det(J_f(a))$.

2.4.10 Composition of differentiable functions

Theorem 2.4.8. *Let $f : U \to \mathbb{R}^q$ and $g : V \to \mathbb{R}^s$ (U and V are open subsets of $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively), such that $f(U) \subset V$.*

1. *If f is differentiable at $a \in U$ and if g is differentiable at $b = f(a) \in V$, then $g \circ f$ is differentiable at a, and we have:*

$$d(g \circ f)_a = d(g)_{f(a)} \times df_a(h), \; h \in U, \tag{2.17}$$

or:

$$J_{g \circ f}(a)h = J_g(f(a)) \times J_f(a)h. \tag{2.18}$$

2. *If f is differentiable on U and if g is differentiable on V, then $g \circ f$ is differentiable on U.*

3. *If f is of class C^1 on U and g is of class C^1 on V, then $g \circ f$ is of class C^1 on U.*

Proof. 1. Assume that f is differentiable at a. By definition:

$$f(a + h) = f(a) + df_a(h) + \|h\| \, \varepsilon_1(h), \text{ with } \varepsilon_1(h) \underset{h \to o_p}{\longrightarrow} o_q.$$

Similarly, g is differentiable at $b = f(a)$:

$$g(b+k) = g(b) + dg_b(k) + \|k\|\,\varepsilon_2(k), \text{ with } \varepsilon_2(k) \underset{k\to o_q}{\longrightarrow} o_s.$$

$g \circ f$ is differentiable at $a \Longleftrightarrow \exists?$ a linear application L, such that:

$$(g\circ f)(a+h) = (g\circ f)(a) + L(h) + \|h\|\,\varepsilon(h), \text{ with } \varepsilon(h) \underset{h\to o_p}{\longrightarrow} o_s.$$

We have

$$(g\circ f)(a+b) = g[f(a+b)] = g[b + f(a+b) - b] = g[b + k(h))],$$

such that: $k(h) = f(a+h) - f(a)$. Then

$$(g\circ f)(a+h) = g(b+k(h))) = g(b) + dg_b(k(h)) + \|k(h)\|\,\varepsilon_2(k(h)).$$

It is clear that: $\lim\limits_{h\to o_p} k(h) = \lim\limits_{h\to o_p} f(a+h) - f(a) = o_q$. Then $\lim\limits_{k(h)\to o_q} \varepsilon_2(k(h)) = o_s$. Since dg_b is linear, we can write:

$$\begin{aligned}(g\circ f)(a+h) &= g(b) + dg_b df_a(h) + \|h\|\,dg_b(\varepsilon_1(h)) + \|k(h)\|\,\varepsilon_2(k(h))\\ &= g(b) + dg_b(df_a(h)) + \|h\|\,\varepsilon(h),\end{aligned}$$

where $\varepsilon(h) = dg_b(\varepsilon_1(h)) + \dfrac{\|k(h)\|}{\|h\|}\varepsilon_2(k(h)), \forall h \neq o_p$. It remains to be proven that $\lim\limits_{h\to o_p} \varepsilon(h) = o_s$. We have

$$\begin{aligned}\lim_{h\to op} dg_b(\varepsilon_1(h)) &= d_{gb}(\lim_{h\to o_p} \varepsilon(h)) \quad (\text{ because } dg_b \text{ is continuous})\\ &= d_{gb}(o_q) = o_s \quad (\text{ because } dg_b \text{ is linear}).\end{aligned}$$

On the other hand

$$\frac{\|k(h)\|}{\|h\|} = \frac{\left\|df_a(h) + \|h\|\,\varepsilon_1(h)\right\|}{\|h\|} \leq \frac{\left\|df_a(h)\right\|}{\|h\|} + \|\varepsilon_1(h)\|\,.$$

We choose the norm $\|h\|_\infty = \sup\limits_{i=\overline{1,q}} |h_i|$ and

$$\left\|df_a(h)\right\|_\infty = \sup_{1\leq i\leq q}\left|\sum_{j=1}^{p} \frac{\partial f_i}{\partial f_j}(a)h_j\right| = \left|\sum_{j=1}^{p} \frac{\partial f_{i_0}}{\partial x_j}(a)h_j\right|.$$

So:

$$0 \leq \left|\frac{\|k(h)\|}{\|h\|}\varepsilon_2(k(h))\right| \leq \left[\sum_{j=1}^{p}\left|\frac{\partial f_{i_0}}{\partial x_j}\right|(a) + \|\varepsilon_1(h)\|\right]|\varepsilon_2(k(h))| \to 0, \text{ when } h \to 0_p$$

Then $g \circ f$ is differentiable at a, and moreover:

$$d(g \circ f)_a = dg_{f(a)} \times df_a.$$

Let $h = \left(h_1, ..., h_p\right)^T$, we then have:

$$\begin{aligned} J_{g\circ f}(a)h &= d(g \circ f)_a(h) = \left(d(g)_{f(a)} \times d(f)_a\right)(h) \\ &= J_g(f(a))h, \forall h \neq 0, \end{aligned}$$

which gives $J_{g\circ f}(a) = J_g(f(a)) \times J_f(a)$.

2. Paragraphs (2) and (3) are then obvious.

□

Special cases

1) If $p = q = s = 1$, we have:

$$(g \circ f)'\ (a)h = g'(f(a))f'(a)h, \forall h \in \mathbb{R}^*$$

That is

$$(g \circ f)'(a) = g'(f(a))f'(a).$$

2) If $p = s = 1$ and q arbitrary. i.e;

$f : u \subset \mathbb{R} \longrightarrow \mathbb{R}^q$ and $g : v \subset \mathbb{R}^q \longrightarrow \mathbb{R}$

$x \mapsto f(x) = (f_1(x), \ldots, f_q(x))$ and $g(f(x)) = g(f_1(x), \ldots, f_q(x))$.

So $J_f(a) = \begin{pmatrix} f_1'(a) \\ \vdots \\ f_q'(a) \end{pmatrix}$ (vector matrix), and

$J_g(f(a)) = (\frac{\partial g}{\partial f_1}(f(a)) \ldots\ldots \frac{\partial g}{\partial f_q}(f(a)))$(Line matrix). Then

$$J_{g\circ f}(a) = J_g(f(a)) \times J_f(a) = \sum_{i=1}^{q}\frac{\partial g}{\partial f_i}(f(a)) \times f_i'(a). \quad (2.19)$$

Example 2.4.2. *Consider the functions $f : \mathbb{R} \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}$ defined, respectively, by:*

$$f(x) = (\cos x, \sin x, \tan x)$$

and

$$g(x, y, z) = x^2 + y + z^3.$$

We'll try to find $J_{g\circ f}(0)$. It is clear that f and g are differentiable. Moreover:

$$J_f(0) = \begin{pmatrix} f_1'(0) \\ f_2'(0) \\ f_3'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

and

$$J_g(x, y, z) = \begin{pmatrix} 2x & 1 & 3z^2 \end{pmatrix}.$$

Since $f(0) = (1, 0, 0)$, we have: $J_g(f(0)) = \begin{pmatrix} 2 & 1 & 0 \end{pmatrix}$, and thus

$$J_{g\circ f}(0) = \begin{pmatrix} 2 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 1.$$

3) If $p = q$ and s arbitrary:
$X = (x_{1,......,x_p}) \overset{f}{\mapsto} f(X) = (f_1(X), ..., f_p(X))$ and $g(f(X)) = (g_1(f(X), ..., g_s(f(X)))$
We have $J_F(a) = J_g(f(a))J_f(a)$. So:

$$\begin{pmatrix} \frac{\partial F_1}{\partial x_1}(a) & . & . & . & \frac{\partial F_1}{\partial x_p}(a) \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ \frac{\partial F_s}{\partial x_1}(a) & . & . & . & \frac{\partial F_s}{\partial x_p}(a) \end{pmatrix} = \begin{pmatrix} \frac{\partial g_1}{\partial f_1}(f(a)) & . & . & . & \frac{\partial g_1}{\partial f_p}(f(a)) \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ \frac{\partial g_s}{\partial f_1}(f(a)) & . & . & . & \frac{\partial g_s}{\partial f_p}(f(a)) \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & . & . & . & \frac{\partial f_1}{\partial x_p}(a) \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ \frac{\partial f_p}{\partial x_1}(a) & . & . & . & \frac{\partial f_p}{\partial x_p}(a) \end{pmatrix}$$

So

$$\frac{\partial F_i}{\partial x_j}(a) = \sum_{k=1}^{p} \frac{\partial g_i}{\partial f_k}(f(a)) \frac{\partial f_k}{\partial x_j}, i = \overline{1, s} \text{ and } j = \overline{1, p}, \ J_F(a) = \left[\frac{\partial F_i}{\partial x_j}(a)\right]. \quad (2.20)$$

2.4.11 Differentiation with respect to a vector

Definition 2.4.10. *Let f be an application defined on an open subset U of $\mathbb{R}^p$ with values in $\mathbb{R}^q$ and let $v \in \mathbb{R}^p$ be a non nul vector. We say that f is differentiable at $a \in U$ (or admits a derectional derivative) with respect to the vector v, if:*

$$\lim_{t\to 0} \frac{f(a+tv)-f(a)}{t} \text{ exists.} \tag{2.21}$$

This limit is denoted by $d_v f_a$.

Remark 2.4. 1. *Existence of directional derivative does not depend on a vector, but only on its direction. Indeed, if f is differentiable at a with respect to a vector $v \in \mathbb{R}^p - \{0\}$, then f is also differentiable at a with respect to all vectors of the form λv, $\lambda \in \mathbb{R}^*$, because*

$$\lim_{t\to 0} \frac{f(a+t\lambda v)-f(a)}{t} = \lambda \lim_{s\to 0} \frac{f(a+sv)-f(a)}{s},$$

and thus

$$\lambda \in \mathbb{R}^* \;\; d_{\lambda v} f_a = \lambda d_v f_a.$$

2. *if $n = 1$, we have only two directions corresponding to the values $v = 1$ and $v = -1$. in this case, a function f defined on an open interval I of $\mathbb{R}$ with values in $\mathbb{R}^p$, is differentiable at $a \in I$ with respect to $v = 1$ if:*

$$\lim_{t\to 0} \frac{f(a+t)-f(a)}{t} \text{ exists,}$$

and it is differentiable at $a \in I$ with respect to $v = -1$, if

$$\lim_{t\to 0} \frac{f(a-t)-f(a)}{t} = -\lim_{t\to 0} \frac{f(a+t)-f(a)}{t} \text{ exists.}$$

In other words, in the case where $n = 1$, there is an equivalence between differentiable and differentiable with respect to a vector.

Theorem 2.4.9. *Let f be an application defined on the open U of $\mathbb{R}^p$ with values in $\mathbb{R}^q$. If f is differentiable at $a \in U$ then it has at a a derivative with respect to any vector $v \in \mathbb{R}^p - \left\{0_p\right\}$ and we have $d_v f_a = df_a(v)$.*

Proof. Since f is differentiable at a, we then have:

$$f(a+h) = f(a) + df_a(h) + \|h\|\,\varepsilon(h), \text{ where} \lim_{h\to 0_p} \varepsilon(h) = 0. \tag{2.22}$$

Let $h = tv$ where t is a real number belonging to a neighborhood of o.
The expression (2.22) becomes:

$$f(a+tv) = f(a) + tdf_a(v) + |t|\ \ \|v\|\,\varepsilon_1(t), \tag{2.23}$$

where $\lim_{t\to 0}\varepsilon_1(t) = \lim_{t\to 0}\varepsilon(tv) = 0$.

If $t \neq 0$, it is rewritten in the form:

$$\frac{f(a+tv) - f(a)}{t} = df_a(v) + \frac{|t|}{t}\|v\|\,\varepsilon_1(t) \tag{2.24}$$

We pass to the limit as t tends towards 0, we find the result. □

Remark 2.5. *Note that theorem(2.4.9) does not admit a converse: a function can admit derivatives with respect to all vectors without being differentiable.*

Example 2.4.3. *Let us consider the application $f : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y}, & \text{if } y \neq -x^2 \\ 0, & \text{otherwise.} \end{cases}$$

Since $\lim_{x\to 0} f(x, x^3 - x^2) = -1 \neq 0$, the application f is not continuous at $(0,0)$. This implies that f is not differentiable at $(0,0)$.
On the other hand, f has a directional derivatives at $(0,0)$ with respect to all vectors. Indeed. Let $v = (\alpha, \beta) \in \mathbb{R}^2_*$, we then have

$$\frac{f((0,0)+tv) - f(0,0)}{t} = \frac{f((t\alpha, t\beta))}{t} = \frac{\alpha\beta}{t\alpha^2 + \beta}$$

This quantity always has a limit as t tends towards 0, which equals 0 if $\beta = 0$ and α otherwise.

2.4.12 Mean Value inequality

Definition 2.4.11. *Let a and b be two elements of $\mathbb{R}^p$. The segment with endpoints a and b is the set $[a,b]$ defined by:*

$$[a,b] = \left\{X \in \mathbb{R}^P;\ \exists t \in [0,1]\ \ X = a + t(b-a)\right\}. \tag{2.25}$$

Definition 2.4.12. *A subset E of $\mathbb{R}^p$ is called convex, if and only if:*

$$\forall (a,b) \in E^2, \ [a,b] \subset E. \tag{2.26}$$

Theorem 2.4.10. *(Mean Value Theorem) Let f be a function of class C^1 on an open set $U \subset \mathbb{R}^P$ with values on $\mathbb{R}$. Then for all vector a, b of U, such that $[a,b] \subset U$, there exists $c \in]a,b[$ verifying:*

$$f(b) = f(a) + df_c(b-a). \tag{2.27}$$

Or, bysetting $b = a + h$, there exists $\theta \in]0,1[$ verifying:

$$f(a+h) = f(a) + df_{a+\theta h}(h). \tag{2.28}$$

Remark 2.6. *Given the expression $df_c(h) = \sum_{i=1}^{p} \frac{\partial f}{\partial x_i}(c)h_i$. The two preceding formulas can be written as*

$$f(b) = f(a) + \sum_{i=1}^{p} \frac{\partial f}{\partial x_i}(c)(b_i - a_i), \tag{2.29}$$

or

$$f(a+h) = f(a) + \sum_{i=1}^{p} \frac{\partial f}{\partial x_i}(a+\theta h)h_i. \tag{2.30}$$

Proof. (**Proof of Theorem (2.4.10)) :**
Let $t \mapsto F(t)$ be the function defined on $[0,1]$ by $F(t) = f(g(t))$, where

$$g(t) = a + t(b-a).$$

It is clear that g is of class C^1 on $[0,1]$. So F is of class C^1 on $[0,1]$.
According to the composition theorem:

$$F'(t) = (f \circ g)'(t) = \sum_{i=1}^{p} \frac{\partial f}{\partial x_i}(a + t(b-a))(b_i - a_i).$$

Since F is of class C^1 on $[0,1]$, then we can apply the mean value theorem established for numerical functions of a real variable, $\exists \theta \in]0,1[$ such that:

$$F(1) = F(0) + F'(\theta),$$

which implies that there exists $c \in]a, b[$, such that:

$$f(b) = f(a) + \sum_{i=1}^{p} \frac{\partial f}{\partial x_i}(c)(b_i - a_i).$$

□

Remark 2.7. *Note that this Mean Value Theorem is only valid if the arrival space has dimension* 1.
Indeed, if we consider the function $f : \mathbb{R} \to \mathbb{R}^3$*, defined by* $f(t) = (\cos t, \sin t, t)$.
We have

$$f(2\pi) - f(0) = (0, 0, 2\pi) \text{ and } f^{'}(t) = (-\sin t, \cos t, 1).$$

So

$$\forall t \in \mathbb{R},\ f(2\pi) - f(0) \neq 2\pi \dot{f}(t).$$

Theorem 2.4.11. *(Mean Value Inequality) Let* $f : U \longrightarrow \mathbb{R}^q$ *be a function of class* C^1 *on* U *(*U *is an open convex of* $\mathbb{R}^p$*).*
Assume that there exists a constant $M \geq 0$*, such that* $\left\| d_f(x) \right\| \leq M$ *, for all* $x \in U$*.*
Then:

$$\left\| f(x) - f(y) \right\| \leq M \left\| x - y \right\|. \tag{2.31}$$

Proof. Let $x, y \in U$, so $x + t(y - x) \subset U, \forall t \in [0, 1]$ (because U is convex) and let $F : [0, 1] \to \mathbb{R}^p$ be a function defined by: $F(t) = x + t(y - x)$.
It is clear that F is of class C^1 on $[0, 1]$, and moreover:

$$\dot{F}(t) = d_f(x + t(y - x))(y - x).$$

By the fundamental theorem of calculus, we can writte:

$$F(1) - F(0) = \int_0^1 \dot{F}(t)dt.$$

So

$$\begin{aligned}\left\|f(x)-f(y)\right\| &= \|F(1)-F(0)\| \\ &= \left\|\int_0^1 \dot{F}(t)dt\right\| \\ &= \left\|\int_0^1 (y-x)d_f(x+t(y-x))dt\right\| \\ &\leq \int_0^1 \left\|(y-x)\right\| \left\|d_f(x+t(y-x))\right\| dt \\ &\leq M\left\|y-x\right\|\end{aligned}$$

□

2.5 Taylor's formula

2.5.1 Successive derivatives of the function

$$t \mapsto F(t) = f(a+t(b-a))$$

Definition 2.5.1. *We say that the function F is of class C^n, when all the derivatives of F up to the order n exist and are continuous on the set where we work.*

Definition 2.5.2. *Now, let $f : U \longrightarrow \mathbb{R}^q$ be a function of class C^n on U (U be an open on $\mathbb{R}^p$). So F defined by $F(t) = f(a+t(b-a))$ is also of class C^n on $\mathbb{R}$.*

We have

$$\dot{F}(t) = \sum_{i=1}^{p} \frac{\partial f}{\partial x_i}(a+t(b-a))(b_i-a_i). \tag{2.32}$$

Let $h = (h_1, h_2, ..., h_p) = b-a$. So $\dot{F}(t) = \sum_{i=1}^{p} \frac{\partial f}{\partial x_i}(a+th)h_i$.

Therefore

$$\ddot{F}(t) = \sum_{j=1}^{p}\sum_{i=1}^{p} \frac{\partial^2 f}{\partial x_j \partial x_i}(a+th)h_i h_j. \tag{2.33}$$

Recall that:

$$\left(\sum_{j=1}^{p} a_i\right)^2 = \sum_{j=1}^{p} a_i^2 + 2\sum_{1\leq i<j\leq p}^{p} a_i a_j. \tag{2.34}$$

Since $\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial xj}$ (becuse f is of class C^2 on U), we have:

$$\ddot{F}(t) = \sum_{i=1}^{p} \frac{\partial^2 f}{\partial x_i^2}(a+th)h_i^2 + 2\sum_{1\leq i<j\leq p} \frac{\partial^2 f}{\partial x_j \partial x_i}(a+th)h_i h_j. \tag{2.35}$$

By induction, it can be shown that:

$$F^n(t) = \;= \sum_{i_n=1}^{p} \ldots \sum_{i_1=1}^{p} \frac{\partial^r f}{\partial x_{i_n} \ldots \partial x_{i_1}}(a+th)h_{i_1}\ldots h_{i_n}, \text{ for all } h_{i_j} \in U, j = 1, \ldots, n.$$

2.5.2 Higher-order differentials

Definition 2.5.3. *(Differential of order 2) We will begin by viewing the second differential as a bilinear function..*

Let U be an open of $\mathbb{R}^p$, $f : U \to \mathbb{R}$, be an application and let $a \in U$. We assume that the second-order partial derivatives of f at a exist. The second differential of f at the point a, denoted $d^2 f_a$, is defined as the bilinear form:

$$\begin{array}{ccc} (\mathbb{R}^p)^2 & \to & \mathbb{R} \\ (h,k) & \mapsto & d_h(d_k f_a). \end{array}$$

In terms of coordinates:

$$d^2 f_a(h,k) = d_h(d_k f_a) = \sum_{j=1}^{p}\sum_{i=1}^{p} \frac{\partial^2 f}{\partial x_j \partial x_i}(a)h_i k_j, \tag{2.36}$$

where d_h is the directional derivative in the direction h.

2.5.3 Taylor's formula

Theorem 2.5.1. *Let $f : U \subset \mathbb{R}^p \to \mathbb{R}$ be an application $r-$times differentiable at $a \in U$, then it admits a Taylor-Young expansion of order r at the point a; i.e., there* *exists a function* $\varepsilon : \mathbb{R}^p \to \mathbb{R}$, *with* $\lim \varepsilon(h) = 0$, *such that:*

$$f(a+h) = f(a) + \sum_{k=1}^{r} d^k f_a h^k + \|h\|^r \varepsilon(h) \tag{2.37}$$

Remark 2.8. *(Taylor formula of order 02) Let* f *be a function of class* C^2 *on a convex set* U *of* $\mathbb{R}^p$ *with values in* $\mathbb{R}$, *and let* $a, b \in U$. *So* $\exists c \in]a, b[$ *such that:*

$$\begin{aligned} f(b) &= f(a) + \sum_{i=1}^{p} \frac{\partial f}{\partial x_i}(a)(b_i - a_i). + \\ & \frac{1}{2}\left[\sum_{i=1}^{p} \frac{\partial^2 f}{\partial x_i^2}(c)(b_i - a_i)^2 + 2 \sum_{1\le i<j\le p} \frac{\partial^2 f}{\partial x_j \partial x_i}(c)(b_i - a_i)(b_j - a_j)\right]. \end{aligned}$$

Remark 2.9. *(Writing in dimension* $p = 2$*)*

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + (x-x_0)\frac{\partial f}{\partial x}(x_0,y_0) + (y-y_0)\frac{\partial f}{\partial y}(x_0,y_0) + \\ &+\frac{1}{2}\left[\begin{array}{c} (x-x_0)^2\frac{\partial^2 f}{\partial x^2}(x_0+\theta(x-x_0), y_0+\theta(y-y_0)) + 2(x-x_0)(y-y_0)\frac{\partial^2 f}{\partial x \partial y}(x_0+\theta(x-x_0), y_0 + \\ +\theta(y-y_0)) + 2(y-y_0)^2\frac{\partial^2 f}{\partial y^2}(x_0+\theta(x-x_0), y_0+\theta(y-y_0)), \end{array}\right] \end{aligned}$$

where $\theta \in]0, 1[$.

2.5.4 Quadratic form and Hessian matrix

Definition 2.5.4. *In the expression (2.35), the application:*

$$q_a = \dot{F}(0) = \sum_{i=1}^{p} \frac{\partial^2 f}{\partial x_i^2}(a)h_i^2 + 2 \sum_{1\le i<j\le p} \frac{\partial^2 f}{\partial x_j \partial x_i}(a)h_i h_j. \tag{2.38}$$

is a quadratic form on $\mathbb{R}^p$.
The symmetric matrix that represents $d^2 f_a$ *in the canonical basis of* $\mathbb{R}^p$, *is called the Hessian matrix of* f *at the point* a *and it is denoted by* $H_f(a)$
The coefficients of the matrix $H_f(a)$ *are the second partial derivatives* $\frac{\partial^2 f}{\partial x_j \partial x_i}(a)$, *for*

all $1 \le i, j \le p$. *So*

$$H_f(a) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(a) & \cdot\ \cdot\ \cdot & \frac{\partial^2 f}{\partial x_1 \partial x_p}(a) \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \frac{\partial^2 f}{\partial x_p \partial x1}(a) & & \frac{\partial^2 f}{\partial x_p^2}(a) \end{pmatrix}, \tag{2.39}$$

and for all $1 \le i, j \le p$, *we have* $\frac{\partial^2 f}{\partial x_j \partial x_i}(a) = \frac{\partial^2 f}{\partial x_i \partial x_j}(a)$

Definition 2.5.5. *(Differential of order* ≥ 2*) Let* $f : U \subset \mathbb{R}^p \to \mathbb{R}$ *be an application of class* C^{r-1}. *By induction, we can define the differential of order* r *at a point* $a \in U$, *the continuous* $r-$*linear application* $d^r f_a$ *by:*

$$\begin{aligned} d^r f_a(h_1, h_2, ..., h_r) &= d_{h_1}(...d_{h_r} f_a) \\ &= \sum_{i_r=1}^{p} ... \sum_{i_1=1}^{p} \frac{\partial^r f}{\partial x_{i_r} ... \partial x_{i_1}}(a) h_{i_1} ... k_{i_r}, \text{for all } h_{i_j} \in U, j = \overline{1, r}. \end{aligned}$$

Definition 2.5.6. *Let* $f : U \subset \mathbb{R}^p \to \mathbb{R}$ *be an application* $r-$*times differentiable in* $a \in U$. *The application* $\varphi : \mathbb{R}^p \to \mathbb{R}^q$ *defined by:*

$$\varphi(h) = d^r f_a(h, h, ..., h),$$

is called the symbolic power of the order r *of the differential of* f *at* a.

Notation 2.5.1. *We denote by:*

$$d^r f_a(h, h, ..., h) = d^r f_a h^r \tag{2.40}$$

2.5.5 Second-order Taylor expansion

Since the partial derivatives are continuous, we can write:

$$\frac{\partial^2 f}{\partial x_i \partial x_i}(c) = \frac{\partial^2 f}{\partial x_i \partial x_i}(a + \theta(b-a)) = \frac{\partial^2 f}{\partial x_i \partial x_i}(a) + n_{i,j}(b-a), \tag{2.41}$$

where $n_{i,j}(b-a) \to 0$ when $b \to a$, and since

$$2\left|(b_i - a_i)(b_j - a_j)\right| \le |(b_i - a_i)|^2 + \left|(b_j - a_j)\right|^2 \le \|b - a\|^2,$$

we can finally state that there exists a function $(b-a) \mapsto \epsilon(b-a)$ defined in the neighborhood of 0, such that:

$$f(b) = f(a)+\sum_{i=1}^{p}\frac{\partial f}{\partial x_i}(a)(b_i-a_i).+\frac{1}{2}\left[\begin{array}{c}\sum_{i=1}^{p}\frac{\partial^2 f}{\partial x_i^2}(a)(b_i-a_i)^2 \quad + \\ +2\sum_{1\leq i<j\leq p}^{p}\frac{\partial^2 f}{\partial x_j \partial x_i}(a)(b_i-a_i)(b_j-a_j)\end{array}\right]+\|b-a\|^2\,\epsilon(b-a),$$

where $\epsilon(b-a) \longrightarrow 0$, when $b \longrightarrow a$.

Example 2.5.1. *Let's consider the function*

$$f(x,y) = ln(1+x+y).$$

We computr its Taylor expansion of order 2 *at* $(x_0, y_0) = (0,0)$
Since

$$f(0,0) = 0, \frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 1$$

and

$$\frac{\partial^2 f}{\partial x^2}(0,0) = \frac{\partial^2 f}{\partial y^2}(0,0) = \frac{\partial^2 f}{\partial x \partial y}(0,0) = -1,$$

then:

$$f(x,y) = x+y-\frac{(x+y)^2}{2}+(x^2+y^2)\epsilon(x,y),$$

where $\epsilon(x,y) \longrightarrow 0$*, when* $(x,y) \longrightarrow (0,0)$.

2.5.6 Another Taylor formula for an arbitrary order of a function of two variables

Very practical symbolic notation

We denote by $\left[x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right]^{(n)} = \sum_{k=0}^{n} c_n^k\, x^k\, y^{n-k}\frac{\partial^n f}{\partial x^k\, \partial y^{n-k}}.$

Derivative of order n of the function $F(t) = f(x_0 + th_1, y_0 + th_2)$

It is easy to see by induction that if f is of class C^n, then F is also of class C^n and that its derivative of order n is given by:

$$F^{(n)}(t) = \sum_{k=0}^{n} C_n^k h_1^k \, h_2^{n-k} \frac{\partial^n f}{\partial x^k \, \partial y^{n-k}}(x_0 + th_1, y_0 + th_2). \tag{2.42}$$

Taylor's formula of order n

Let f be a function of class C^n, then we can apply the Taylor-Lagrange formula to the order n on the function $t \in [0,1] \mapsto F(t)$ between 0 and 1, we then get:

$$F(1) = F(0) + \sum_{k=1}^{n-1} \frac{1}{k!} \frac{F^{(k)}(0)}{(k)!} + \frac{F^{(n)}(\theta)}{n!}, \text{ where } \theta \in]0,1[\,. \tag{2.43}$$

That is

$$\begin{aligned} f\,(x_0 + h_1, y_0 + h_2) \quad &= \quad f\,(x_0, y_0) + \sum_{k=1}^{n-1} \frac{1}{k!} \left[h_1 \frac{\partial f}{\partial x} + h_2 \frac{\partial f}{\partial y} \right]^{(k)} (x_0, y_0) + \\ &\quad + \frac{1}{n!} \left[h_1 \frac{\partial f}{\partial x} + h_2 \frac{\partial f}{\partial y} \right]^{(n)} (x_0 + \theta h_1, y_0 + \theta h_2). \end{aligned}$$

Taylor expansion of order n

$$\begin{aligned} f(x, y) \quad &= \quad f(x_0, y_0) + \sum_{k=1}^{n} \frac{1}{k!} \left[(x - x_0) \frac{\partial f}{\partial x} + (y - y_0) \frac{\partial f}{\partial y} \right]^{(k)} (x_0 - y_0) + \\ &\quad + \|(x - x_0), (y - y_0)\|^n \varepsilon \left[(x - x_0), (y - y_0) \right], \end{aligned}$$

with $\varepsilon \left[(x - x_0), (y - y_0) \right] \longrightarrow 0$, when (x, y) tends to (x_0, y_0).

2.6 Diffeomorphism

Definition 2.6.1. *If f is a bijection of class C^1 from an open U of $\mathbb{R}^p$ to an open V of $\mathbb{R}^p$, and if the inverse bijection f^{-1} is also of class C^1, we then say that f is a C^1-diffeomorphism from U to V.*

Remark 2.10. *Note that V is necessarily an open set. Indeed, it is the inverse image of the open set U under the continuous application f^{-1}. It follows from the composition theorem that the Jacobian matrix of f is invertible at every point a of U and that the Jacobian of f^{-1} at the point $f(a)$ is the Jacobian of the inverse matrix, i.e.*

$$J_{f^{-1}}(f(a)) = \left[J_f(a)\right]^{-1}. \tag{2.44}$$

Example 2.6.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a function defined by $f(x, y) = (x + y, x - y)$. So f^{-1}:$\mathbb{R}^2 \to \mathbb{R}^2$, is defined by $f^{-1}(x, y) = \frac{1}{2}(x + y, x - y)$.*
It is clear that f and f^{-1} are of class C^1 on $\mathbb{R}^2$. So f is a diffeomorphisme of class C^1 on $\mathbb{R}^2$.

Remark 2.11. *If f is a bijection of class C^k and if f^{-1} is also of class C^k, we say that f is a diffeomorphisme of class C^k $(k \geq 1)$.*

Theorem 2.6.1. *(**Local inversion theorem**) Let f be a function of class C^k $(k \geq 1)$, defined in a neighborhood of a point $a \in \mathbb{R}^p$, with values in $\mathbb{R}^p$, and let $b = f(a)$.*
If d_f is invertible, so there exists a neighborhood A of a and a neighborhood B of b such that f is a local C^k-diffeomorphism from A to B.

Theorem 2.6.2. *(**Global inversion theorem**) Let f be a function of class C^k $(k \geq 1)$ on $U \subset \mathbb{R}^p$(U is an open), injective and such that the differential $d_f(x)$ is invertible for all $x \in U$, then f is a C^k-diffeomorphism from the open U onto the open $f(U)$.*

Proof. Since $d_f(x)$ is inversible, then $det J_f(x) \neq 0$ and since f is injective, then f is bijective from U onto $f(U)$.
So, f is a diffeomorphisme, and moreover $f(U)$ is an open set. □

2.6.1 Partial differential equations (PDE)

The particular characteristic of a partial differential equation (PDE) is that it involves functions of several variables.

Definition 2.6.2. *A PDE is then a mathematical relation that connects a function with its partial derivatives with respect to multiple variables.*

Example 2.6.2. *Here are some examples of equations:*
1. $\frac{\partial f}{\partial t}(t,x) + c\frac{\partial f}{\partial x} f(t,x) = 0$*: One dimensional transport equation, where x is the spatial variable represents location along a line(for example, distance in meters), t is the time variable, f is the transported quantity (for example, temperature, concentration of polluttant in a pip, density of a fluid) and the constnt c is transport velocity .*
2. $\frac{\partial^2 f}{\partial t^2}(t,x) = c^2\frac{\partial^2 f}{\partial x^2}(t,x)$*: One dimensional wave equation, where x is the spatial variable (position along a line), t is the time variable,* $f(t,x)$ *is the wave displacement (or pressure, voltage, etc.) and the constant c is the wave propagation speed.*
3. $\frac{\partial f}{\partial t}(t,x) = k\frac{\partial^2}{\partial x^2} f(t,x)$*: One dimensional heat diffusion equation, where* $f(t,x)$ *is the temperature, x is the spatial variable, t is the time variable and k is the thermal diffusivity constant*

2.6.2 Change of variables and PDEs

In this part, we aim to determine all functions of class C^k on an open $U \subset \mathbb{R}^P$, which satisfy a PDE. For this, we can use the diffeomorphism

$$\phi : (u_1, u_2, ..., u_p) \mapsto \phi(u_1, ..., u_p).$$

So, f is of class C^k on $U \Longrightarrow$ there exists an unique function g of class C^k on U, such that: $f = g \circ \phi$

Example 2.6.3. *Using the change of variables* $u = xy$ *and* $v = y$*, find all functions* f *of class* C^1 *on*

$$U = \left\{(x,y) \in \mathbb{R}^2 \diagup 0 < x < 1 \text{ and } 0 < y < 1\right\}, \tag{2.45}$$

verifying:

$$x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} = y. \tag{2.46}$$

Solution: *Let* $\phi : U \to \phi(U)$*, such that* $\phi(x,y) = (xy, y) = (u, v)$*.*
1. Let's show that $\phi : U \to \phi(U)$ *is a diffeomorphism of class* C^1 *on U.*
It is clear that U is an open. Moreover

$$\phi(U) = \left\{(u,v) \in \mathbb{R}^2 \;\; u = xy \in]0,1[\text{ and } v = y \in]0,1[\right\} = U$$

So $\phi(U)$ *is also an open.*
ϕ *is of class* C^1 *on* U *(obvious).*
ϕ *injective?.*
Let $x, y \in U$, *such that* $\phi(x,y) = \phi(x',y')$. *So* $\begin{cases} xy = x'y' \Longrightarrow x = x' \\ \quad\text{and } y = y' \end{cases}$
$J_\phi(x,y) = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \Longrightarrow \det J_\phi = y > 0.$
Therefore, ϕ *is a diffeomorphisme of class* C^1 *on* U*.*
Now, we solve the equation:

$$x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} = y. \tag{2.47}$$

f *is of class* C^1 *on* $U \Longrightarrow$ *there exists an unique function* g *of class* C^1 *on* U, *such that:* $f = g \circ \phi$.
The Jacobian product method leads us to:

$$J_f(x,y) = J_g(\phi(x,y).J\phi(x,y)) = J_g(u,v).J\phi(x,y).$$

So

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix},$$

That is:

$$\begin{cases} \frac{\partial f}{\partial x} = y\frac{\partial g}{\partial u} \\ \frac{\partial f}{\partial y} = x\frac{\partial g}{\partial u} + \frac{\partial g}{\partial v}. \end{cases}$$

The equation (2.47) *reduces to*

$$xy\frac{\partial g}{\partial u} - y(x\frac{\partial g}{\partial u} + \frac{\partial g}{\partial v}) = y.$$

Consequently: $\frac{\partial g}{\partial v} = -1$. *So* $g(u,v) = -v + h(u)$, *where* h *is a function of a single variable of class* C^1 *on* $]0,1[$.
Finally, we have:

$$\begin{aligned} f(x,y) &= g\left[\phi(x,y)\right] \\ &= g(u,v) \\ &= -v + h(u) = -y + h(xy). \end{aligned}$$

Example 2.6.4. *(Case of polar coordinates):*
Let ϕ:$]0,+\infty[\times\mathbb{R}\to\mathbb{R}^2\setminus\{(0,0)\}$ *such that* $\phi(r,\theta)=(r\cos\theta, r\sin\theta)$.
Let $F=f\circ\phi$, *where* f *is of class* C^1 *on* $\mathbb{R}^2\setminus\{(0,0)\}$.
1. *Express the partial derivatives of F in terms of those of* f.
2. *Find all functions* f *of class* C^1 *on* $\mathbb{R}^2\setminus\{(0,0)\}$, *such that:*

$$y\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial y}=0. \tag{2.48}$$

3. *Find all functions* f *of class* C^1 *on* $\mathbb{R}^2-\{(0,0)\}$, *such that:*

$$y\frac{\partial f}{\partial x}+x\frac{\partial f}{\partial y}=kf,\ (k=cst). \tag{2.49}$$

4. *Find all functions* f *of class* C^1 *on* $\mathbb{R}_*\times\mathbb{R}$, *such that:*

$$x\frac{\partial f}{\partial x}-y\frac{\partial f}{\partial y}=0. \tag{2.50}$$

Solution: *It is clear that* ϕ *is a diffeomorphism of class* $C^{+\infty}$ *on* $]0,+\infty[\times\mathbb{R}$.
1. *We have*

$$\begin{pmatrix}\frac{\partial F}{\partial r} & \frac{\partial F}{\partial\theta}\end{pmatrix}=\begin{pmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\end{pmatrix}\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix}, \tag{2.51}$$

So

$$\begin{cases}\frac{\partial F}{\partial r}=\cos\theta\frac{\partial f}{\partial x}+\sin\theta\frac{\partial f}{\partial y}\\ \frac{\partial F}{\partial\theta}=-r\sin\theta\frac{\partial f}{\partial x}+r\cos\theta\frac{\partial f}{\partial y}\end{cases} \tag{2.52}$$

2. *From Equ. (2.52), we deduce:*

$$\begin{cases}\frac{\partial f}{\partial x}=\cos(\theta)\frac{\partial F}{\partial r}-\frac{\sin(\theta)}{r}\frac{\partial F}{\partial\theta}\\ \frac{\partial f}{\partial y}=\sin(\theta)\frac{\partial F}{\partial r}+\frac{\cos(\theta)}{r}\frac{\partial F}{\partial\theta}.\end{cases} \tag{2.53}$$

Thus

$$y\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial y}=0\Leftrightarrow\frac{\partial F}{\partial\theta}=0. \tag{2.54}$$

Consequently: $F(r,\theta)=h(r)$, *where* h *is of class* C^1 *on* $]0,+\infty[$.
Finally

$$f(x,y)=F\left[\phi^{-1}(x,y)\right]=F(r,\theta)=h(r)=h(\sqrt{x^2+y^2}). \tag{2.55}$$

3. *We have*

$$y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y} = kf \Longleftrightarrow \frac{\partial F}{\partial \theta} = -kF. \tag{2.56}$$

Recall that the solutions of the differential equation $\dot{y}(x) = -ky(x)$ *are of the form:*

$$y(x) = c\exp(-kx), \tag{2.57}$$

Then:

$$F(r, \theta) = c(r)\exp(-k\theta),$$

where $c(r)$ *is a function of class* C^1 *on* $]0, +\infty[$.

And since $\theta = \arctan(\frac{y}{x})$, *we have*

$$f(x, y) = c(\sqrt{x^2 + y^2})\exp\left(-k\arctan(\frac{y}{x})\right). \tag{2.58}$$

4. *We have*

$$\begin{aligned} x\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} &= 0 \Longleftrightarrow \frac{\partial F}{\partial r} = 0 \\ &\Leftrightarrow F = h(\theta), \end{aligned}$$

where h *is of class* C^1 *on* $\mathbb{R}$.

Consequenty

$$f(x, y) = h\left(\arctan\left(\frac{y}{x}\right)\right). \tag{2.59}$$

Example 2.6.5. *(Wave equation)*

Using the change of variables $u = x + cy$, $v = x - cy$, *find all functions* f *of class* C^2 *on* $\mathbb{R}^2$, *verifying:*

$$\frac{\partial^2 f}{\partial y^2}(x, y) - c^2\frac{\partial^2 f}{\partial x^2}(x, y) = 0, \tag{2.60}$$

where c *is a non-zero positive real constant.*

Solution: *Let* ϕ: $\mathbb{R}^2 \to \mathbb{R}^2$, *such that* $\phi\,(x, y) = (u = x + cy, v = x - cy)$.

Let $f = g \circ \phi$. *The formula for partial derivatives gives us:*

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} \begin{pmatrix} 1 & c \\ 1 & -c \end{pmatrix}, \tag{2.61}$$

So:

$$\begin{cases} \frac{\partial f}{\partial x} = \frac{\partial g}{\partial u} + \frac{\partial g}{\partial v}, \\ \frac{\partial f}{\partial x} = c\frac{\partial g}{\partial u} - c\frac{\partial g}{\partial v}. \end{cases}$$

It follows that:

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial u} + \frac{\partial g}{\partial v}\right)\\ &= \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial g}{\partial u} + \frac{\partial g}{\partial v}\right) = \frac{\partial^2 g}{\partial u^2} + 2\frac{\partial^2 g}{\partial u\,\partial v} + \frac{\partial^2 g}{\partial v^2}, \qquad (2.62)\end{aligned}$$

and:

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \left(c\frac{\partial}{\partial u} - c\frac{\partial}{\partial v}\right)\left(c\frac{\partial g}{\partial u} - c\frac{\partial g}{\partial v}\right) = c^2\frac{\partial^2 g}{\partial u^2} - 2c^2\frac{\partial^2 g}{\partial u\,\partial v} + c^2\frac{\partial^2 g}{\partial v^2}. \qquad (2.63)$$

or

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \left(\frac{\partial g}{\partial u} + \frac{\partial g}{\partial v}\right)^{(2)} = \sum_{k=0}^{2} C_2^k 1^k.1^{2-k}\frac{\partial^2 g}{\partial u^k\,\partial v^{2-k}} = \frac{\partial^2 g}{\partial u^2} + 2\frac{\partial^2 g}{\partial u\,\partial v} + \frac{\partial^2 g}{\partial v^2}.\\ \frac{\partial^2 f}{\partial y^2} &= \left(c\frac{\partial g}{\partial u} - c\frac{\partial g}{\partial v}\right)^{(2)} = \sum_{k=0}^{2} C_2^K C^k(-C)^{2-K}\frac{\partial^2 g}{\partial u^K \partial v^{2-K}} = C^2\left(\frac{\partial^2 g}{\partial u^2} - 2\frac{\partial^2 g}{\partial u\,\partial v} + \frac{\partial^2 g}{\partial v^2}\right)\end{aligned}$$

The expression (2.75) becomes:

$$\frac{\partial^2 g}{\partial u\,\partial v} = 0 \Leftrightarrow g(u,v) = h(u) + k(v), \qquad (2.64)$$

where h and k are two functions of class C^1 on $\mathbb{R}$.
Finally:

$$f(x,y) = g(\phi(x,y)) = g(u,v) = h(u) + k(v) = h(x+cy) + k(x-cy). \qquad (2.65)$$

Example 2.6.6. *Let $g : U = \left\{(x,y) \in \mathbb{R}^2 \ tq\ y > |x|\right\} \to \mathbb{R}$ and $\varphi : U \to \varphi(U)$ with $\varphi(x,y) = (u = x-y, v = x+y)$. Let g be a function of class C^2. Let $f = g \circ \varphi$.*
1. Show that φ is a C^2-diffeomorphism between U and an open $\varphi(U)$ of $\mathbb{R}^2$ which will be specified.
2. Express the partial derivatives of f in terms of that of g.
3. Show that if f is a solution of the PDE.

$$\frac{\partial^2 f}{(\partial x)^2}(x,y) - \frac{\partial^2 f}{(\partial y)^2}(x,y) = \sqrt{(y^2 - x^2)}, \qquad (2.66)$$

then g is solution of $4\dfrac{\partial^2 g}{\partial u \partial v}(x,y) - \sqrt{-uv} = 0$
4. Deduce the general solution of (2.75).

***Solution:* 1.** *φ is injective and it is of class $C^{+\infty}$, and moreover*

$u = x - y$ and $v = x + y \Rightarrow \det j_\varphi(x, y) = \det\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = -2 \neq 0$. i.e j_φ is revertible.

According to the global inversion theorem, φ is a $C^{+\infty}$ diffeomorphism between U and $\varphi(U)$.

Calculate $\varphi(U)$:

** If $x > 0$, then $y - x > 0 \Leftrightarrow u = x - y < 0$ and $y > x > 0 \Leftrightarrow v = x + y > 0$*

** If $x < 0$, then $v = x + y > 0$ and $u = x - y < 0$.*

Then $\varphi(U) =]-\infty, 0[\times]0, +\infty[$

2. We have $f = g \circ \varphi$. We find

$$\left(\frac{\partial f}{\partial x}(x, y) \quad \frac{\partial f}{\partial y}(x, y) \right) = \left(\frac{\partial g}{\partial u}(u, v) \quad \frac{\partial g}{\partial v}(u, v) \right)\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{2.67}$$

That is

$$\begin{cases} \frac{\partial f}{\partial x}(x, y) = \frac{\partial g}{\partial u}(u, v) + \frac{\partial g}{\partial v}(u, v) \\ \frac{\partial f}{\partial y}(x, y) = -\frac{\partial g}{\partial u}(u, v) + \frac{\partial g}{\partial v}(u, v) \end{cases} \tag{2.68}$$

3. So

$$\begin{cases} \frac{\partial^2 f}{(\partial x)^2}(x, y) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial g}{\partial u}(u, v) + \frac{\partial g}{\partial v}(u, v)\right) \\ \quad = \frac{\partial^2 g}{(\partial u)^2}(u, v) + 2\frac{\partial^2 g}{\partial u \partial v}(u, v) + \frac{\partial^2 g}{(\partial v)^2}(u, v) \\ \frac{\partial^2 f}{(\partial y)^2}(x, y) = \left(-\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial g}{\partial u}(u, v) + \frac{\partial g}{\partial v}(u, v)\right) \\ \quad = \frac{\partial^2 g}{(\partial u)^2}(u, v) - 2\frac{\partial^2 g}{\partial u \partial v}(u, v) + \frac{\partial^2 g}{(\partial v)^2}(u, v). \end{cases}$$

Therefore the equation

$$\frac{\partial^2 f}{(\partial x)^2}(x, y) - \frac{\partial^2 f}{(\partial y)^2}(x, y) = \sqrt{(y^2 - x^2)},$$

becomes:

$$4\frac{\partial^2 g}{\partial u \partial v}(u, v) = \sqrt{-u}\,\sqrt{v}.$$

Which implies that:

$$g(u, v) = -\frac{1}{9}(-uv)^{\frac{3}{2}} + F(u) + H(v)), \tag{2.69}$$

where F, H are of class C^2 on $\mathbb{R}_-^*$ and $\mathbb{R}_+^*$, respectively.
Consequently:

$$f(x,y) = g(u,v) = -\frac{1}{9}(y^2 - x^2)^{\frac{3}{2}} + F(x-y) + H(x+y).$$

2.7 Local (relative) extrema

2.7.1 Free extrema

Let $f : U \longrightarrow \mathbb{R}$ (U is an open of $\mathbb{R}^p$) and let $a \in U$.

Definition 2.7.1. *We say that f has a local maximum at $a \in U$, if there exists an open ball $B(a,r) \subset U$, such that for all $x \in B$, we have $f(x) \leq f(a)$.*

Definition 2.7.2. *We say that f presents a local minimum at $a \in U$, if there exists an open ball $B(a,r) \subset U$ such that for all $x \in B$, we have $f(x) \geq f(a)$.*

Definition 2.7.3. *We say that f presents a local extremum at $a \in U$, if $f(a)$ is a local maximum or local minimum.*

Critical points (Stationary)

Definition 2.7.4. *Let $f : U \longrightarrow \mathbb{R}$ (U is an open of $\mathbb{R}^p$) be an application of class C^1 on U.*
A point $a \in U$ is said to be a critical point for f when the differential of f at a point a is zero. That is to say, each of the partial derivatives $\frac{\partial f}{\partial x_1}(a), \ldots, \frac{\partial f}{\partial x_p}(a)$ is zero.

Definition 2.7.5. *The gradient of f at point a is the vector $\nabla f(a)$ of $\mathbb{R}^p$ whose components are the first partial derivatives of f at the point a.*
That is:

$$\nabla f(a) = \left(\frac{\partial f}{\partial x_1}(a), \ldots, \frac{\partial f}{\partial x_p}(a)\right). \tag{2.70}$$

Definition 2.7.6. *The point a is a critical point for f, if and only if $\nabla f(a) = o_{\mathbb{R}^p}$*

Example 2.7.1. *Cnsider the function* $(x, y) \mapsto f(x, y) = x^3 + y^3 - 3xy$.
It is clear that f *is of class* C^1 *on* $\mathbb{R}^2$. *Then*

$$\begin{cases} \frac{\partial f}{\partial x}(x, y) = 3(x^2 - y) = 0 \\ \frac{\partial f}{\partial y}(x, y) = 3(y^2 - x) = 0 \end{cases} \Leftrightarrow x^2 = y \text{ and } y^2 = x.$$

So $x^4 = x$.
Finally, the critical points are $(0, 0)$ *and* $(1, 1)$.

Example 2.7.2. $(x, y) \mapsto f(x, y) = x^4 + y^3 + 2y\cos(x) + 5y$.
f *is of class* C^1 *on* $\mathbb{R}^2$. *Then*

$$\begin{cases} \frac{\partial f}{\partial x}(x, y) = 4x^3 - 2y\sin x. \\ \frac{\partial f}{\partial y}(x, y) = 3y^2 + 2\cos(x) + 5. \end{cases}$$

Since $\frac{\partial f}{\partial y}(x, y) \geq 5 - 2 = 3 > 0$, f *has no critical point.*

Necessary condition of local extremum

Theorem 2.7.1. *Let* $f : U \longrightarrow \mathbb{R}$ *(*U *is an open of* $\mathbb{R}^p$*) be a function of* C^1 *on* U.
If f *presents a local extremum at a point* $a \in U$, *then* a *is a critical point for* f.

Proof. Suppose, for example, that f has a local maximum at a. Then:

$$\exists r > 0, \text{ such that for all } \|x - a\| < r, \text{ we have } f(x) \leq f(a)$$

Let F be the function of variable t defined in a neighborhood of 0 by:

$$F(t) = f(a + th); \forall h \neq 0_p.$$

Let $x = a + th$. So

$$\|x - a\| = \|th\| < r \Rightarrow |t - 0| < \frac{r}{\|h\|} = \alpha > 0,$$

and

$$F(t) = f(x) \leq f(a) = F(0).$$

So
$$\exists \alpha > 0, \text{ such that for all } |t-0| < \alpha, \ F(t) \leq F(0);$$
which implies that F presents a local maximum at 0, and moreover $\dot{F}(0) = 0$.
On the other hand:
$$\dot{F}(t) = \sum_{i=1}^{p} h_i \frac{\partial f}{\partial x_i}(a + th).$$
Then
$$0 = \dot{F}(0) = \sum_{i=1}^{p} h_i \frac{\partial f}{\partial x_i}(a) = df_a(h). \tag{2.71}$$

□

Quadratic forms

Definition 2.7.7. *A quadratic form on $\mathbb{R}^p$ is any function $Q : \mathbb{R}^p \to \mathbb{R}$ defined by $Q(x) = x^T Ax$, where $x = \left(x_1, ..., x_p\right)$, A is an $n \times n$ symetric real matrix ($A = A^T$), and x^T is the transpose of the vector x. It can also be written as polynomial form: $Q(x) = \sum\limits_{i=1}^{p} a_{ii}x_i^2 + 2\sum\limits_{i \neq j} a_{ij}x_ix_j$, where a_{ij} are real coffecients.*
** If $Q(x) > 0$,$\forall x \neq 0_{\mathbb{R}^n}$, we then say that Q is a positive definite quadratic form. This is therefore equivalent to saying that all the egeinvalues of the matrix A are strictly positive.*
** If $Q(x) < 0$,$\forall x \neq 0_{\mathbb{R}^n}$, we then say that Q is a negative definite quadratic form. This is therefore equivalent to saying that all the egeinvalues of the matrix A are strictly negative.*

2.7.2 Sufficient condition for a local extremum

Theorem 2.7.2. *Let f be a function of class C^2 in a neighborhood of a critical point $a \in \mathbb{R}^p$.*
1. If the quadratic form $H(u) = \sum\limits_{i,j=1}^{p} \frac{\partial^2 f}{\partial x_j \partial x_j}(a)u_iu_j$ is positive definite, then f has a local minimum at a.
2. If H is negative definite, then f has a local maximum at a.

Proof. 1. Let's demonstrate this in the case of a positive-definite quadratic form. Then there exists a positive real number m such that for every vector $U = (u_1, ..., u_p)$, we have:

$$H(u) \geq m \|u\|^2,$$

where m denotes the smallest eigenvalue of the hessian matrix.
Now, let $u = x - a$. Using the second Taylor expansion, we get:

$$f(x) - f(a) = \nabla f(a)(x-a) + \frac{1}{2}H(x-a) + \|x-a\|^2 \varepsilon(x-a).$$

Since $\varepsilon(x-a) \longrightarrow 0$, when $x \longrightarrow a$, we have:

$$\forall m > 0, \exists \delta > 0 \text{ such that } \forall x \in B(a,\delta),\ |\varepsilon(x-a)| < \frac{m}{2},$$

which implies that $\varepsilon(x-a) > -\frac{m}{2}$. Then

$$f(x) - f(a) > \frac{1}{2}m\|x-a\|^2 - \frac{m}{2}\|x-a\|^2 = 0.$$

2. The proof is identical for a local maximum at a, by choosing:

$$H(u) \leq -m\|u\|^2 \quad \text{and} \quad \varepsilon(x-a) < \frac{m}{2}.$$

□

Case of the two-dimensional Let $(x,y) \longmapsto f(x,y)$ be a function of class C^2. The critical points are therefore obtained by solving the system of equations:

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{cases}$$

The quadratic form H at a critical point (a,b) is of the form:

$$H(x,y) = (x-a)^2\frac{\partial^2 f}{(\partial x)^2}(a,b) + 2(x-a)(y-b)\frac{\partial^2 f}{\partial y \partial x}(a,b) + (y-b)^2\frac{\partial^2 f}{(\partial y)^2}(a,b).$$

Then H is a second-degree polynomial that maintains a constant sign if and only if its discriminant is negative.

Let $A = \dfrac{\partial^2 f}{(\partial x)^2}(a,b)$, $B = \dfrac{\partial^2 f}{\partial y \partial x}(a,b)$ et $C = \dfrac{\partial^2 f}{(\partial y)^2}(a,b)$. The Hessian H behaves in the following way:

1. $H(x,y) > 0 \iff B^2 - AC < 0$ and $A > 0$, This is a local minimum.
2. $H(x,y) < 0 \iff B^2 - AC < 0$ and $A < 0$, This is a local maximum.
3. H changes its sign $\iff B^2 - AC > 0$. No extremes.
4. Doubtful case $\iff B^2 - AC = 0$.

Example 2.7.3. $f(x,y) = x^3 + y^3 - 3xy$.
The critical points are $(0,0)$ *and* $(1,1)$.
* *At* $(0,0)$*:*
$A = C = 0$ *and* $B = -3$ *and* $B^2 - AC = 9 > 0$. *So no extremes, and since* $A = 0$, *this is a saddle point.*
* *At* $(1,1)$*:*
$A = C = 6$ *and* $B = -3$. *So* $B^2 - AC = -27 < 0$ *and since* $A > 0$, *this is a local minimum.*

Example 2.7.4. *(Case of the three-dimensional)*
$f(x,y,z) = 15 - 8x + 8x^2 - 4x^3 + x^4 + 12y + 4y^2 + 12z + 4yz + 4z^2$. *This function is of class* C^2 *on* $\mathbb{R}^3$.

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y,z) = 0 = -8 + 16x - 12x^2 + 4x^3 \iff x = 1. \\ \left.\begin{array}{l} \dfrac{\partial f}{\partial y}(x,y,z) = 0 = 12 + 8y + 4z \\ \dfrac{\partial f}{\partial z}(x,y,z) = 0 = 12 + 8z + 4y \end{array}\right\} \iff y = z \iff y = z = -1. \end{cases}.$$

The only critical point is therefore $(1,-1,-1)$.
We have $\dfrac{\partial^2 f}{(\partial x)^2}(x,y,z) = 12x^2 - 24x + 16$, $\dfrac{\partial^2 f}{(\partial y)^2}(x,y,z) = 8$, $\dfrac{\partial^2 f}{(\partial z)^2}(x,y,z) = 8$.
$\dfrac{\partial^2 f}{\partial x \partial y}(x,y,z) = \dfrac{\partial^2 f}{\partial x \partial z}(x,y,z) = 0$, $\dfrac{\partial^2 f}{\partial z \partial y}(x,y,z) = 4$.
So

$$H(x,y,z) = 4x^2 + 8y^2 + 8(\frac{y+z}{2})^2 + 6z^2 \geq 0.$$

Therefore, $(1,-1,-1)$ *is a local maximum.*

2.7.3 Constrained extrema

Theorem 2.7.3. *(Lagrange's Theorem) Let $U \subset \mathbb{R}^p$ be an open set, $1 \leq n < p$, $a \in U$ and let $f, g_1, g_2, ..., g_n : U \to \mathbb{R}$ be a functions of class C^1, such that:*

$$rang\begin{pmatrix} (\nabla g_1(a) \\ . \\ . \\ . \\ (\nabla g_n(a) \end{pmatrix} = n.$$

So, for the restriction of the function f to the set $\{x \in U,$ such that $g_1(x) = ... = g_n(x) = 0\}$ admit a local extremum at a, there must exist n scalars $\lambda_1, \lambda_2, ..., \lambda_n$, such that:

$$\nabla\left(f + \sum_{k=1}^{n} \lambda_k g_k\right)(a) = 0.$$

By definition, the scalars $\lambda_1, \lambda_2, ..., \lambda_n$ are called Lagrange multipliers and

$$L : f + \sum_{k=1}^{n} \lambda_k g_k : U \to \mathbb{R},$$

is called Lagrange function.

Remark 2.12. *Since the condition is necessary but not sufficient, the existence of the scalars $\lambda_1, \lambda_2, ..., \lambda_n$ does not guarantee the existence of the local extremum at a.*

Remark 2.13. *If $n = 1$, then*

$$rang(\nabla g_1(a)) = 1 \Leftrightarrow \nabla g_1(a) \neq 0.$$

2.8 Implicit function theorem

Here, we seek the condition under which the equation $f(x) = 0$ can be locally solved as $y = \varphi(x)$, where φ is a function defined in a neighborhood of a given point.

We denote by f a function of class C^n of an opent U of $\subset \mathbb{R}^p$, with values in $\mathbb{R}$.

The following theorem allows us to determine the derivative of the desired function even if we cannot explicitly state it.

Theorem 2.8.1. *Let $a = (a_1, a_2, ..., a_p)$ be a point of U, such that*

$$f(a) = 0 \text{ and } \frac{\partial f}{\partial x_p}(a) \neq 0.$$

Then, there exists a unique continuous function $\varphi : B((a_1, a_2, ..., a_{p-1}), \delta) \to \mathbb{R}$ verifying:
1. $\varphi(a_1, a_2, ..., a_{p-1}) = a_p$.
2. For all $(x_1, x_2, ..., x_{p-1}) \in B$, we have $f(x_1, x_2, ..., x_{p-1}, \varphi(x_1, x_2, ..., x_{p-1})) = 0$.
Moreover φ is of class C^n on B.

Remark 2.14. *1. Around the point (x_0, y_0), we can therefore solve locally the equation $f(x, y) = 0$ in the form $y = \varphi(x)$.*
2. We will obviously take I and J to be small enough so that the rectangle $I \times J$ remains included in the open set U

Proof. (**Proof of theorem 2.8.1**)
Without loss of generality, assume that $p = 2$.
a) Existence of the solution $y = \varphi(x)$: For the purposes of the proof, assume that $\frac{\partial f}{\partial y}(x_0, y_0) > 0$, and case $\frac{\partial f}{\partial y}(x_0, y_0) < 0$ is treated in a similar way.
Since $\frac{\partial f}{\partial y}(x, y)$ is a continuous function that is non-zero at the point (x_0, y_0), we can find a rectangle $[x_0 - h, x_0 + h] \times [y_0 - k, y_0 + k]$ on which $\frac{\partial f}{\partial y}(x_0, y_0) > 0$.
Let $g : [y_0 - k, y_0 + k] \to \mathbb{R}$ be a function defined by $g(y) = f(x, y)$, for all x fixed in $[x_0 - h, x_0 + h]$.
So $\dot{g}(y) = \frac{\partial f}{\partial y}(x, y) > 0 \Rightarrow g$ is increasing on $[y_0 - k, y_0 + k]$.
In particular, the function $y \to g(y) = f(x_0, y)$ is strictly increasing on $[y_0 - k, y_0 + k]$.
And since $g(y_0) = f(x_0, y_0) = 0$, we have:

$$f(x_0, y_0 - k) = g(y_0 - k) < 0 \text{ and } f(x_0, y_0 + k) = g(y_0 + k) > 0. \tag{2.72}$$

By continuity, there therefore exists a small segment $]x_0 - a, x_0 + a[\subset]x_0 - h, x_0 + h[$, such that for all $x \in]x_0 - a, x_0 + a[$, we have

$$f(x, y_0 - k) < 0 \text{ and } f(x_0, y_0 + k) > 0. \tag{2.73}$$

And since f is continuous, then there exists one and only one point $y \in]y_0 - k, y_0 + k[$, such that:

$$f(x, y) = 0, \forall x \in]x_0 - a, x_0 + a[\,. \tag{2.74}$$

We have therefore defined an application φ from $I =]x_0 - a, x_0 + a[$ to $J =]y_0 - k, y_0 + k[$, such that: $f(x, \varphi(x)) = 0$.

b) Continuity of the function φ: It results from this construction.

Let's repeat the same reasoning, replacing k with ε (ε smaller), but ε arbitrary. So

$$\forall \varepsilon > 0, \exists a > 0, \text{ such that } |x - x_0| < a \Longrightarrow \left|\varphi(x) - \varphi(x_0)\right| < \varepsilon.$$

Consequently φ is continuous.

c) Differentiability of the function φ. Let $(x, \varphi(x))$ and $(x + t, \varphi(x + t))$ two points of U (t small enough so that $(x + t, \varphi(x + t)) \subset U$).

The Mean Value Theorem tells us that there exists a point p on segment $(x, \varphi(x)), (x + t, \varphi(x + t))$, such that:

$$\begin{aligned} 0 &= f((x + t, \varphi(x + t)) - f(x, \varphi(x)) \\ &= ((x + t) - x)\frac{\partial f}{\partial x}(p) + (\varphi(x + t) - \varphi(x))\frac{\partial f}{\partial y}(p) \\ &= t\frac{\partial f}{\partial x}(p) + (\varphi(x + t) - \varphi(x))\frac{\partial f}{\partial y}(p). \end{aligned}$$

Since $\dfrac{\partial f}{\partial y}(p) \neq 0$, we can write:

$$\frac{\varphi(x + t) - \varphi(x)}{t} = \frac{-\dfrac{\partial f}{\partial x}(p)}{\dfrac{\partial f}{\partial y}(p)}.$$

We then have:

$$\dot{\varphi}(x) = \lim_{t \to 0} \frac{\varphi(x+t) - \varphi(x)}{t} = \frac{-\frac{\partial f}{\partial x}(x, \varphi(x))}{\frac{\partial f}{\partial y}(x, \varphi(x))} \text{ (because } p \underset{t \to 0}{\to} (x, \varphi(x))).$$

And since f is of class C^1, then $\dot{\varphi}$ is continuous. In other words, φ is of class C^1 on U. □

2.8.1 Case of dimension $p = 2$

Let $(a, b) \in U$, where $b = \varphi(a)$. In the neighborhood of a, the function φ verifies $f(x, \varphi(x)) = 0$. Then the theorem of differentiation of composite functions gives us:

$$\begin{pmatrix} \frac{\partial f}{\partial x}(x, \varphi(x)) & \frac{\partial f}{\partial y}(x, \varphi(x)) \end{pmatrix} \begin{pmatrix} 1 \\ \dot{\varphi}(x) \end{pmatrix} = 0.$$

Which implies that:

$$\frac{\partial f}{\partial x}(x, \varphi(x)) + \dot{\varphi}(x) \frac{\partial f}{\partial y}(a, \varphi(a)) = 0.$$

So:

$$\dot{\varphi}(x) = \frac{-\frac{\partial f}{\partial x}(x, \varphi(x))}{\frac{\partial f}{\partial y}(a, \varphi(a))}.$$

Assume that a is a critical point of φ. So $\dot{\varphi}(a) = 0$, we then have:

$$\ddot{\varphi}(a) = \frac{-\frac{\partial^2 f}{\partial x^2}(a, \varphi(a))}{\frac{\partial f}{\partial y}(a, \varphi(a))}.$$

2.8.2 Case of dimension $p = 3$

Let $(a, b, c) \in U$, where $c = \varphi(a, b)$. in the neighborhood of (a, b), the function φ verifies $f(x, y, \varphi(x, y)) = 0$. Then the theorem of differentiation of composite functions gives us:

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} \end{pmatrix} = 0.$$

Which implies that:

$$\frac{\partial \varphi}{\partial x}(x,y) = \frac{-\frac{\partial f}{\partial x}(x,y,\varphi(x,y))}{\frac{\partial f}{\partial z}(x,y,\varphi(x,y))}$$

and

$$\frac{\partial \varphi}{\partial y}(x,y) = \frac{-\frac{\partial f}{\partial y}(x,y,\varphi(x,y))}{\frac{\partial f}{\partial z}(x,y,\varphi(x,y))}.$$

Furthermore, if (a,b) is a critical point of φ, we then have:

$$\frac{\partial^2 \varphi}{(\partial x)^2}(a,b) = \frac{-\frac{\partial^2 f}{\partial x^2}(a,b,\varphi(a,b))}{\frac{\partial f}{\partial z}(a,b,\varphi(a,b))},$$

$$\frac{\partial^2 \varphi}{(\partial y)^2}(a,b) = \frac{-\frac{\partial^2 f}{\partial y^2}(a,b,\varphi(a,b))}{\frac{\partial f}{\partial z}(a,b,\varphi(a,b))}$$

and

$$\frac{\partial^2 \varphi}{(\partial x \partial y)}(a,b) = \frac{-\frac{\partial^2 f}{\partial x \partial y}(a,b,\varphi(a,b))}{\frac{\partial f}{\partial z}(a,b,\varphi(a,b))}.$$

2.9 Exercises about chapter 2

Exercise 2.9.1. *Determine, whenever they exist, the following limits:*

1. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$, 2. $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{\sqrt{x^2+y^2}}$, 3. $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x)\sin(y)}{\tan\left(\sqrt{x^2+y^2}\right)}$,

4. $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin\left(x^2\right)\sin(y)}{x^2+\sinh^2(y^2)}$, 5. $\lim\limits_{(x,y)\to(0,0)} \left(x^2+y^2\right)^x$, 6. $\lim\limits_{(x,y)\to(0,0)} \dfrac{\ln(1+|xy|^{\alpha})}{x^2+y^2}$, $\alpha > 1$.

Exercise 2.9.2. *Studying the continuity of functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ defined by:*

$$1.f(x,y) = \begin{cases} y + \dfrac{1}{y}\arctan(x^2 y), & \text{if } y \neq 0 \\ 0, & \text{if } y=0 \end{cases}$$

$$2.g(x,y) = \begin{cases} x\exp(\arctan(\dfrac{y}{x})), & \text{if } x \neq 0 \\ 0, & \text{if } x=0 \end{cases}$$

Exercise 2.9.3. *Let $h : \mathbb{R} \to \mathbb{R}$ be a function of class C^1 and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by:*

$$f(x,y) = \begin{cases} \dfrac{h(x)-h(y)}{x-y}, & \text{if } x \neq y \\ \dot{h}(x), & \text{if } x=y, \end{cases}$$

Show that f is continuous on $\mathbb{R}^2$.

Exercise 2.9.4. *Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ and γ five constants positive and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by :*

$$f(x,y) = \begin{cases} \dfrac{|x|^{\alpha_1}|y|^{\alpha_2}}{\left(|x|^{\beta_1}+|y|^{\beta_2}\right)^{\gamma}}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

Show that f is continuous at $(0,0)$ if and only if $\dfrac{\alpha_1}{\beta_1} + \dfrac{\alpha_2}{\beta_2} > \gamma$.

Exercise 2.9.5. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by:*

$$f(x,y) = \begin{cases} xy\ln(|x|+|y|), & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

Show that f is of classe C^1 on $\mathbb{R}^2$.

Exercise 2.9.6. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of class C^1.*

1. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(t) = f\left(\sin(t), \exp(t^2)\right)$. Show that g is class C^1 and calculate the first derivative of g as a function of the partial derivatives of f.

2. Define $h : \mathbb{R}^2 \to \mathbb{R}$ by $h(u, v) = f(uv, u^2 + v^2)$. Show that h is of class C^1 and express the first partial derivatives of h in terms of those of f.

Exercise 2.9.7. *1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by:*

$$f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

1. 1. Is f continuous on $\mathbb{R}^2$.

1. 2. Is f of class C^1 on $\mathbb{R}^2$.

1. 3. Show in two different ways the differentiability of f on $\mathbb{R}^2$.

1.4. Calculate $\dfrac{\partial^2 f}{\partial x \partial y}(0, 0)$ and $\dfrac{\partial^2 f}{\partial y \partial x}(0, 0)$. Conclude.

2. (Supplementary part) Same questions for the function $g : \mathbb{R}^2 \to \mathbb{R}$, which is defined by:

$$g(x, y) = \begin{cases} x^2 y \ln\left(x^2 + y^2\right), & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

Exercise 2.9.8. *1. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be a functions defined by:*

$$f(x, y) = \begin{cases} \dfrac{y^2 \sin(x^2)}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases} \qquad g(x, y) = \begin{cases} \dfrac{y^2 \sin(x)}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

Are the functions f and g differentiable at $(0, 0)$?.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be two function defined by :$f(x, y) = \cos(x) \exp(y)$.

2.1. Calculate the gradient of f at every point $(x, y) \in \mathbb{R}^2$.

2.2. Calculate the Hessian of f at the point $(0, 0)$.

2.3. Find, using two different methods, the second-order Taylor expansion of the function f in a neighborhood of the point $(0, 0)$.

3. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be a function defined by:

$$\varphi(x, y) = (x - \sin(\alpha y),\ y - \sin(\alpha x)),\ \alpha \in \left]-1, 1\right[.$$

Is the function φ a C^1-diffeomorphism of $\mathbb{R}^2$ on $\varphi(\mathbb{R}^2)$?

Exercise 2.9.9. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ and $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ with $\varphi(x, y) = (x - y, x + y)$. Suppose that g is of class C^2. Let $f = g \circ \varphi$.*

1. Show that φ determines a C^2-diffeomorphism.

2. Express the second partial derivatives of f in terms of that of g.

3. Show that if f is a solution of the PDE:

$$\frac{\partial^2 f}{(\partial x)^2}(x, y) - \frac{\partial^2 f}{(\partial y)^2}(x, y) = 4(x^2 - y^2), \tag{2.75}$$

then g is solution of $\frac{\partial^2 g}{\partial u \partial v}(x, y) - uv = 0$

4. Deduce the general solution of (2.75).

Exercise 2.9.10. *1. Show that the equation $x^2 + 2\exp(y) + \sin(xy) - 2 = 0$ defines in the neighborhood of the point 0 an implicit function $y = \varphi(x)$, such that $\varphi(0) = 0$. Show that φ has a local maximum at 0.*

2. Show that the equation $3x^2 + 6y^2 + z^5 - 2z^4 + 1 = 0$ defines in the neighborhood of the point $(0, 0)$ an implicit function $z = \varphi(x, y)$, such that $\varphi(0, 0) = 1$. Verify that $(0, 0)$ is a stationary point of φ. Study its nature.

Exercise 2.9.11. *Find the stationary points of the following functions and study their natures:*

1. $f(x, y) = x + y^2 - \sinh(x + y)$.

2. $g(x, y) = x + y - \sinh(x + y)$.

Exercise 2.9.12. *Find on E_i the extrema of the functions $f_i : E_i \subset \mathbb{R}^2 \to \mathbb{R}$, $(i = 1, 2, 3)$ defined by:*

1. $f_1(x, y) = x(1 + y) + \ln(\sqrt{2 + x^2 + y^2})$ and $E_1 = \overline{B((0, 0), 1)}$.

2. $f_2(x, y) = \frac{x + y}{1 + x^2 + y^2}$ and $E_2 = \overline{B((0, 0), 1)}$.

3. $f_3(x, y) = \frac{y^2}{x + y}$ and $E_3 = \left\{(x, y) \in \mathbb{R}^2 : 0 \le x^2 + y^2 \le 9,\ 0 \le x \le y\right\}$.

2.10 Solutions of exercices of chapter 2

Solution of exercise 2.9.1

Let's determine, whenever they exist, the limit of f, when $(x, y) \to 0_{\mathbb{R}^2}$

1. $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$.
We have $f(0,y) = -1 \neq 1 = f(x,0)$.

Therefore, f has no limit at the point $(0,0)$

2. $\left|f(x,y)\right| = \left|\dfrac{\sin(xy)}{\sqrt{x^2+y^2}}\right| \leq \dfrac{|xy|}{\sqrt{x^2+y^2}} \leq |x| \to 0$, when $(x,y) \to 0_{\mathbb{R}^2}$.
Therefore $\lim_{(x,y)\to 0_{\mathbb{R}^2}} f(x,y) = 0$.

3. $f(x,y) = \dfrac{\sin(x)\sin(y)}{\tan\left(\sqrt{x^2+y^2}\right)} = \left(\dfrac{\dfrac{\sin(x)}{x} \times \dfrac{\sin(y)}{y}}{\dfrac{\tan\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}}\right) \times \left(\dfrac{xy}{\sqrt{x^2+y^2}}\right) \to 0$, when $(x,y) \to 0_{\mathbb{R}^2}$.

Therefore $\lim_{(x,y)\to 0_{\mathbb{R}^2}} f(x,y) = 0$.

4. $\left|f(x,y)\right| = \left|\dfrac{\sin\left(x^2\right)\sin(y)}{x^2 + \sinh^2(y^2)}\right| \leq \left|\dfrac{x^2\left|y\right|}{x^2}\right| \leq \left|y\right| \to 0$, when $(x,y) \to 0_{\mathbb{R}^2}$,
Therefore $\lim_{(x,y)\to 0_{\mathbb{R}^2}} f(x,y) = 0$.

5. Let $(x,y) = (r\cos\theta, r\sin\theta)$.

We then have

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \left(x^2+y^2\right)^x &= \lim_{(x,y)\to(0,0)} \exp\left(x\ln\left(x^2+y^2\right)\right) \\ &= \exp(2\cos(\theta)\lim_{r\to 0} r\ln(r)) = 1. \end{aligned}$$

6. Since $\alpha > 1$ and

$$0 \leq \left|f(x,y)\right| = \left|\frac{\ln(1+\left|xy\right|^\alpha)}{x^2+y^2}\right| = \left|\frac{\ln(1+\left|xy\right|^\alpha)}{\left|xy\right|^\alpha}\right| \times \frac{\left|xy\right|^\alpha}{x^2+y^2} \leq \frac{1}{2}\left|xy\right|^{\alpha-1} \to 0,$$

then $\lim_{(x,y)\to 0_{\mathbb{R}^2}} f(x,y) = 0$.

Solution of exercise 2.9.2

Let's study the continuity of the following functions:

1.$f(x,y)= y + \dfrac{1}{y}\arctan(x^2 y)$, if $y \neq 0$ and $f(x,0) = 0$.

1.1. For all $(a,b) \in \mathbb{R} \times \mathbb{R}^*$, f is continuous.

1.2. For all $a \in \mathbb{R}^*$, f is discontinuous at the points $(a,0)$, because

$$\lim_{y\to 0} f(a,y) = a^2 \neq 0 = f(a,0).$$

1.3. At the point $(0, 0)$, f is continuous, because

$$\lim_{(x,y)\to 0_{\mathbb{R}^2}} f(a, y) = 0 = f(0, 0).$$

2.$g(x, y)$=$x\exp(\arctan(\frac{y}{x}))$, if $x \neq 0$ and $g(0, y) = 0$.
2.1. For all $(a, b) \in \mathbb{R}^* \times \mathbb{R}$, g is continuous.
2.2. At the points $(0, b)$ $(b \in \mathbb{R})$, g is continuous, because

$$\left|g(x, y) - g(0, b)\right| \leq |x| \exp\left(\frac{\pi}{2}\right) \to 0,$$

when $(x, y) \to (0, b)$. Therefore $\lim_{(x,y)\to(0,b)} g(x, y) = 0$.

Solution of exercise 2.9.3

1. For all $(x, y) \in \mathbb{R}^2 (x \neq y)$, f is continuous.
2. At the points $(a, a) \in \mathbb{R}^2$ $(a \in \mathbb{R})$. Since h is of class C^1, according to the mean value theorem, at each element $(x, y) \in \mathbb{R}^2 (x \neq y)$, we can associate $\theta_{x,y} \in]0, 1[$, such that

$$h(x) - h(y) = \dot{h}(y + \theta_{x,y}(x - y))(x - y);$$

which allows us to write:

$$f(x, y) - f(a, a) = \begin{cases} \dot{h}(y + \theta_{x,y}x(x - y)) - \dot{h}(a), \text{ if } x \neq y \\ 0, \text{ if } x = y. \end{cases}$$

Consequently $\lim_{(x,y)\to(a,a)} f(x, y) = f(a, a)$.

Solution of exercise 2.9.4

Let α_1, α_2, β_1, β_2 and γ be five positive constants and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by:

$$f(x, y) = \begin{cases} \dfrac{|x|^{\alpha_1} \left|y\right|^{\alpha_2}}{\left(|x|^{\beta_1} + \left|y\right|^{\beta_2}\right)^{\gamma}}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

Let's show that f is continuous at $(0, 0)$ if and only if $\dfrac{\alpha_1}{\beta_1} + \dfrac{\alpha_2}{\beta_2} > \gamma$.

1. Suppose that f is continuous at $(0, 0)$.

Since $\lim_{(x,y)\to(0,0)} f(x,y) = 0$, we must have that:

$$f\left(\frac{1}{\sqrt[\beta_1]{n}}, \frac{1}{\sqrt[\beta_2]{n}}\right) = \frac{1}{2^\gamma}\left(\frac{1}{n}\right)^{\frac{\alpha_1}{\beta_1}+\frac{\alpha_2}{\beta_2}-\gamma},$$

quantity tends towards 0, when $n \to +\infty$, if $\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} > \gamma$.

2. Conversely. Suppose that $\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} > \gamma$.

We have for all $(x, y) \neq (0, 0)$:

$$\begin{aligned}
\left|f(x,y) - f(0,0)\right| &= \left|f(x,y)\right| = \frac{\left(|x|^{\beta_1}\right)^{\frac{\alpha_1}{\beta_1}}\left(|y|^{\beta_2}\right)^{\frac{\alpha_2}{\beta_2}}}{\left(|x|^{\beta_1} + |y|^{\beta_2}\right)^\gamma} \\
&\leq \frac{\left(|x|^{\beta_1} + |y|^{\beta_2}\right)^{\frac{\alpha_1}{\beta_1}}\left(|x|^{\beta_1} + |y|^{\beta_2}\right)^{\frac{\alpha_2}{\beta_2}}}{\left(|x|^{\beta_1} + |y|^{\beta_2}\right)^\gamma} \\
&= \left(|x|^{\beta_1} + |y|^{\beta_2}\right)^{\frac{\alpha_1}{\beta_1}+\frac{\alpha_2}{\beta_2}-\gamma} \to 0, \text{ when } (x,y) \to (0,0).
\end{aligned}$$

Solution of exercise 2.9.5

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by:

$$f(x,y) = \begin{cases} xy\ln(|x| + |y|), & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

Let' show that f is of class C^1 on $\mathbb{R}^2$.

Recall that the derivative of the absolute value function $|t|$, $t \neq 0$ is $\frac{|t|}{t}$. So:

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} y\ln(|x| + |y|) + \dfrac{|x|\,y}{|x| + |y|}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

and

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} x\ln(|x| + |y|) + \dfrac{x\,|y|}{|x| + |y|}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}.$$

The two functions $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continueous on $\mathbb{R}^2 - \{(0,0)\}$.
Furthermore, for all $(x,y) \in \mathbb{R}^2 - \{(0,0)\}$, we have:

$$\begin{aligned}
\left|\frac{\partial f}{\partial x}(x,y)\right| &\leq \left(|x| + |y|\right)\left|\ln(|x| + |y|)\right| + |y| \to 0 = \frac{\partial f}{\partial x}(0,0), \text{when } (x,y) \to (0,0) \\
\left|\frac{\partial f}{\partial y}(x,y)\right| &\leq \left(|x| + |y|\right)\left|\ln(|x| + |y|)\right| + |x| \to 0 = \frac{\partial f}{\partial y}(0,0), \text{when } (x,y) \to (0,0).
\end{aligned}$$

So $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continueous at $(0,0)$. Consequently f is of class C^1 on $\mathbb{R}^2$.

Solution of exercise 2.9.6

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of class C^1.

1. We define $g : \mathbb{R} \to \mathbb{R}$ by $g(t) = f(\sin(t), \exp(t^2))$.

1.1. Let's demonstrate that g is of class C^1:

Let $h : \mathbb{R} \to \mathbb{R}^2$ be a function defined by:

$$h(t) = \left(\sin(t), \exp(t^2)\right).$$

We then have $g = f \circ h$.

Since f and h are of class C^1, we deduce that g is also of class C^1.

1.2. Let us calculate the first derivative of g based on the partial derivatives of f.

Applying the formula for the derivative of a composite function, we can write:

$$\begin{aligned} \dot{g}(t) &= \begin{pmatrix} \frac{\partial f}{\partial x}\left(\sin(t),\exp(t^2)\right) & \frac{\partial f}{\partial y}\left(\sin(t), \exp(t^2)\right) \end{pmatrix} \begin{pmatrix} \cos t \\ 2t\exp(t^2) \end{pmatrix} \\ &= \cos(t)\frac{\partial f}{\partial x}\left(\sin(t),\exp(t^2)\right) + 2t\exp(t^2)\frac{\partial f}{\partial y}\left(\sin(t), \exp(t^2)\right). \end{aligned}$$

2. Now, we define $h : \mathbb{R}^2 \to \mathbb{R}$ by $h(u, v) = f(uv, u^2 + v^2)$.

2.1. Let's demonstrate that h is of class C^1.

Let $k : \mathbb{R}^2 \to \mathbb{R}^2$ be a function defined by:

$$h(u, v) = \left(uv, u^2 + v^2\right).$$

We then have $h = f \circ k$.

Since f and k are of class C^1, we deduce that h is also of class C^1.

2.2. Let's express the first partial derivatives of h in terms of those of f.

Applying the formula for the derivative of a composite function, we can write:

$$\begin{pmatrix} \frac{\partial h}{\partial u}(u, v) & \frac{\partial h}{\partial v}(u, v) \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}\left(uv, u^2 + v^2\right) & \frac{\partial f}{\partial y}\left(uv, u^2 + v^2\right) \end{pmatrix} \begin{pmatrix} v & u \\ 2u & 2v \end{pmatrix},$$

which implies that:

$$\begin{cases} \frac{\partial h}{\partial u}(u, v) = v\frac{\partial f}{\partial x}\left(uv, u^2 + v^2\right) + 2u\frac{\partial f}{\partial y}\left(uv, u^2 + v^2\right) \\ \frac{\partial h}{\partial v}(u, v) = u\frac{\partial f}{\partial x}\left(uv, u^2 + v^2\right) + 2v\frac{\partial f}{\partial y}\left(uv, u^2 + v^2\right) \end{cases}$$

Solution of exercise 2.9.7

1. Let $f:\mathbb{R}^2\to\mathbb{R}$ be a function defined by:

$$f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0)\end{cases}$$

1. 1. Let's show that f is continuous on $\mathbb{R}^2$.

* For all $(x,y)\in\mathbb{R}^2-\{(0,0)\}$, f is continuous.

* At the point $(0,0)$, f is also continuous, because $\left|f(x,y)\right|\leq\frac{1}{2}\left|x^2-y^2\right|\to 0$, when $(x,y)\to(0,0)$.

So $\lim_{(x,y)\to(0,0)} f(x,y)=0=f(0,0)$.

1. 2. Let's show that f is of class C^1 on $\mathbb{R}^2$.

We have:

$$\frac{\partial f}{\partial x}(x,y)=\begin{cases} \dfrac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0)\end{cases}$$

and

$$\frac{\partial f}{\partial y}(x,y)=\begin{cases} \dfrac{-y^4x-4y^2x^3+x^5}{(x^2+y^2)^2}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if}(x,y)=(0,0)\end{cases}$$

The two functions $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are continueous on $\mathbb{R}^2-\{(0,0)\}$.

Furthermore, for all $(x,y)\in\mathbb{R}^2-\{(0,0)\}$, we have

$$\begin{aligned}\left|\frac{\partial f}{\partial x}(x,y)\right| &\leq 6\left|y\right|\to 0=\frac{\partial f}{\partial x}(0,0)\text{, as }(x,y)\to(0,0)\\ \left|\frac{\partial f}{\partial y}(x,y)\right| &\leq 6\left|x\right|\to 0=\frac{\partial f}{\partial y}(0,0)\text{, as }(x,y)\to(0,0)\text{.}\end{aligned}$$

So $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are continuous at $(0,0)$.

Consequently f is of class C^1 on $\mathbb{R}^2$.

1. 3. Let us show, using two methods, the differentiability of f on $\mathbb{R}^2$.

*The first method. Since f is of class C^1 on $\mathbb{R}^2$, we deduce that f is differentiable on $\mathbb{R}^2$.

* The second method:. For all $(x,y)\in\mathbb{R}^2-\{(0,0)\}$, f is differentiable.
At the point $(0,0)$, we have

$$L(x,y)=x\frac{\partial f}{\partial x}(0,0)+y\frac{\partial f}{\partial x}(0,0)=0.$$

So

$$\begin{aligned}\frac{\left|f(x,y)-g(0,0)-L(x,y)\right|}{\left\|(x,y)\right\|} &= \left|xy\frac{x^2-y^2}{(x^2+y^2)^{\frac{3}{2}}}\right| \\ &\leq |x|\longrightarrow 0,\text{when } x\longrightarrow 0.\end{aligned}$$

Then, f is est differentiable at $(0,0)$.
1.4. We have:

$$\frac{\frac{\partial f}{\partial x}f(t,0)-\frac{\partial f}{\partial x}f(0,0)}{t}=1 \text{ and } \frac{\frac{\partial f}{\partial y}f(0,t)-\frac{\partial f}{\partial y}f(0,0)}{t}=-1,$$

which implies that:

$$\begin{cases}\dfrac{\partial^2 f}{\partial y\partial x}(0,0)=\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)(0,0)=\lim_{t\to 0}\dfrac{\frac{\partial f}{\partial x}f(0,t)-\frac{\partial f}{\partial x}f(0,0)}{t}=1\\ \dfrac{\partial^2 f}{\partial x\partial y}(0,0)=\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)(0,0)=\lim_{t\to 0}\dfrac{\frac{\partial f}{\partial y}f(t,0)-\frac{\partial f}{\partial y}f(0,0)}{t}=-1\end{cases}$$

Since $\frac{\partial^2 f}{\partial x\partial y}(0,0)\neq\frac{\partial^2 f}{\partial y\partial x}(0,0)$, according to the Schwarz's theoerm, we deduce that f does not of class C^2 on $\mathbb{R}^2$.

Solution of exercise 2.9.8

1. Let $f,g:\mathbb{R}^2\longrightarrow\mathbb{R}$ be a functions defined by:

$$f(x,y)=\begin{cases}\dfrac{y^2\sin(x^2)}{x^2+y^2}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0)\end{cases}\qquad g(x,y)=\begin{cases}\dfrac{y^2\sin(x)}{x^2+y^2}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0)\end{cases}$$

Let's show that the functions f and g are differentiabes at $(0,0)$?.

1. 1. $$\frac{f(t,0)-f(0,0)}{t}=\frac{f(0,t)-f(0,0)}{t}=0\Rightarrow\begin{cases}\dfrac{\partial f}{\partial x}(0,0)=\lim_{t\to 0}\dfrac{f(t,0)-f(0,0)}{t}=0\\ \dfrac{\partial f}{\partial y}(0,0)=\lim_{t\to 0}\dfrac{f(0,t)-f(0,0)}{t}=0\end{cases}$$

$L_1(x,y) = \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y = 0.$

So

$$\frac{\left|f(x,y) - f(0,0) - L_1 x, y)\right|}{\left\|(x,y)\right\|} \overset{N(y,0)}{\simeq} \left|\frac{y^2x^2}{(x^2+y^2)^{\frac{3}{2}}}\right| \leq \sqrt{x^2+y^2} \to 0, \text{ as } (x,y) \to (0,0).$$

Then, f is differentiable at $(0,0)$.

1. 2. On the other hand: $\begin{cases} \frac{\partial g}{\partial x}(0,0) = \lim_{t\to 0} \frac{g(t,0)-g(0,0)}{t} = 0 \\ \frac{\partial g}{\partial y}(0,0) = \lim_{t\to 0} \frac{g(0,t)-g(0,0)}{t} = 0 \end{cases}$

$L_2(x,y) = \frac{\partial g}{\partial x}(0,0)x + \frac{\partial g}{\partial y}(0,0)y = 0.$

So

$$\frac{\left|g(x,y) - g(0,0) - L_2(x,y)\right|}{\left\|(x,y)\right\|} = \frac{y^2\sin(x)}{(x^2+y^2)^{\frac{3}{2}}} = G(x,y).$$

We have $G(x,x) = \frac{x^2\sin(x)}{(2x^2)^{\frac{3}{2}}} \overset{N(0)}{\simeq} \frac{x^3}{(2x^2)^{\frac{3}{2}}} = \frac{1}{(2)^{\frac{3}{2}}} \nrightarrow 0.$

Therefore, g is not differentiable at $(0,0)$.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by: :

$$f(x,y) = \cos(x)\exp(y).$$

2.1. The gradient of f at all points $(x,y) \in \mathbb{R}^2$ is given by:

$$\nabla_f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) & \frac{\partial f}{\partial x}(x,y) \end{pmatrix} = \begin{pmatrix} -\sin(x)\exp(y) & \cos(x)\exp(y) \end{pmatrix}.$$

2.2. The Hessian of f at the point $(0,0)$ is given by:

$$H_f(0,0) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(0,0) & \frac{\partial^2 f}{\partial x \partial y}(0,0) \\ \frac{\partial^2 f}{\partial y \partial x}(0,0) & \frac{\partial^2 f}{\partial y^2}(0,0) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

2.3. Let us find, using two different methods, the second-order Taylor expansion of f in a neighborhood of $(0,0)$.

<u>The first method</u>: It is clear that f is of class C^2 on $\mathbb{R}^2$.

Since

$$\frac{\partial f}{\partial x}(0,0) = \frac{\partial^2 f}{\partial x \partial y}(0,0) = 0, \frac{\partial f}{\partial y}(0,0) = 1, \frac{\partial^2 f}{\partial x^2}(0,0) = -1 \text{ and } \frac{\partial^2 f}{\partial y^2}(0,0) = 1,$$

we then have

$$f(x,y) = \cos(x)\exp y = 1 + y - \frac{x^2}{2} + \frac{y^2}{2} + (x^2+y^2)\epsilon(x,y),$$

where $\epsilon(x,y) \to 0$, as $(x,y) \to (0,o)$.

The second method: We have:

$$f(x,y) = \cos(x)\exp y = \left(1 - \frac{x^2}{2} + x^2\epsilon(x)\right)\left(1 + y + \frac{y^2}{2} + y^2\epsilon(y)\right).$$

By creating the product, and retaining only the terms in x, y, xy, x^2, y^2 and encompassing the rest of the form $(x^2+y^2)\epsilon(x,y)$, we get the same result.

3. We have

$$\det J_\varphi(x,y) = \begin{vmatrix} 1 & -\alpha\cos(\alpha y) \\ -\alpha\cos(\alpha x y) & 1 \end{vmatrix} = 1-\alpha^2\cos(\alpha x)\cos(\alpha y) \geq 1-\alpha^2 > 0.$$

Therefore J_φ is invertible at the point (x,y).

On the other hand, the function φ is of class C^∞ (obvious).

The function φ is also injective.

Indeed, let (x_1,y_1) and $(x_2,y_2) \in \mathbb{R}^2$, such that $\varphi(x_1,y_1) = \varphi(x_2,y_2)$. ie;

$$\begin{cases} x_1 - \sin(\alpha y_1) = & x_2 - \sin(\alpha y_2) \\ y_1 - \sin(\alpha x_1) = & y_2 - \sin(\alpha x_2) \end{cases}$$

So

$$\begin{aligned} |x_1 - x_2| &= \left|\sin(\alpha y_1) - \sin(\alpha y_2)\right| \\ &\leq \alpha\left|y_1 - y_2\right| \\ &= \alpha\left|\sin(\alpha x_1) - \sin(\alpha x_2)\right| \\ &\leq \alpha^2\left|x_1 - x_2\right|. \end{aligned}$$

Since $1-\alpha^2 > 0$, we then have $x_1 = x_2$, and thus $y_1 = y_2$.

According to the global inversion theorem, φ is a C^1-diffeomorphism from $\mathbb{R}^2$ to $\varphi(\mathbb{R}^2)$.

Solution of exercise 2.9.9

1. φ is injective and it is of class C^2 on $\mathbb{R}^2$ (obvious).

Let $u = x - y$ and $v = x + y$.

So

$$\det j_\varphi(x,y) = \det\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = 2 \neq 0.$$

Therefore j_φ is invertible.

According to the global inversion theorem, φ is a $C^{+\infty}$ diffeomorphism .

2. Let $f = g \circ \varphi$. We find

$$\begin{pmatrix} \frac{\partial f}{\partial x}(x,y) & \frac{\partial f}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} \frac{\partial g}{\partial u}(u,v) & \frac{\partial g}{\partial v}(u,v) \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

That is

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = \frac{\partial g}{\partial u}(u,v) + \frac{\partial g}{\partial v}(u,v) \\ \frac{\partial f}{\partial y}(x,y) = -\frac{\partial g}{\partial u}(u,v) + \frac{\partial g}{\partial v}(u,v) \end{cases}.$$

So

$$\begin{cases} \frac{\partial^2 f}{(\partial x)^2}(x,y) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial g}{\partial u}(u,v) + \frac{\partial g}{\partial v}(u,v)\right) \\ \quad = \frac{\partial^2 g}{(\partial u)^2}(u,v) + 2\frac{\partial^2 g}{\partial u \partial v}(u,v) + \frac{\partial^2 g}{(\partial v)^2}(u,v) \\ \frac{\partial^2 f}{(\partial y)^2}(x,y) = \left(-\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial g}{\partial u}(u,v) + \frac{\partial g}{\partial v}(u,v)\right) \\ \quad = \frac{\partial^2 g}{(\partial u)^2}(u,v) - 2\frac{\partial^2 g}{\partial u \partial v}(u,v) + \frac{\partial^2 g}{(\partial v)^2}(u,v) \end{cases}.$$

3. So, the equation: $\frac{\partial^2 f}{(\partial x)^2}(x,y) - \frac{\partial^2 f}{(\partial y)^2}(x,y) = 4(x^2 - y^2)$ becomes:

$$\frac{\partial^2 g}{\partial u \partial v}(u,v) = uv.$$

4. By integrating the previous equation we find

$$g(u,v) = \frac{1}{4}u^2v^2 + F(u) + H(v),$$

where F, H are of class C^2 on $\mathbb{R}$.

Consequently

$$f(x,y) = g(u,v) = \frac{1}{4}(x^2 - y^2)^2 + F(x-y) + H(x+y).$$

Solution of exercise 2.9.10

1.1. Let $f : R^2 \to R$ be a function defined by

$$f(x,y) = x^2 + 2\exp(y) + \sin(xy) - 2.$$

Then, for all $(x, y) \in R^2$, we have:

$$\frac{\partial f}{\partial y}(x, y) = 2\exp(y) + x\cos(xy).$$

Since $f(0, 0) = 0$ and $\frac{\partial f}{\partial y}(0, 0) = 2 \neq 0$, The implicit function theorem allows us to state that there exists a unique continuous function $\varphi :]-\sigma, \sigma[\to \mathbb{R}$, such that $\varphi(0) = 0$ and for all $x \in]-\sigma, \sigma[: f(x, \varphi(x)) = 0$, and moreover $\varphi \in C^\infty$.

1.2. For all $(x, y) \in \mathbb{R}^2$, we have :

$$\frac{\partial f}{\partial x}(x, y) = 2x + y\cos(xy).$$

So: $\dot{\varphi}(0) = -\dfrac{\frac{\partial f}{\partial x}(x, y)}{\frac{\partial f}{\partial y}(x, y)} = 0$. i.e; 0 is a critical point of φ.

Nature of the critical point 0. For all $(x, y) \in \mathbb{R}^2$:

$$\ddot{\varphi}(0) = -\frac{\frac{\partial^2 f}{(\partial x)^2}(0, \varphi(0))}{\frac{\partial f}{\partial y}(0, \varphi(0))} = -1 < 0,$$

so the function φ has a local maximum at 0.

2.1. Let $f : R^3 \to R$ be a function defined by:

$$f(x, y, z) = 3x^2 + 6y^2 + z^5 - 2z^4 + 1.$$

For all $(x, y, z) \in R^3$: $\frac{\partial f}{\partial z}(x, y, z) = 5z^4 - 8z^3$.

Since $f(0, 0, 1) = 0$ and $\frac{\partial f}{\partial z}(0, 0, 1) = -3 \neq 0$, The implicit function theorem allows us to state that there exists a unique continuous function $\varphi : B((0, 0), \sigma) \to R$, such that $\varphi(0, 0) = 1$ and for all $(x, y) \in B((0, 0), \sigma)$: $f(x, y, \varphi(x, y)) = 0$, and moreover $\varphi \in C^\infty$.

2.2. For all $(x, y, z) \in \mathbb{R}^3$, we have: $\frac{\partial f}{\partial x}(x, y, z) = 6x$ and $\frac{\partial f}{\partial y}(x, y, z) = 12y$. So

$$\frac{\partial \varphi}{\partial x}(0, 0) = \frac{\partial \varphi}{\partial y}(0, 0) = 0.$$

Therefore $(0,0)$ is a critical point of φ.

Nature of the point of $(0,0)$. For all $(x,y,z) \in \mathbb{R}^3$:
$\frac{\partial^2 f}{(\partial x)^2}(0,0,1) = 6$, $\frac{\partial^2 f}{\partial x \partial y}(0,0,1) = 0$ and $\frac{\partial^2 f}{(\partial y)^2}(0,0,1) = 12$, we then have $A = 2, B = 0$ and $C = 4$. So $B^2 - AC = -8 < 0$ and $A = 2 > 0$; which implies that the function φ has a local minimum at $(0,0)$.

Solution of exercise 2.9.11

1.1. Let $(a,b) \in \mathbb{R}^2$ a critical point of the function

$$f(x,y) = x + y^2 - \sinh(x+y).$$

We then have

$$\begin{cases} \frac{\partial f}{\partial x}(a,b) = 1 - \cosh(a+b) = 0 \\ \frac{\partial f}{\partial y}(a,b) = 2b - \cosh(a+b) = 0 \end{cases} \Rightarrow \begin{cases} a+b = 0 \\ 2b - 1 = 0 \end{cases} \Rightarrow (a,b) = (-\frac{1}{2}, \frac{1}{2}).$$

1.2 Nature of the point $(-\frac{1}{2}, \frac{1}{2})$.

We have

$$\begin{cases} \frac{\partial^2 f}{(\partial x)^2}(x,y) = -\sinh(x+y) \\ \frac{\partial^2 f}{(\partial y)^2}(x,y) = 2 - \sinh(x+y) \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) = -\sinh(x+y); \end{cases}$$

So

$$\begin{cases} A = \frac{\partial^2 f}{(\partial x)^2}(-\frac{1}{2}, \frac{1}{2}) = 0 \\ C = \frac{\partial^2 f}{(\partial y)^2}(-\frac{1}{2}, \frac{1}{2}) = 2 \\ B = \frac{\partial^2 f}{\partial x \partial y}(-\frac{1}{2}, \frac{1}{2}) = 0 \end{cases}$$

Since $B^2 - Ac = 0$, we cannot conclude.

Let $\delta > 0$. We then have:

$$f(-\frac{1}{2} + \delta, \frac{1}{2} - \delta) = -\frac{1}{4} + \delta^2 > -\frac{1}{4} = f(-\frac{1}{2}, \frac{1}{2})$$

and

$$\begin{aligned} f(-\frac{1}{2}+\delta,\frac{1}{2}) &= -\frac{1}{4}+\delta-\sinh\delta \\ &< -\frac{1}{4} = f(-\frac{1}{2},\frac{1}{2}) \quad (\text{because } \sinh\delta > \delta \end{aligned}$$

Therefore,

$$\exists x_1 = (-\frac{1}{2}+\delta, \frac{1}{2}-\delta) \in B\Big((-\frac{1}{2},\frac{1}{2}),\delta\Big)$$

and

$$\exists x_2 = (-\frac{1}{2}+\delta, \frac{1}{2}) \in B\Big((-\frac{1}{2},\frac{1}{2}),\delta\Big),$$

verifying

$$f(x_1) > f(-\frac{1}{2},\frac{1}{2}) \text{ and } f(x_2) < f(-\frac{1}{2},\frac{1}{2}).$$

So f does not admit a local extremum at $(-\frac{1}{2},\frac{1}{2})$.
2.1. Let $(a,b) \in \mathbb{R}^2$ a critical point of the function

$$g(x,y) = x + y - \sinh(x+y).$$

We then have:

$$\begin{cases} \frac{\partial g}{\partial x}(a,b) = 1 - \cosh(a+b) = 0 \\ \frac{\partial g}{\partial y}(a,b) = 1 - \cosh(a+b) = 0, \end{cases}$$

which implies that:

$$a + b = 0 \Rightarrow (a,b) = (-a,a).$$

2.2 Nature of the point $(-a,a)$.
We have:

$$\frac{\partial^2 g}{(\partial x)^2}(x,y) = \frac{\partial^2 g}{(\partial y)^2}(x,y) = \frac{\partial^2 g}{\partial x \partial y}(x,y) = -\sinh(x+y)$$

which implies that $\Rightarrow A = B = C = 0$. Since $B^2 - Ac = 0$, we can not conclude.
Let $\delta > 0$. We have:

$$g(a+\delta, -a) = \delta - \sinh\delta < 0 = g(-a,a)$$

and

$$g(a-\delta, -a) = -\delta + \sinh\delta > 0 = g(-a,a)$$

Therefore $\exists x_1 = (a+\delta, -a) \in B((-a,a),\delta)$ and $\exists x_2 = (a-\delta, -a) \in B((-a,a),\delta)$ verifying $g(x_1) < g(-a,a)$ and $g(x_2) > g(-a,a)$.
So g does not admit a local extremum at $(-a,a)$.

Solution of exercise 2.9.12

Let's find the extrema of the function $f_1 : E_1 \subset \mathbb{R}^2 \to \mathbb{R}$, $(i = 1,2,3)$ defined by $f_1(x,y) = x(1+y) + \ln(\sqrt{2+x^2+y^2})$ and $E_1 = \overline{B((0,0),1)}$.
We have

$$\begin{aligned} \overline{B((0,0),1)} &= \left\{(x,y) \in \mathbb{R}^2;\ x^2+y^2 \leq 1\right\} \\ &= \mathring{B}((0,0),1) \cup \partial B((0,0),1). \end{aligned}$$

The function f_1 is continuous on the compact $\overline{B((0,0),1)}$, therefore it attains its extrema. Let (a,b) denote one of these points.
1.1. On $\mathring{B}((0,0),1)$, we have:

$$\begin{cases} \dfrac{\partial f}{\partial x}(a,b) = 1 + b + \dfrac{a}{2+a^2+b^2} \\ \dfrac{\partial f}{\partial y}(a,b) = a + \dfrac{b}{2+a^2+b^2} \end{cases}.$$

Assume that $\nabla_f(a,b) = (0,0)$. So

$$\begin{cases} 1 + b + \dfrac{a}{2+a^2+b^2} = 0 \\ a + \dfrac{b}{2+a^2+b^2} = 0, \end{cases}.$$

which implies that:

$$a^2 = b(b+1) \tag{2.76}$$

and since $(a,b) \in \mathring{B}((0,0),1)$, we then have:

$$a^2 + b^2 < 1 \tag{2.77a}$$

From (2.76) and (2.77a), we find $2b^2 + b - 1 < 0$, contradiction.
So $\nabla_f(a,b) \neq (0,0)$, for all $(a,b) \in \mathring{B}((0,0),1)$, and therefore f does not admit extrema on $\mathring{B}((0,0),1)$
We are looking for the extrema on $\partial B((0,0),1)$.
We have

$$\partial B((0,0),1) = \left\{(x,y) \in \mathbb{R}^2;\ x^2+y^2-1 = 0 = g(x,y)\right\}$$

So

$$\begin{cases} \frac{\partial g}{\partial x}(a,b) = 2a \\ \frac{\partial g}{\partial y}(a,b) = 2b \end{cases}.$$

Let $(a,b) \in \partial B\left((0,0),1\right)$, we then have $a^2+b^2=1$.

If we suppose that $\nabla g(a,b) = (0,0)$, i.e; $a=b=0$, then $a^2+b^2=0$ contradiction with $a^2+b^2=1$. So $\nabla(g(a,b)) \neq (0,0)$, so $rang\nabla g(a,b) = 1$. According to the Lagrange's theorem, $\exists \lambda \in \mathbb{R}$, such that

$$\begin{cases} \nabla(f_1+\lambda g)(a,b) = (0,0) \\ g(a,b) = 0 \end{cases} \tag{2.78a}$$

On the other hand,

$$(a,b) \in \partial B\left((0,0),1\right),$$

we can write:

$$\begin{aligned} f_1(a,b) &= a(1+b) + \ln(\sqrt{2+a^2+b^2}) \\ &= a(1+b) + \frac{\ln 3}{2} \end{aligned} \tag{2.79}$$

From (2.78a) and (2.79), we get:

$$\begin{cases} (1+b)+2\lambda a = 0 \\ a+2\lambda b = 0 \\ a^2+b^2 = 1 \end{cases} \Rightarrow \begin{cases} a^2+b^2 = (1+b)b+b^2 \\ a^2+b^2 = 1 \end{cases} \Rightarrow 2b^2+b-1=0$$

So $b_1 = -1$ and $b_2 = \frac{1}{2}$.

For $b_1 = -1$, we get $a_1 = 0$, and for $b_2 = \frac{1}{2}$, we get $a_2 = \pm\frac{\sqrt{3}}{2}$.

We then have

$$f_1(0,-1) = \frac{\ln 3}{2} \text{ and } f_1(\pm\frac{\sqrt{3}}{2},\frac{1}{2}) = \pm\frac{3\sqrt{3}}{4} + \frac{\ln 3}{2}..$$

Consequently:

$$\max_{x\in\overline{B}} f_1(x,y) = \frac{3\sqrt{3}}{4} + \frac{\ln 3}{2} \text{ and } \min_{x\in\overline{B}} f_1(x,y) = -\frac{3\sqrt{3}}{4} + \frac{\ln 3}{2}.$$

Chapter 3

Multiple integrals

The multiple integral is a natural generalization of the single integral, which represents the integral of a function of one real variable. In multivariable calculus, this concept is extended to functions depending on two or more variables. In particular, double integrals and triple integrals allow the integration of functions defined over regions in two-dimensional or three-dimensional spaces. These integrals play an essential role in the analysis of quantities such as area, volume, mass, and other physical properties distributed over multidimensional domains.

3.1 Double integrals

3.1.1 Basic idea of double integral of a continuous function on a rectangle

Let $f : R = [a, b] \times [c, d] \to \mathbb{R}$ be a continuous function on the rectangle R. When we divide R into $n \times m$ subrectangles, we obtain a **grid of small rectangles** q. More precisely, we define a subdivision of the segment $[a, b]$

into n subintervals

$$[x_{i-1}, x_i], \quad i = 1, \dots, n, \quad \text{with } x_0 = a \text{ and } x_n = b,$$

and a subdivision of the segment $[c, d]$ into m subintervals

$$[y_{j-1}, y_j], \quad j = 1, \dots, m, \quad \text{with } y_0 = c \text{ and } y_m = d.$$

The grid of small rectangles$q$ is therefore composed of the $n \times m$ subrectangles

$$R_{i,j} = [x_{i-1}, x_i] \times [y_{j-1}, y_j], \quad i = 1, \dots, n, \; j = 1, \dots, m.$$

The lower Darboux sum is defined by:

$$S_{\inf}(q) = \sum_{i=1}^{n} \sum_{j=1}^{m} (x_i - x_{i-1}) \left(y_j - y_{j-1} \right) m_{i,j}, \tag{3.1}$$

where $m_{i,j}$ represents the minimum of f on $R_{i,j}$.
Let M now be the upper bound of f on the rectangle R, we can then write:

$$S_{\inf}(q) \leq (b - a)(d - c) M. \tag{3.2}$$

Similarly, the upper Darboux sum is defined by:

$$S_{\sup}(q) = \sum_{i=1}^{n} \sum_{j=1}^{m} (x_i - x_{i-1}) \left(y_j - y_{j-1} \right) M_{i,j}, \tag{3.3}$$

where $M_{i,j}$ represents the maximum of f on $R_{i,j}$.
Now, let m be the lower bound of f on the rectangle R, we can then write:

$$S_{\sup}(q) \geq (b - a)(d - c) m. \tag{3.4}$$

From (3.2) and (3.4), it follows that the set of lower Darboux sums possesses a supremum, which is always less than or equal to the infimum of the set of upper Darboux sums. Moreover, it can be shown that these two bounds are in fact equal. As the partition of the domain becomes increasingly fine—so that the diagonal of each sub-rectangle approaches zero—the corresponding Darboux sums converge to a single value. This limit coincides with the limit of the Riemann sums, leading to the following rigorous definition of the double integral:

Definition 3.1.1. *For a continuous function $f : R = [a,b] \times [c,d] \to \mathbb{R}$ to be integrable on R, it is necessary and sufficient that for all $\epsilon > 0$, there exists a grid of small rectangles $q = \left\{R_{i,j} = [x_{i-1}, x_i] \times \left[y_{j-1}, y_j\right], i = 1, ..., n, and j = 1, ..., m\right\}$, such that:*

$$\begin{cases} 1. S_{\inf}(q) \leq \int\int_R f(x,y)dxdy \leq S_{\sup}(q). \\ 2. \left|S_{\sup}(q) - S_{\inf}(q)\right| \leq \epsilon (b-a)(d-c), \end{cases} \tag{3.5}$$

where the number $\int\int_R f(x,y)dxdy$ represents the volume of $\mathbb{R}^3$ located between the plane xoy, the surface of equation and the four vertical planes $x = a$, $x = b$, $y = c$ and $y = d$.

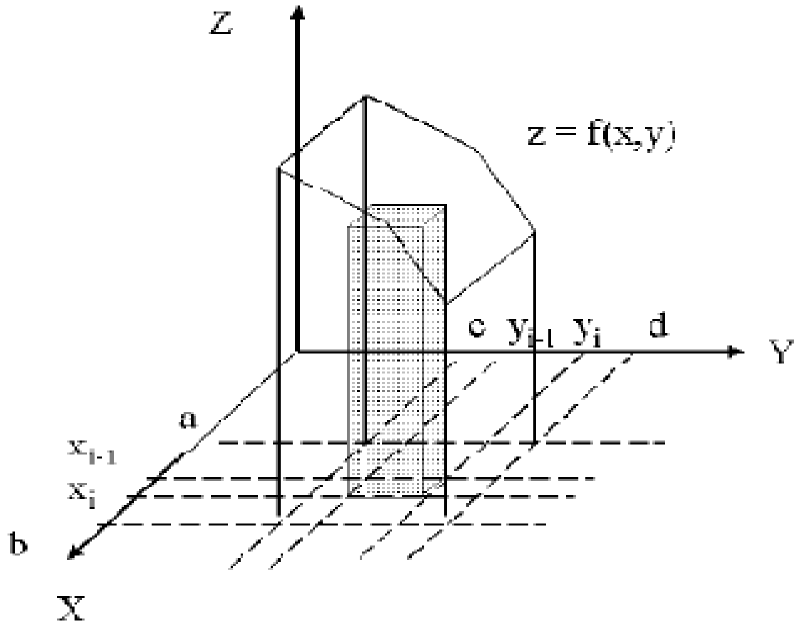


Figure 3.1:

Example 3.1.1. *Using the definition of the double integral, calculate*

$$I = \int\int_R f(x,y)dxdy,$$

where $f(x,y) = x + y$ *and* $R = [0,1] \times [0,1]$.
We will take as a grid of small squares, a regular cut of R into n^2 *small squares with vertices* $\left(\frac{i}{n}, \frac{j}{n}\right)$, *for* $i = 1, ..., n$ *and* $j = 1, ..., n$.
First, let's calculate $m_{i,j}$ *and* $M_{i,j}$ *on the small squares* $\left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right]$.
Since $\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y) = 1 > 0$, *we then have*

$$m_{i,j} = \frac{i-1}{n} + \frac{j-1}{n} \text{ and } M_{i,j} = \frac{i}{n} + \frac{j}{n}.$$

We have:

$$\begin{aligned} S_{\sup}(q) &= \sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_{i-1})\left(y_j - y_{j-1}\right)M_{i,j} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{n^2}\left(\frac{i}{n} + \frac{j}{n}\right) = \frac{n+1}{n}. \end{aligned}$$

Similarly, we can find:

$$\begin{aligned} S_{\inf}(q) &= \sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_{i-1})\left(y_j - y_{j-1}\right)m_{i,j} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{n^2}\left(\frac{i-1}{n} + \frac{j-1}{n}\right) = \frac{n-1}{n}. \end{aligned}$$

Consequently:

$$\frac{n-1}{n} \leq I = \iint_R f(x,y)dxdy \leq \frac{n+1}{n}.$$

We pass to the limit when $n \longrightarrow +\infty$, *we get* $I = 1$.

3.1.2 Fubini's theorems

Fubini's theorem on a rectangle

Theorem 3.1.1. *Let f be a continuous function on a rectangle $R = [a, b] \times [c, d]$, we then have:*

$$\iint_R f(x, y)dxdy = \int_{[a,b]} dx \left(\int_{[c,d]} f(x, y)dy \right) = \int_{[c,d]} dy \left(\int_{[a,b]} f(x, y)dx \right). \quad (3.6)$$

Remark 3.1. *The interest of Fubini's theorem is therefore that calculating a double integral on a rectangle reduces to calculating two simple integrals: We fix y and integrate with respect to x over the segment $[a, b]$, then we integrate this expression for y on the segment $[c, d]$. Alternatively, we can do the same: integrate with respect to y and then with respect to x.*

Example 3.1.2. *Let's calculate the following integral:*

$$I = \iint_R f(x, y)dx,$$

where

$$f(x, y) = \frac{1}{(x + y + 1)^2},$$

and

$$R = [0, 1] \times [0, 1].$$

We have:

$$\begin{aligned} I &= \iint_R f(x, y)dx = \int_{[0,1]} dy \left(\int_{[0,1]} \frac{dx}{(x + y + 1)^2} \right) \\ &= -\int_{[0,1]} \left[\frac{1}{x + y + 1} \right]_0^1 dy = \int_{[0,1]} \frac{dy}{y + 1} - \int_{[0,1]} \frac{dy}{y + 2} dy \\ &= \left[\ln \left(\frac{y + 1}{y + 2} \right) \right]_0^1 = \ln \frac{4}{3}. \end{aligned}$$

Remark 3.2. *If the function f can be writen as:*

$$f(x, y) = h(x) \times g(y),$$

where $h : [a, b] \to \mathbb{R}$ and $g : [c, d] \to \mathbb{R}$ are the continuous functions, then:

$$\iint_{[a,b]\times[c,d]} f(x, y)dxdy = \left(\int_{[a,b]} h(x)dx \right) \times \left(\int_{[c,d]} g(y)dy \right). \quad (3.7)$$

Fubini's theorem on an elementary compact set of $\mathbb{R}^2$

Definition 3.1.2. *We call a compact element of $\mathbb{R}^2$, any part C of $\mathbb{R}^2$ verifying:*

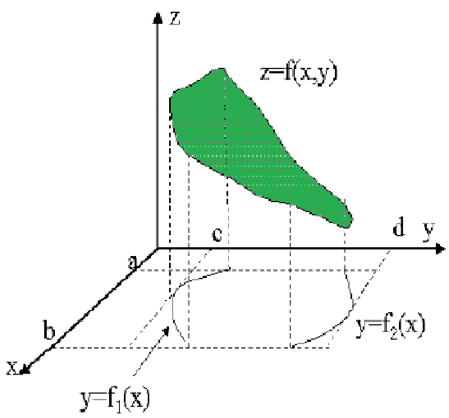


Figure 3.2:

$$C = \Big\{(x,y) \in \mathbb{R}^2 / \ a \leq x \leq b \text{ and } \ f_1(x) \leq y \leq f_2(x)\Big\} \text{ (see Figure 3.2)}, \qquad (3.8)$$

where $f_1, f_2 : [a,b] \to \mathbb{R}$ are continuous functions.
or:

$$C = \Big\{(x,y) \in \mathbb{R}^2 / \ c \leq y \leq d \text{ and } \ g_1(y) \leq x \leq g_2(y)\Big\} \text{ (see Figure 3.3)}, \qquad (3.9)$$

where $g_1, g_2 : [c,d] \to \mathbb{R}$ are continuous functions.

Theorem 3.1.2. *Let C be an elementary compact of $\mathbb{R}^2$ and let f be a continuous function on C.*
1. If the compact C can be represented by the formula (3.8), then

$$\iint_C f(x,y)dxdy = \int_a^b dx \times \left(\int_{f_1(x)}^{f_2(x)} f(x,y)dy\right). \qquad (3.10)$$

1. If the compact C can be represented by the formula (3.9), then

$$\iint_C f(x,y)dxdy = \int_c^d dy \times \left(\int_{g_1(x)}^{g_2(x)} f(x,y)dx\right). \qquad (3.11)$$

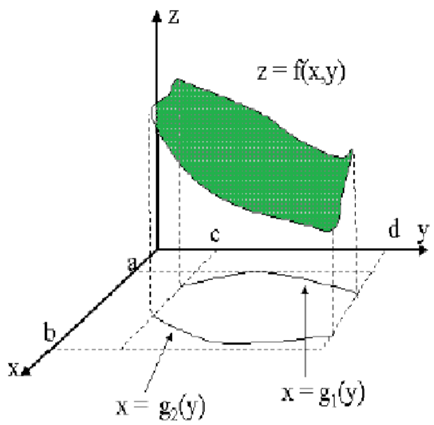


Figure 3.3:

Example 3.1.3. *Let's calculate*

$$I = \iint_C f(x,y)dx,$$

where

$$f(x,y) = 1, et\ C = \left\{(x,y) \in \mathbb{R}^2 / \ x \geq 0 \ and \ \ x-1 \leq y \leq 1-x\right\}. \qquad (3.12)$$

According to the figure 3.4, C can be represented by:

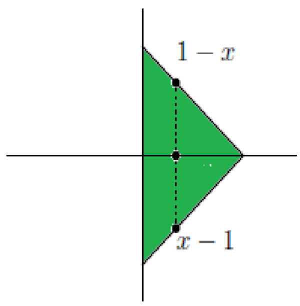


Figure 3.4:

$$C = \left\{(x, y) \in \mathbb{R}^2 / \ 0 \le x \le 1 \ and \ \ x - 1 \le y \le 1 - x\right\}. \qquad (3.13)$$

We then have:

$$\iint_C f(x, y)dxdy = \int_0^1 dx \times \left(\int_{x-1}^{1-x} dy\right) = \int_0^1 (2 - 2x)\, dx = 1.$$

3.1.3 Properties of the double integral

1. The double integral on a compact C of $\mathbb{R}^2$ is linear; that is to say:

$$\iint_C (\alpha f(x, y) + \beta g(x, y))\, dxdy = \alpha \iint_C f(x, y)dxdy + \beta \int\int_C g(x, y)dxdy.$$

2. If $f(x, y) \ge 0$, for all $x, y \in C$, then $\int\int_C f(x, y)dxdy \ge 0$.
3. If $f(x, y) \ge g(x, y)$, for all $x, y \in C$, then $\int\int_C f(x, y)dxdy \ge \int\int_C g(x, y)dxdy$.

4. $\left|\iint_C f(x, y)dxdy\right| \le \iint_C \left|f(x, y)\right| dxdy$.

5. $\left|\int\int_C (f \times g)\,(x, y)dxdy\right| \le \left(\sqrt{\int\int_C f^2(x, y)dxdy}\right) \times \left(\sqrt{\iint_v g^2(x, y)dxdy}\right)$.

6. Let C_1 and C_2 be two simple compact verifying $\mathring{C}_1 \cap \mathring{C}_2 = \varnothing$, then:

$$\int\int_{C_1 \cup C_2} f(x, y)dxdy = \int\int_{C_1} f(x, y)dxdy + \int\int_{C_2} f(x, y)dxdy. \qquad (3.14)$$

3.1.4 Change of variables

Let D and $\hat{D}$ be two elementary compacts of $\mathbb{R}^2$ and let $\varphi : \hat{D} \to D$ be a bijection of class C^1, such that:

$$\text{for all } (u, v) \in \hat{D}, \varphi(u, v) = (x, y) \in D.$$

We therefore have the following theorem.

Theorem 3.1.3.

$$\iint_D f(x, y)dxdy = \iint_{\hat{D}} f\left(\varphi(u, v)\right) \left|\det J_\varphi(u, v)\right| dudv, \qquad (3.15)$$

where $\left|\det J_{\varphi}(u,v)\right| = \left|\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\right|$ *(The absolute value of the determinant of the Jacobian matrix of φ does not vanish inside D.*

Special cases

Conversion to polar coordinates Let $\varphi : \mathbb{R}_+^* \times [0, 2\pi] \to \mathbb{R}^2$, be a function defined by

$$\varphi(\rho, \theta) = (x = \rho \cos\theta, y = \rho \sin\theta).$$

We then have:

$$\left|\det J_{\varphi}(\rho,\theta)\right| = \left|\begin{pmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{pmatrix}\right| = \rho. \tag{3.16}$$

Therefore:

$$\iint_D f(x,y)dxdy = \iint_{\hat{D}} \rho f(\rho\cos\theta, \rho\sin\theta)d\rho d\theta. \tag{3.17}$$

Example 3.1.4. *Let's calculate the following integral:*

$$\iint_D f(x,y)dxdy,$$

where:

$$f(x,y) = \sqrt{1 + x^2 + y^2}$$

and

$$D = \left\{(x,y) \in \mathbb{R}^2; 1 \leq x^2 + y^2 \leq 2,\ x \geq 0,\ y \geq 0\right\}$$

By performing the change of variables $x = \rho\cos\theta, y = \rho\sin\theta$, we get:

$$\hat{D} = \left\{(\rho,\theta) \in \mathbb{R}^2; 1 \leq \rho \leq \sqrt{2},\ 0 \leq \theta \leq \frac{\pi}{2}\right\}.$$

So

$$\begin{aligned} \int_D f(x,y)dxdy &= \int_0^{\frac{\pi}{2}} d\theta \int_1^{\sqrt{2}} \rho\sqrt{1+\rho^2}d\rho \\ &= \frac{\pi}{6}\left[\left(1+\rho^2\right)^{\frac{3}{2}}\right]_1^{\sqrt{2}} \\ &= \frac{\pi}{6}\left((3)^{\frac{3}{2}} + (2)^{\frac{3}{2}}\right). \end{aligned}$$

Change of affine variables Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$, be a function defined by:

$$\varphi(u,v) = (x = \alpha_1 u + \beta_1 v + \gamma_1, y = \alpha_2 u + \beta_2 v + \gamma_2),$$

So

$$\left|\det J_\varphi(u,v)\right| = \left|\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}\right| = \left|\alpha_1\beta_2 - \beta_1\alpha_2\right| \neq 0.$$

Therefore

$$\iint_D f(x,y)dxdy = \iint_{\hat{D}} \left|\alpha_1\beta_2 - \beta_1\alpha_2\right| f(\alpha_1 u + \beta_1 v + \gamma_1, \alpha_2 u + \beta_2 v + \gamma_2)dudv.$$

Example 3.1.5. *Let's calculate the following integral:*

$$I = \iint_D f(x,y)dxdy,$$

where

$$f(x,y) = \exp(x+y)$$

and

$$D = \left\{(x,y) \in \mathbb{R}^2; 0 \leq x+y \leq 1 \text{ and } 0 \leq x-y \leq 1\right\}.$$

Let

$$\hat{D} = \left\{(u,v) \in \mathbb{R}^2; u = x+y, v = x-y \text{ and } (x,y) \in D\right\}.$$

So

$$\hat{D} = ([0,1])^2.$$

On the other hand

$$\begin{cases} x = \dfrac{u+v}{2}, \\ y = \dfrac{u-v}{2} \end{cases},$$

Whitch implies that:

$$\left|\det J_\varphi(u,v)\right| = \left|\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\right| = \frac{1}{2}.$$

We then have:

$$I = \frac{1}{2}\int_0^1 \exp(u)du \int_0^1 dv = \frac{e-1}{2}.$$

3.2 Triple integrals

3.2.1 Basic idea of the triple integral of a continuous function on a domain D of $\mathbb{R}^3$

The principle idea of the triple integral is the same as for the double integral, simply replacing a small area element with a small volume element. Let f be a continuous function of three variables (x, y, z) on a domain D of $\mathbb{R}^3$. We define the following integral:

$$\iiint_D f(x, y, z)dxdydz$$

as the limit of the sum of the form

$$\sum_i \sum_j \sum_k (x_i - x_{i-1})\left(y_j - y_{j-1}\right)(z_k - z_{k-1})\, f(u_i, v_j, w_k),$$

in which, (u_i, v_j, w_k) is a point on the small parallelepiped $[x_i - x_{i-1}]\times\left[y_j - y_{j-1}\right]\times [z_k - z_{k-1}]$.

3.2.2 Fubini's theorem

Fubini's Theorem on a parallelepiped

Theorem 3.2.1. *Let f be a function on a parallelepiped $P = [a, b] \times [c, d] \times [e, f]$, we then have:*

$$\begin{aligned}\iiint_P f(x, y, z)dxdydz &= \int_{[e,f]} dz\left(\iint_{[a,b]\times[c,d]} f(x, y, z)dxdy\right)\\ &= \int_{[a,b]} dx\left(\iint_{[c,d]\times[e,f]} f(x, y, z)dydz\right)\\ &= \dots \end{aligned} \tag{3.18}$$

Example 3.2.1. *Let's calculate the following integral:*

$$I = \iiint_P f(x, y, z)dxdydz,$$

where

$$f(x, y, z) = 2(xy + yz + xz)$$

and

$$P = [0, 1]^3.$$

We have

$$\int_0^1 2\,(xy + yz + xz)\,dx = y + 2yz + z.$$

Then

$$\int_0^1 (y + 2yz + z)\,dy = \frac{1}{2} + 2z,$$

and finally

$$\int_0^1 \left(\frac{1}{2} + 2z\right)dz = \frac{3}{2}.$$

Fubini's theorem on a domain D of $\mathbb{R}^3$

The idea is to take one of the three variables x, y, and z that varies between two bounds with extremes a and b. Suppose, for example, that it is z (we can interchange the roles of x, y, and z), such that the planar domain obtained by cutting the volume D with a plane $z = constant$ is a domain D_z sufficiently simple for us to calculate the double integral $\iint_{D_z} f(x, y, z)dxdy$. We then have

$$\iiint_P f(x, y, z)dxdydz = \int_a^b dz\left(\iint_{D_z} f(x, y, z)dxdy\right). \qquad (3.19)$$

Example 3.2.2. *Let's calculate the following integral:*

$$\iint_D f(x, y, z)dxdydz,$$

where

$$f(x, y, z) = 1$$

and

$$D = \left\{(x, y, z) \in \mathbb{R}^3; x, y, z \geq 0 \text{ and } x + y + 2z \leq 1\right\}.$$

The goal is to calculate the volume of D. We cut D by a horizontal plane $z = z_0$,

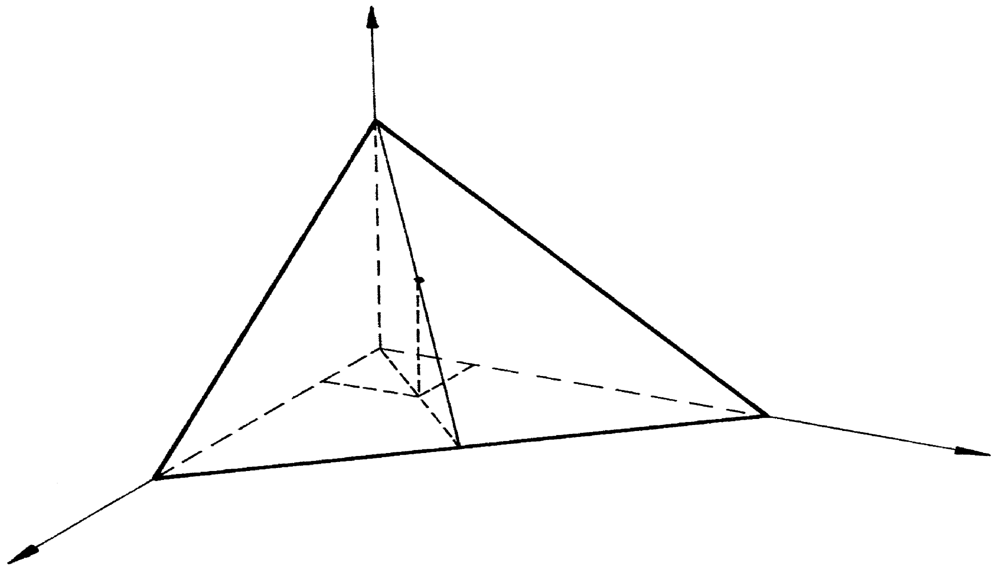


Figure 3.5: The compact D

which gives us a triangle D_z defined by x and y bounded by axes $x = y = 0$ and $x + y = 1 - 2z_0$, such that $z_0 \in \left[0, \frac{1}{2}\right]$. Therefore, we have:

$$\begin{aligned}
\int_0^{\frac{1}{2}} dz \left(\iint_{D_z} f(x,y,z)dxdydz \right) &= \int_0^{\frac{1}{2}} dz \left(\int_0^{1-2z} dx \left(\int_0^{1-2z-x} f(x,y,z)dy \right) \right) \\
&= \int_0^{\frac{1}{2}} dz \left(\int_0^{1-2z} (1 - x - 2z)\, dx \right) \\
&= \int_0^{\frac{1}{2}} \left(\frac{1}{2} - 2z + 2z^2 \right) dz = \frac{1}{12}.
\end{aligned}$$

3.2.3 Change of variables in a triple integral

Let D and $\hat{D}$ be two elementary compacts of $\mathbb{R}^3$ and let $\varphi : \hat{D} \to D$ be a bijection of class C^1defined on $\hat{D}$ by:

$$\varphi(u, v, k) = (x, y, z) \in D.$$

We therefore have the following theorem.

Theorem 3.2.2.

$$\iiint_D f(x,y,z)dxdydz = \iiint_{\tilde{D}} f(\varphi(u,v,k))\left|\det J_\varphi(u,v,k)\right| dudvdk,$$

where $\left|\det J_\varphi(u,v,k)\right| = \left|\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial k} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial k} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial k} \end{pmatrix}\right|$ *(The absolute value of the determinant of the Jacobian matrix of* φ *does not vanish inside* D*.*

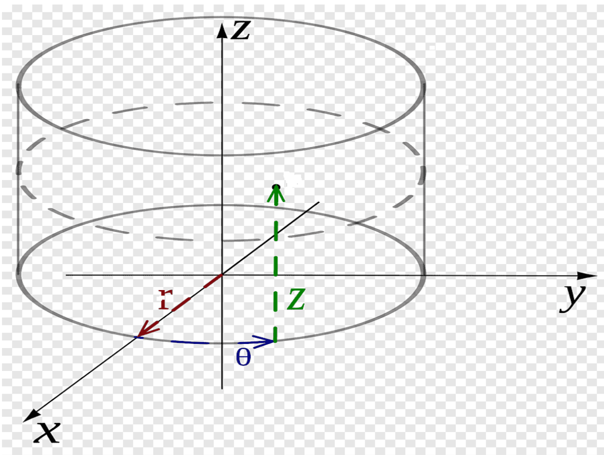
Conversion to cylindrical coordinates

Let

$$\varphi : \mathbb{R}_+^* \times [0, 2\pi] \times \mathbb{R} \to \mathbb{R}^3, \text{ be a function defined on } \mathbb{R}^3 \text{ by}$$

$$\varphi(\rho, \theta, z) = (x = \rho\cos\theta, y = \rho\sin\theta, z = z).$$

We then have



$$\left|\det J_\varphi(\rho,\theta,z)\right| = \left|\begin{pmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\right| = \rho.$$

So

$$\iiint_D f(x,y,z)dxdydz = \iiint_{\hat{D}} \rho f(\rho\cos\theta, \rho\sin\theta, z)d\rho d\theta dz.$$

Example 3.2.3. *Let's calculate the following integral:*

$$\iiint_D f(x,y,z)dxdydz,$$

where

$$f(x,y,z) = \frac{1}{x^2+y^2+z^2}$$

and

$$D = \left\{(x,y,z) \in \mathbb{R}^3; 1 \le x^2+y^2 \le 4,\ x \ge 0,\ y \ge 0, 0 \le z \le 1\right\}.$$

By performing the change of variables

$$x = \rho\cos\theta, y = \rho\sin\theta, z = z,$$

we get:

$$\hat{D} = \left\{(\rho,\theta,z) \in \mathbb{R}^3; 1 \le \rho \le 2,\ 0 \le \theta \le \frac{\pi}{2} \text{ and } 0 \le z \le 1\right\}.$$

So

$$\iiint_D f(x,y,z)dxdydz = \int_0^1 dz \int_0^{\frac{\pi}{2}} d\theta \int_1^2 \frac{d\rho}{\rho} = \frac{\pi}{2}\left[\ln\rho\right]_1^2 = \frac{\pi}{2}\ln 2.$$

Conversion to spherical coordinates

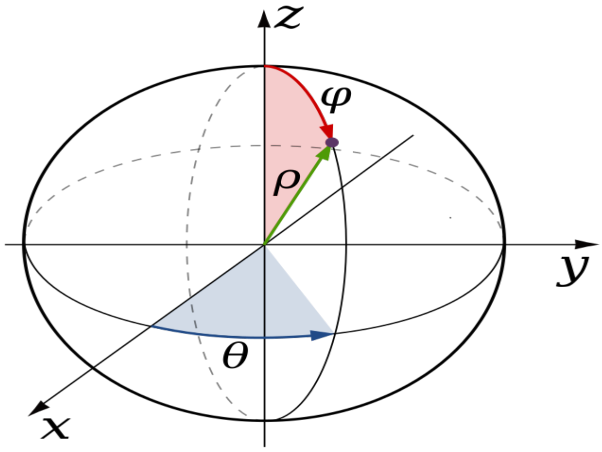
The spherical coordinates are given by:

$$\begin{cases} x = \rho\sin\theta\cos\varphi, \\ y = \rho\sin\theta\sin\varphi, \\ z = \rho\cos\theta, \end{cases} \tag{3.20}$$

where $\theta \in [0,\pi]$ and $\varphi \in [0,2\pi]$.

In this cas, we have:

$$\left|\det J_\varphi(\rho,\theta,\varphi)\right| = \left|\begin{pmatrix} \sin\theta\cos\varphi & \rho\cos\theta\cos\varphi & -\rho\sin\theta\sin\omega \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\theta & -\rho\sin\theta & 0 \end{pmatrix}\right| = \rho^2\sin\theta. \tag{3.21}$$



Example 3.2.4. *Let's calculate the following integral:*

$$\iiint_D f(x, y, z)dxdydz,$$

where

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

and

$$D = \left\{(x, y, z) \in \mathbb{R}^3; 1 \leq x^2 + y^2 + z^2 \leq 4\right\}.$$

Using spherical coordinates, we get

$$\hat{D} = \left\{(\rho, \theta, \varphi) \in \mathbb{R}^3; 1 \leq \rho \leq 2,\ 0 \leq \varphi \leq 2\pi \text{ and } 0 \leq \theta \leq \pi\right\}.$$

So

$$\iiint_D f(x, y, z)dxdydz = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \int_1^2 \rho d\rho = 6\pi.$$

3.2.4 Applications

Calculation of some volumes

Definition 3.2.1. *The volume V of a field is given by:*

$$V = \iiint_D dxdydz, \tag{3.22}$$

where D represents the domain delimited by this field.

Volume of a cylinder: In this case

$$D = \left\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 \leq R^2 \text{ and } 0 \leq z \leq h\right\}.$$

We then have

$$V = \iiint_D dxdydz = \int_0^h dz \int_0^{2\pi} d\theta \int_0^R \rho d\rho = \pi R^2 h \tag{3.23}$$

Volume of a sphere: In this case

$$D = \left\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 \leq R^2 \right\}.$$

So

$$V = \iiint_D dxdydz = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^R \rho \sin\theta d\rho = \frac{1}{3}\pi R^3 \tag{3.24}$$

Volume of an ellipsoid: In this case

$$D = \left\{(x, y, z) \in \mathbb{R}^3; \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1, (a, b, c) \in (\mathbb{R}_+^*)^3 \right\}.$$

By performing the change of variable

$$y = b\sqrt{1 - \frac{x^2}{a^2}} \sin\theta,$$

where $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, we get

$$\begin{aligned} V &= \iiint_D dxdydz = \int_{-a}^{a} dx \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} dy \int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz \\ &= 2c \int_{-a}^{a} dx \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} dy \\ &= 2bc \int_{-a}^{a} dx \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1-\frac{x^2}{a^2}\right) \cos^2\theta d\theta \\ &= 8bc \int_0^a \left(1-\frac{x^2}{a^2}\right) dx \int_0^{\frac{\pi}{2}} \cos^2\theta d\theta = \frac{4}{3}\pi abc. \end{aligned}$$

Volume of the ellipsoid of the equation $x^2 + a^2y^2 + b^2z^2 = c^2$,

where $a, b, c > 0$

Let

$$\begin{aligned} D &= \left\{(x, y, z) \in \mathbb{R}^3; x^2 + a^2y^2 + b^2z^2 = c^2 \right\} \\ &= \{(c\rho \sin\theta \cos\varphi, b\rho \sin\theta \sin\varphi, a\rho \cos\theta) : 0 < \rho < 1, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi \}. \end{aligned}$$

So

$$V = \iiint_D dxdydz = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^1 c^2\rho^2 \sin\theta d\theta = \frac{4}{3}\pi c^2.$$

Volume common to the two cylinders with respective equations $x^2 + y^2 = R^2$ and $x^2 + z^2 = R^2$, $R > 0$. In this case:

$$D = \left\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 < R^2 \text{ and } x^2 + z^2 < R^2\right\}.$$

So

$$\begin{aligned} V &= \iiint_D dxdydz = \int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dz \\ &= 8 \int_{-R}^{R} \left(R^2 - x^2\right) dx = \frac{16}{3} R^2. \end{aligned} \tag{3.25}$$

3.3 Exercises about chapter 3

Exercise 3.3.1. *1. Let* $D_1 = [0,1] \times [0,2]$. *Calculate* $I_1 = \iint_{D_1} y \frac{\exp(2x+y^2)}{1+\exp(x)} dxdy$.

2. Let $D_2 = \left\{(x,y) \in \mathbb{R}^2 : x > 1,\ y > 1,\ x+y < 3\right\}$. *Calculate* $I_2 = \iint_{D_2} \frac{dxdy}{(x+y)^3}$.

3. Let $D_3 = [0, +\infty[$. *Calculate the generalized integral* $I_3 = \int_{D_3} \exp(-x^2)dx$.

4. Let $D_4 = \left\{(x,y) \in \mathbb{R}^2 : 0 < x < y < 2x,\ xy < 4,\ x^2+y^2 > 4\right\}$.
Calculate $I_4 = \iint_{D_4} x^2 y dxdy$.

5. Let $D_5 = \left\{(x,y) \in \mathbb{R}^2 : 1 < x^2+y^2 < 4\right\}$. *Calculate* $I_5 = \iint_{D_5} \frac{\cos(x^2+y^2)}{2+\sin(x^2+y^2)} dxdy$.

6. Let $D_6 = \left\{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1,\ x+y > 1\right\}$. *Calculate* $I_6 = \iint_{D_6} \frac{dxdy}{(x^2+y^2)^2}$.

7. Let $D_7 = \left\{(x,y) \in \mathbb{R}^2; x^2+y^2-2\sqrt{2}x < 0 \text{ and } x > \sqrt{2}\right\}$.
Calculate $I_7 = Area(D_7)$

8. Let $D_8 =]0,1[\times]0,1[$. *Using the change of variables* $x = u^2,\ y = \frac{v}{u}$, *calculate*
$I_8 = \iint_{D_8} \frac{dxdy}{(1+x)(1+xy^2)}$.

9. Let $D_9 = \left\{(x,y) \in \mathbb{R}^2 : x < y < 2x \text{ and } x < y^2 < 2x\right\}$.

9.1. Calculate $I_9 = \iint_{D_9} \frac{y}{x} dxdy$.

9.2. Using the change of variables $u = \frac{x}{y}$ *and* $v = \frac{y^2}{x}$, *recalculate the value of* I_9.

10. Let $D_{10} = \left\{(x,y) \in \mathbb{R}^2 / \ x^2 < y < 2-x^2\right\}$ *and let* $I_{10} = \iint_{D_{10}} dxdy$.
Represent D_{10}, *then compute the value of* I_{10}.

Exercise 3.3.2. *1. Let* $D_1 = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 < 1, x^2+y^2+z^2 < 4,\ z > 0\right\}$.
Calculate $I_1 = \iiint_{D_1} z dxdydz$.

2. Let $D_2 = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 < z < 2-x^2-y^2\right\}$. *Calculate* $I_2 = \iiint_{D_2} dxdydz$.

3. Let $D_3 = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 < 2z,\ x^2+y^2+z^2 < 3\right\}$.
Calculate $I_3 = \iiint_{D_3} dxdydz$.

4. Calculate the volumes of a cylinder, of a sphere, of an ellipsoid, and of an ellipsoid *of equation:*

$$a^2x^2 + b^2y^2 + c^2z^2 < R^2, a; b, c \in \mathbb{R}_*^+.$$

3.4 Solutions of exercices about chapter 3

Solution of exercise 3.3.1

1. We have:

$$\begin{aligned}
I_1 &= \iint_{D_1} y\frac{\exp(2x+y^2)}{1+\exp(x)}dxdy = \left(\int_0^1 \frac{\exp(2x)}{1+\exp(x)}dx\right)\times\left(\int_0^2 y\exp(y^2)dy\right) \\
&= \left(\int_1^e \frac{z}{1+z}dz\right)\times\left(\int_0^2 y\exp(y^2)dy\right) \\
&= [z-\ln(1+z)]_1^e \times \left[\frac{\exp(y^2)}{2}\right]_0^2 \\
&= \left(e-1+\ln\left(\frac{2}{1+e}\right)\right)\times\left(\frac{\exp(4)-1}{2}\right).
\end{aligned}$$

2. As illustrated in Figure 3.6, the compact:

$$D_2 = \left\{(x,y)\in\mathbb{R}^2 : x>1, y>1, x+y<3\right\}$$

can be rewritten in the form:

$$D_2 = \left\{(x,y)\in\mathbb{R}^2 : 1<x<2, 1<y<3-x\right\},$$

then

$$\begin{aligned}
I_2 &= \iint_{D_2} \frac{dxdy}{(x+y)^3} = \int_1^2 dx \int_1^{3-x} \frac{dy}{(x+y)^3} = \frac{-1}{2}\int_1^2 \left[\frac{1}{(x+y)^2}\right]_1^{3-x} dx \\
&= \frac{-1}{2}\int_1^2 \left(\frac{1}{9}-\frac{1}{(x+1)^2}\right)dx = \frac{-1}{2}\left[\frac{1}{9}x+\frac{1}{x+1}\right]_1^2 = \frac{1}{36}
\end{aligned}$$

3. First, let's show that the integral $I_3 = \int_0^\infty \exp(-x^2)dx$ exists.

Since

$$\exp(-x^2) = O\left(\exp(-x)\right), \text{ when } x \to +\infty,$$

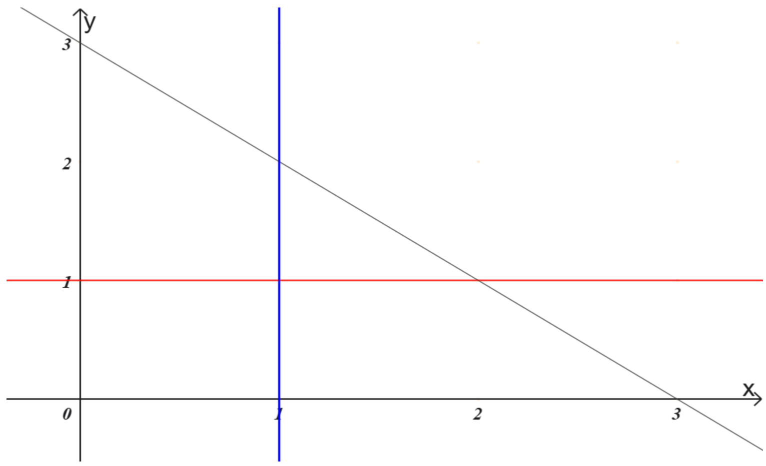


Figure 3.6: The triangle D_2

and since $\int_0^{\infty} \exp(-x)dx = 1$, then, the generalized integral $I_3 = \int_0^{\infty} \exp(-x^2)dx$ is convergent. Let $I_3^a = \int_0^{a} \exp(-x^2)dx$. So

$$
\begin{aligned}
(I_3^a)^2 &= \int_0^a \exp(-x^2)dx \times \int_0^a \exp(-y^2)dy \\
&= \iint_{[0,a]^2} \exp(-(x^2+y^2))dxdy.
\end{aligned}
$$

Let $x = \rho\cos(\theta)$ and $y = \rho\sin(\theta)$. So

$$(\rho, \theta) \in \left[0, \sqrt{2}a\right] \times \left[0, \frac{\pi}{2}\right].$$

Consequently

$$
\begin{aligned}
(I_3^a)^2 &= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\sqrt{2}a} \rho\exp(-\rho^2)d\rho \\
&= \frac{-\pi}{4}\left[\exp(-\rho^2)\right]_0^{\sqrt{2}a} = \frac{\pi}{4}\left[1 - \exp(-2a)\right].
\end{aligned}
$$

We take the limit, when a tends to $+\infty$, we get $(I_3)^2 = \frac{\pi}{4}$, and since exp(.) is a positive function, we then have:

$$I_3 = \int_0^{+\infty} \exp(-x^2)dx = \frac{\sqrt{\pi}}{2}.$$

4. As illustrated in Figure 3.7, the compact:

$$D_4 = \left\{(x, y) \in \mathbb{R}^2 : 0 < x < y < 2x,\ xy < 4,\ x^2 + y^2 > 4\right\}$$

can be rewritten as:

$$D_4 = C_1 \cup C_2,$$

where

$$C_1 = \left\{(x, y) \in \mathbb{R}^2 : \frac{2}{\sqrt{5}} < x < \sqrt{2} \text{ and } \sqrt{4 - x^2} < y < 2x\right\},$$

and

$$C_2 = \left\{(x, y) \in \mathbb{R}^2 : \sqrt{2} < x < 2 \text{ and } x < y < \frac{4}{x}\right\}.$$

So

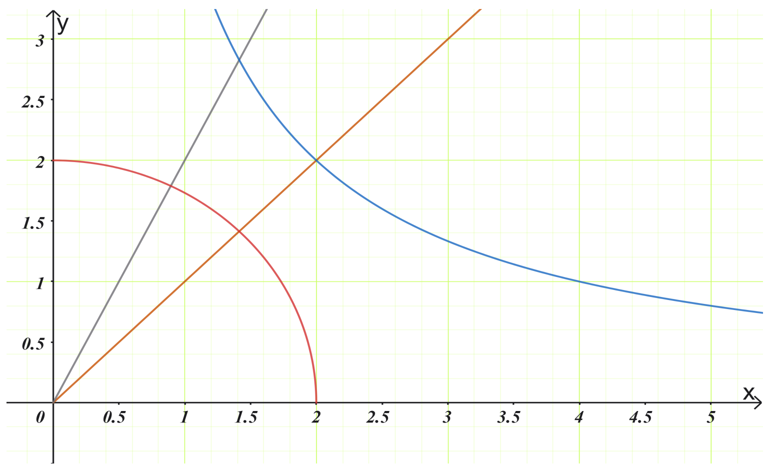


Figure 3.7: The compact D_4

$$\begin{aligned}
\iint_{D_4} x^2 y dx dy &= \iint_{C_1} x^2 y dx dy + \iint_{C_2} x^2 y dx dy \\
&= \int_{\frac{2}{\sqrt{5}}}^{\sqrt{2}} x^2 \left(\int_{\sqrt{4-x^2}}^{2x} y dy \right) dx + \int_{\sqrt{2}}^{2} x^2 \left(\int_{x}^{\frac{4}{x}} y dy \right) dx \\
&= \frac{1}{2} \int_{\frac{2}{\sqrt{5}}}^{\sqrt{2}} \left(5x^4 - 4x^2\right) dx + \frac{1}{2} \int_{\sqrt{2}}^{2} \left(16 - x^4\right) dx \\
&= \frac{1}{2} \left[x^5 - \frac{4}{6} x^3 \right]_{\frac{2}{\sqrt{5}}}^{\sqrt{2}} + \frac{1}{2} \left[16x - \frac{1}{5} x^5 \right]_{\sqrt{2}}^{2} \\
&= -\frac{104}{5} \sqrt{2} + \frac{32}{375} \sqrt{5} + \frac{64}{5}.
\end{aligned}$$

5. Let's calculate $I_5 = \iint_{D_5} \frac{\sin(x^2+y^2)}{2+\cos(x^2+y^2)} dx dy$, where

$$D_5 = \left\{ (x,y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4 \right\}.$$

We set

$$x = \rho \cos \theta \text{ and } y = \rho \sin \theta.$$

So

$$\rho \in]1,2[\text{ and } \theta \in [0, 2\pi].$$

We then have

$$\begin{aligned}
I_5 &= \int_0^{2\pi} \left(\int_1^2 \frac{\rho \cos(\rho^2) d\rho}{2 + \sin(\rho^2)} \right) d\theta \\
&= \pi \left[\ln\left(2 + \sin(\rho^2)\right) \right]_1^2 \\
&= \pi \ln \left(\frac{2 + \sin(4)}{2 + \sin(1)} \right).
\end{aligned}$$

6. Let $D_6 = \left\{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1,\ x + y > 1 \right\}$, and let's calculate

$$I_6 = \iint_{D_6} \frac{dx dy}{(x^2+y^2)^2}.$$

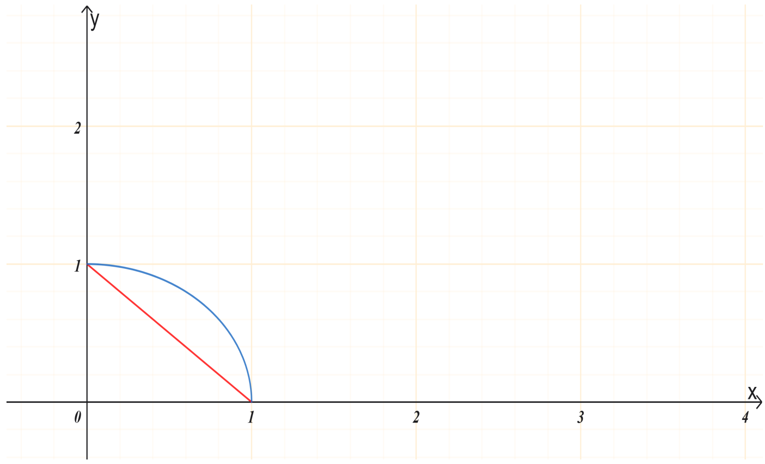


Figure 3.8: The compact D_6

We set $x = \rho\cos\theta$ and $y = \rho\sin\theta$. So

$$\rho \in \left]\frac{1}{\cos(\theta)+\sin(\theta)}, 1\right[\text{ and } \theta \in \left[0, \frac{\pi}{2}\right].$$

We then have:

$$\begin{aligned} I_6 &= \int_0^{\frac{\pi}{2}} \left(\int_{\frac{1}{\cos(\theta)+\sin(\theta)}}^{1} \frac{d\rho}{\rho^3} \right) d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left((\cos(\theta)+\sin(\theta))^2 - 1 \right) d\theta \\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(2\theta)\, d\theta = -\frac{1}{4}\left[\cos(2\theta)\right]_0^{\frac{\pi}{2}} = \frac{1}{2}. \end{aligned}$$

7. We can write

$$D_7 = \left\{(x,y) \in \mathbb{R}^2; (x-\sqrt{2})^2 + y^2 < (\sqrt{2})^2 \text{ and } x > \sqrt{2}\right\}$$

So D_7 is a semi-disc with center $(\sqrt{2}, 0)$ and radius $\sqrt{2}$.

Let $x = \rho\cos(\theta)$ and $y = \rho\sin(\theta)$. We then have

$$(\rho, \theta) \in \left]\frac{\sqrt{2}}{\cos(\theta)}, 2\sqrt{2}\cos(\theta)\right[\times \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[$$

So

$$\begin{aligned}
Air(D_7) &= \int\int_{D_7} dxdy = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\int_{\frac{\sqrt{2}}{\cos(\theta)}}^{2\sqrt{2}\cos(\theta)} \rho d\rho \right) d\theta \\
&= \int_0^{\frac{\pi}{4}} \left(4(1+\cos(2\theta)) - \frac{2}{\cos^2(\theta)} \right) d\theta \\
&= \left(4\left(\theta + \frac{1}{2}\sin(2\theta)\right) - 2\tan(\theta) \right)\Big|_0^{\frac{\pi}{4}} = \pi.
\end{aligned}$$

8. Let $\varphi(u;v) = \left(x = u^2,\ y = \frac{v}{u}\right)$.
Since $(x,\ y) \in]0,1[\times]0,1[$, we then have $(u,v) \in]0,1[\times]0,1[$.
On the other hand

$$\det J_\varphi(u,v) = \begin{vmatrix} 2u & 0 \\ \frac{-v}{u^2} & \frac{1}{u} \end{vmatrix} = 2.$$

So

$$\begin{aligned}
I_8 &= \iint_{D_8} \frac{dxdy}{(1+x)(1+xy^2)} \\
&= \int_0^1 \int_0^1 \frac{2dudv}{(1+u^2)(1+v^2)} \\
&= 2\,(\arctan u)|_0^1\,(\arctan v)|_0^1 \\
&= \frac{\pi^2}{8}.
\end{aligned}$$

9. Let $D_9 = \left\{(x,y) \in \mathbb{R}^2 : x < y < 2x \text{ and } x < y^2 < 2x\right\}$.

9.1. Let's calculate

$$I_9 = \iint_{D_9} \frac{y}{x} dxdy.$$

As illustrated in Figure 3.9, the compact D_9 can be rewritten as:

$$D_9 = C_1 \cup C_2 \cup C_3,$$

where

$$C_1 = \left\{(x,y) \in \mathbb{R}^2 : x_0 < x < x_1 \text{ and } \sqrt{x} < y < 2x\right\},$$
$$C_2 = \left\{(x,y) \in \mathbb{R}^2 : x_1 < x < x_2 \text{ and } \sqrt{x} < y < \sqrt{2x}\right\},$$
$$C_3 = \left\{(x,y) \in \mathbb{R}^2 : x_2 < x < x_3 \text{ and } x < y < \sqrt{2x}\right\}.$$

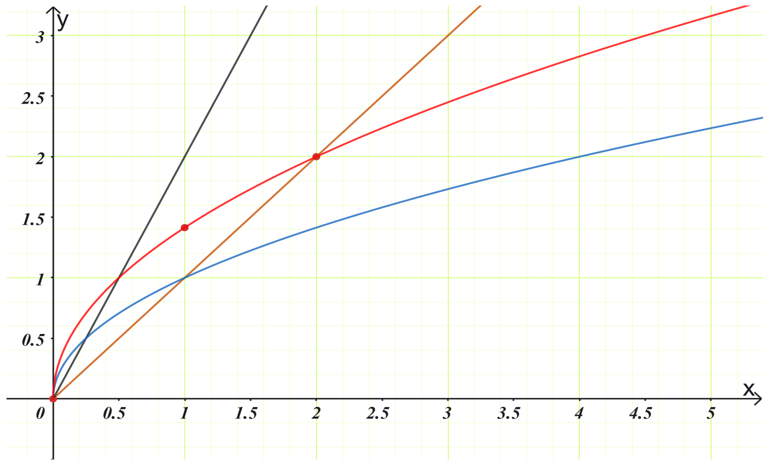


Figure 3.9: The compact D_9

The point x_0 verifies $x_0 = y_0^2$ and $x_0 = y_0$, which implies that $x_0 = \frac{1}{4}$.
The point x_1 verifies $2x_1 = y_1^2$ and $2x_1 = y_1$, which implies that $x_1 = \frac{1}{2}$.
The point x_2 verifies $x_2 = y_2^2$ and $x_2 = y_2$, which implies that $x_2 = 1$.
The point x_3 verifies $2x_3 = y_3^2$ and $x_3 = y_3$, which implies that $x_3 = 2$.

We then have:

$$\begin{aligned} I_9 &= \iint\limits_{D_9} \frac{y}{x} dxdy \\ &= \iint\limits_{C_1} \frac{y}{x} dxdy + \iint\limits_{C_2} \frac{y}{x} dxdy + \iint\limits_{C_3} \frac{y}{x} dxdy \\ &= \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{x} \left(\int_{\sqrt{x}}^{2x} ydy \right) dx + \int_{\frac{1}{2}}^{1} \frac{1}{x} \left(\int_{\sqrt{x}}^{\sqrt{2x}} ydy \right) dx + \int_{1}^{2} \frac{1}{x} \left(\int_{x}^{\sqrt{2x}} ydy \right) dx \\ &= \frac{1}{16} + \frac{1}{4} + \frac{1}{4} \\ &= \frac{9}{16}. \end{aligned}$$

9.2. Let $u = \frac{x}{y}$ and $v = \frac{y^2}{x}$. So $x = u^2 v$ et $y = uv$.
Let $\varphi(u; v) = \left(x = u^2,\ y = \frac{v}{u} \right)$.
Since $(x,\ y) \in D_9$, then $(u, v) \in]\frac{1}{2}, 1[\times]1, 2[$.
On the other hand

$$\det J_\varphi(u, v) = \begin{vmatrix} 2uv & u^2 \\ v & u \end{vmatrix} = u^2.$$

So

$$\begin{aligned} I_9 &= \iint\limits_{D_9} \frac{y}{x} dxdy \\ &= \int_{\frac{1}{2}}^{1} \int_{1}^{2} u^2 v \frac{1}{u} dudv \\ &= \int_{\frac{1}{2}}^{1} udu \int_{1}^{2} vdv \\ &= \frac{9}{16}. \end{aligned}$$

10. As illustrated in Figure 3.10, the compact D_{10} can be rewritten as

$$D_{10} = \left\{ (x, y) \in \mathbb{R}^2 / \ x \in]-1, 1[\text{ and } x^2 < y < 2 - x^2 \right\}.$$

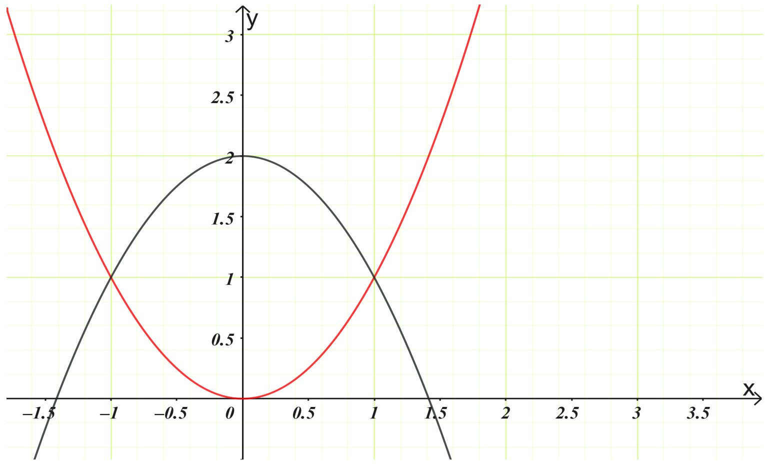


Figure 3.10: The compact D_{10}

So

$$\begin{aligned} I_{10} &= \iint_{D_{10}} dxdy = \int_{-1}^{1}\left(\int_{x^2}^{2-x^2} dy\right)dx \\ &= \int_{-1}^{1}\left(2-2x^2\right)dx = 2x - \frac{2}{3}x^3\Big|_{-1}^{1} = \frac{8}{3}. \end{aligned}$$

Solution of exercise 3.3.2

1. Let $D_1 = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2<1, x^2+y^2+z^2<4,\ z>0\right\}$.

We set:

$$\chi(\rho,\theta,z) = (x=\rho\cos\theta, y=\rho\sin\theta, z=z).$$

So

$$\det J_\chi(\rho,\theta,z) = \rho,$$

where

$$\rho \in \left]0,1\right[, \theta \in \left]0,2\pi\right[\text{ and } 0<z<\sqrt{4-\rho^2}$$

We then have

$$\begin{aligned} I_1 &= \iiint\limits_{D_1} z dx dy dz \\ &= \int_0^{2\pi} d\theta \int_0^1 \rho \left(\int\limits_0^{\sqrt{4-\rho^2}} z dz \right) d\rho \\ &= \pi \int_0^1 \rho \left(4-\rho^2\right) d\rho \\ &= \frac{7\pi}{4}. \end{aligned}$$

2. Let $x = \rho \cos\theta$, $y = \rho \sin\theta$ and $z = z$. So

Figure 3.11: The compact $\tilde{D}_2$

$$(x,y,z) \in D_2 \Leftrightarrow (\rho,\theta,z) \in \tilde{D}_2 = \left\{(\rho,\theta,z) \in \mathbb{R}^3 / \rho \in]0,1[\ \ \theta \in]0,2\pi[\text{ and } \rho^2 < z < 2-\rho^2\right\}$$

Hence

$$\begin{aligned} I_2 &= \iiint\limits_{D_2} dxdydz = \int_0^{2\pi} d\theta \int_0^1 \rho \left(\int_{\rho^2}^{2-\rho^2} dz \right) d\rho \\ &= 2\pi \int_0^1 \left(2\rho - 2\rho^3\right) d\rho \\ &= 2\pi \left(\rho^2 - \frac{1}{2}\rho^4\right)\Big|_0^1 = \pi. \end{aligned}$$

3. Let $D_3 = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 2z,\ x^2 + y^2 + z^2 < 3\right\}$.
We set

$$\chi(\rho, \theta, z) = (x = \rho\cos\theta, y = \rho\sin\theta, z = z).$$

So

$$\det J_\chi(\rho, \theta, z) = \rho.$$

and

$$z \in \left]\frac{\rho^2}{2}, \sqrt{3-\rho^2}\right[, \theta \in]0, 2\pi[$$

and ρ verifies

$$\sqrt{3-\rho^2} = \frac{\rho^2}{2},$$

which implies that $\rho = \sqrt{2}$.
So $\rho \in \left]0, \sqrt{2}\right[$. Finally

$$\begin{aligned} I_3 &= \iiint\limits_{D_3} dxdydz = \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} \rho \left(\int_{\frac{\rho^2}{2}}^{\sqrt{3-\rho^2}} dz \right) d\rho \\ &= 2\pi \int_0^{\sqrt{2}} \rho \left(\sqrt{3-\rho^2} - \frac{\rho^2}{2} \right) d\rho = 2\pi \left[-\frac{1}{3}\left(3-\rho^2\right)^{\frac{23}{2}} - \frac{\rho^2}{8} \right]_0^{\sqrt{2}} = \frac{\pi}{3}\left(6\sqrt{3} - 5\right). \end{aligned}$$

4. Let's calculate the following volumes.

4.1 Volume of a cylinder:

$$C = \left\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 \le R^2 \text{ and } 0 \le z \le h\right\}.$$

Let $x = \rho\cos\theta, y = \rho\sin\theta$ and $z = z$. So

$$\rho \in]0, R] \text{ and } \theta \in [0, 2\pi].$$

Therefore

$$V(C) = \iiint_C dxdydz = int_0^R \rho d\rho \int_0^{2\pi} d\theta \int_0^h dz = \pi R^2 h.$$

4.2 Volume of a sphere: $S = \left\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 \leq R^2\right\}$.
Let

$$\chi(\rho, \theta, \varphi) = (x = \rho\cos\theta\sin\varphi, y = \rho\sin\theta\sin\varphi, z = \rho\cos\varphi).$$

So, $\det J_\chi(\rho, \theta, \varphi) = \rho^2 \sin\varphi$, where

$$\rho \in]0, R], \theta \in [0, 2\pi] \text{ and } \varphi \in [0, \pi].$$

We then have:

$$V(S) = \iiint_S dxdydz = \int_0^R \rho^2 d\rho \int_0^{2\pi} d\theta \int_0^h \sin\varphi d\varphi = \frac{4\pi R^3}{3}.$$

4.3. Volume of an ellipsoid:

$$E = \left\{(x, y, z) \in \mathbb{R}^3; \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq R^2\right\}.$$

Let $au = x, bv = y$ and $cw = z$, so

$$\begin{aligned}(x, y, z) &\in E = \left\{(x, y, z) \in \mathbb{R}^3; \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq R^2\right\} \\ &\Leftrightarrow (u, v, w) \in S = \left\{(u, v, w) \in \mathbb{R}^3; u^2 + v^2 + w^2 < R^2\right\}.\end{aligned}$$

Since $dxdydz = abcdudvdw$, we then have

$$\begin{aligned}V(E) &= \iiint_E dxdydz \\ &= abc \iiint_S dudvdw \\ &= abcV(S) = abc \int_0^R \rho^2 d\rho \int_0^\pi \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \\ &= 4\pi abcR^3 \quad (\text{ using spherical coordinates })\end{aligned}$$

4.4. Volume of an ellipsoid of

$$E: \ a^2x^2 + b^2y^2 + c^2z^2 < R^2, a; b, c \in \mathbb{R}_*^+.$$

Let $u = ax, v = by$ and $w = cz$, So

$$\begin{aligned} (x, y, z) &\in E = \left\{(x, y, z) \in \mathbb{R}^3; a^2x^2 + b^2y^2 + c^2z^2 < R^2\right\} \\ &\Leftrightarrow (u, v, w) \in S = \left\{(u, v, w) \in \mathbb{R}^3; u^2 + v^2 + w^2 < R^2\right\}. \end{aligned}$$

Since

$$dxdydz = \frac{1}{abc}dudvdz,$$

we then have:

$$\begin{aligned} V(E) &= \iiint\limits_E dxdydz = \frac{1}{abc}\iiint\limits_S dudvdw \\ &= \frac{1}{abc}V(S) = \frac{1}{abc}\int_0^R \rho^2 d\rho \int_0^\pi \sin(\varphi)\, d\varphi \int\limits_0^{2\pi} d\theta = \frac{4\pi R^3}{3abc}. \end{aligned}$$

Bibliography

[1] S. Balac et L. Chupin. Analyse et algèbre. Cours de mathématiques de Deuxième année avec exercices corrigés et illustrations avec Maple.

[2] H. Boumaza et all. Mathématiques L3 Analyse. Cours complet avec 600 tests et exercices corrigés.

[3] J. Douchet (2014). Analyse. Recueil d'exercice et aide-mémoire vol.2.

[4] J. Douchet et B. Zwahlen. Calcul différentiel et intégral tome 2, Presses polytechniques et universitaires romandes (PPUR).

[5] J. Genet, et G. Pupion (1974): Analyse moderne, tome 2, Vuibert.

[6] R. Courant (1988). Differential and Integral Calculus. Vol. 2. Wiley Classics Library.

[7] W. Fulks (1978). Advanced calculus.

[8] S. Dineen (2001). Multivariate Calculus and Geometry. Springer Undergraduate Mathematics Series (2 ed.).

[9] M. A. Moskowitz, F. Paliogiannis (2011). Functions of Several Real Variables. World Scientific.