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Contents

Introduction

This textbook is specifically designed for first-year LMD students in Mathematics and Computer Science. Chapter 1 introduces the fundamental concepts of vector spaces, with classical examples from real, complex, and functional spaces. Chapter 2 focuses on linear mappings, including key notions such as kernel, image, and the rank-nullity theorem. Chapter 3 deals with matrices and their standard operations, such as multiplication, transposition, determinant calculation, invertibility, and matrix rank. Finally, Chapter 4 is dedicated to systems of linear equations and presents various methods of solution, including Gaussian elimination and Cramer's rule. This course also includes a wide range of solved exercises to help students reinforce their theoretical understanding through practice. We hope this document meets the students' expectations and effectively supports their academic success. At the end of this course, a list of references is provided for those who wish to explore the subject in greater depth.

Chapter 1

Vector spaces

In this course, the field $(\mathbb{K}, +, \cdot)$ denotes $\mathbb{R}$, $\mathbb{C}$, or any commutative field.

1.1 Definitions

Definition 1.1: Vector space

A vector space over a field $\mathbb{K}$ is a nonempty set E equipped with two operations:

1. **Vector addition (internal operation):** $E \times E \to E$, denoted by $(u, v) \mapsto u + v$.
2. **Scalar multiplication (external operation):** $\mathbb{K} \times E \to E$, denoted by $(\lambda, u) \mapsto \lambda u$.

These operations must satisfy the following axioms for all $u, v \in E$, and $\lambda, \mu \in \mathbb{K}$:

1. $(E, +)$ is an abelian group.
2. Scalar multiplication satisfies the following properties:
   (a) Distributivity over vector addition: $\lambda(u + v) = \lambda u + \lambda v$.
   (b) Distributivity over scalar addition: $(\lambda + \mu)u = \lambda u + \mu u$.
   (c) Compatibility of scalar multiplication: $\lambda(\mu u) = (\lambda\mu)u$.
   (d) Identity element of scalar multiplication: $1_{\mathbb{K}} u = u$, where $1_{\mathbb{K}}$ is the multiplicative identity in $\mathbb{K}$.

Rules of Calculation

The following properties hold in any vector space E over a field $\mathbb{K}$:

1. $\lambda \cdot x = 0_E \quad \Rightarrow \quad \lambda = 0_{\mathbb{K}}$, or $x = 0_E$.
2. For all $x \in E$ and all $\lambda \in \mathbb{K}$:
$$-(\lambda \cdot x) = (-\lambda) \cdot x = \lambda \cdot (-x).$$
3. For all $x \in E \setminus \{0_E\}$, and all $\lambda, \mu \in \mathbb{K}$:
$$\lambda \cdot x = \mu \cdot x \quad \Rightarrow \quad \lambda = \mu.$$
4. For all $x \in E$, and all $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$:
$$\sum_{k=1}^{n} (\lambda_k \cdot x) = \left(\sum_{k=1}^{n} \lambda_k \right) \cdot x.$$
5. For all $x_1, \ldots, x_n \in E$, and all $\lambda \in \mathbb{K}$:
$$\sum_{k=1}^{n} (\lambda \cdot x_k) = \lambda \cdot \left(\sum_{k=1}^{n} x_k \right).$$

Example 1.1: The Vector Space $\mathbb{R}^n$

The set $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\}$ is a vector space over the field $\mathbb{R}$, equipped with the following operations:

- **Vector addition:** For $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$, define:
$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, \ldots, u_n + v_n).$$
- **Scalar multiplication:** For $\lambda \in \mathbb{R}$ and $\mathbf{u} = (u_1, \ldots, u_n)$ in $\mathbb{R}^n$, define:
$$\lambda \cdot \mathbf{u} = (\lambda u_1, \ldots, \lambda u_n).$$

In particular:

- $\mathbb{R}^2$ is the 2-dimensional real vector space (the plane).
- $\mathbb{R}^3$ is the 3-dimensional real vector space.

All the vector space axioms (listed in Definition 1.1) are satisfied under these operations, so $\mathbb{R}^n$ is indeed a vector space over $\mathbb{R}$.

Example 1.2: Important Examples of Vector Spaces

Here are several important examples of vector spaces over common fields:

- **The Vector Space $\mathbb{C}^n$:** The set of all n-tuples of complex numbers:

  $$\mathbb{C}^n = \{(z_1, z_2, \dots, z_n) \mid z_i \in \mathbb{C}\}.$$

  It is a vector space over the field $\mathbb{C}$, with standard component-wise addition and scalar multiplication.

- **The Vector Space of Polynomials $\mathbb{R}[x]$:** The set of all real-coefficient polynomials:

  $$\mathbb{R}[x] = \left\{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mid a_i \in \mathbb{R},\ n \in \mathbb{N}\right\}.$$

  It is a vector space over $\mathbb{R}$, with usual polynomial addition and scalar multiplication.

- **The Vector Space of Continuous Functions $\mathcal{C}([a,b])$:** Let $\mathcal{C}([a,b])$ denote the set of all real-valued continuous functions defined on the closed interval $[a,b]$. Then:

  $$\mathcal{C}([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous}\}.$$

  It is a vector space over $\mathbb{R}$, with pointwise addition and scalar multiplication defined by:

  $$(f+g)(x) = f(x) + g(x),$$
  $$(\lambda f)(x) = \lambda \cdot f(x),$$

for all $f, g \in \mathcal{C}([a,b])$, $\lambda \in \mathbb{R}$, and $x \in [a,b]$.

Definition 1.2: Linear Combination

Let E be a vector space over a field $\mathbb{K}$, and let $x_1, x_2, \ldots, x_n \in E$ be vectors. A vector $x \in E$ is called a *linear combination* of $x_1, x_2, \ldots, x_n$ if there exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{K}$ such that:

$$x = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n.$$

In this case, we say that x is expressed as a linear combination of the vectors $x_1, x_2, \ldots, x_n$ with coefficients $\lambda_1, \ldots, \lambda_n$.

Example 1.3: Examples of Linear Combinations

Here are some examples of linear combinations in different vector spaces:

1. **In $\mathbb{R}^3$:**

   Let $\mathbf{x}_1 = (1, 0, 2)$, $\mathbf{x}_2 = (0, 1, -1)$. Then the vector

   $$\begin{aligned} \mathbf{x} &= 3\mathbf{x}_1 - 2\mathbf{x}_2 \\ &= 3(1,0,2) - 2(0,1,-1) \\ &= (3,0,6) + (0,-2,2) \\ &= (3,-2,8) \end{aligned}$$

   is a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$ with coefficients 3 and -2.

2. **In the space of polynomials $\mathbb{R}[x]$:**

   Let $p_1(x) = 1 + x$, $p_2(x) = x^2$. Then:

   $$\begin{aligned} p(x) &= 2p_1(x) - 5p_2(x) \\ &= 2(1+x) - 5x^2 \\ &= 2 + 2x - 5x^2 \end{aligned}$$

   is a linear combination of p_1 and p_2 in $\mathbb{R}[x]$.

3. **In the space of continuous functions $\mathcal{C}([0,1])$:**

Let $f_1(x) = x$, $f_2(x) = \sin(x)$. Then the function

$$f(x) = 4f_1(x) + \pi f_2(x) = 4x + \pi \sin(x)$$

is a linear combination of f_1 and f_2 in $\mathcal{C}([0,1])$.

1.2 Subspaces

Definition 1.3: Subspace of a Vector Space

Let E be a vector space over a field $\mathbb{K}$. A subset $F \subseteq E$ is called a *vector subspace* of E (or simply a *subspace*) if it satisfies the following conditions:

1. **Non-empty:** $F \neq \emptyset$.
2. **Closed under vector addition:** For all $u, v \in F$, we have $u + v \in F$.
3. **Closed under scalar multiplication:** For all $\lambda \in \mathbb{K}$ and all $u \in F$, we have $\lambda u \in F$.

Proposition 1.1: Equivalent Characterization of Subspaces

Let E be a vector space over a field $\mathbb{K}$ and $F \subseteq E$ a subset. Then F is a subspace of E if and only if:

1. $F \neq \emptyset$,
2. For all $\lambda, \mu \in \mathbb{K}$ and all $u, v \in F$, we have $\lambda u + \mu v \in F$.

Remark 1.1: Zero Vector in Subspaces

Every subspace F contains the zero vector $\mathbf{0}_E$ of E.

Example 1.4: Important Examples of Subspaces

Important examples of subspaces:

- **Trivial subspaces:**

  - The zero subspace $\{\mathbf{0}_E\}$.
  - The entire space E itself.

- **In $\mathbb{R}^3$:**

  The set

  $$F = \{(x, y, z) \in \mathbb{R}^3 \mid 2x + y - z = 0\}$$

  is a subspace of $\mathbb{R}^3$, while the set

  $$G = \{(x, y, z) \in \mathbb{R}^3 \mid 2x - y + 3z = 1\}$$

  is not a subspace (it fails to contain $\mathbf{0}$).

- **Polynomial subspaces:**

  For any field $\mathbb{K}$ and integer $n \geq 0$, the set

  $$\mathbb{K}_n[X] = \{P \in \mathbb{K}[X] \mid \deg(P) \leq n\}$$

  is a subspace of the polynomial space $\mathbb{K}[X]$.

- **Function spaces:**

  The set of all continuous functions $f \in \mathcal{C}([a, b])$ satisfying $f(a) = f(b) = 0$ forms a subspace of $\mathcal{C}([a, b])$.

Proposition 1.2: Properties of Subspaces

Let F be a subspace of a $\mathbb{K}$-vector space E. Then the following properties hold:

1. Any linear combination of vectors in F remains in F.
2. The intersection of any collection of subspaces of E is itself a subspace of E.
3. The union of two subspaces $F \cup G$ is a subspace if and only if $F \subseteq G$ or $G \subseteq F$.

Example 1.5: Union of Subspaces

Let $E_1 = \{(x, 0) \mid x \in \mathbb{R}\}$ and $E_2 = \{(0, y) \mid y \in \mathbb{R}\}$ be two subspaces of $\mathbb{R}^2$. The union $E_1 \cup E_2$ is not a vector space.

Proof 1.1

To verify that $E_1 \cup E_2$ is not a vector space, consider the vectors:

- $\mathbf{u} = (1, 0) \in E_1$.
- $\mathbf{v} = (0, 1) \in E_2$.

Their sum $\mathbf{u} + \mathbf{v} = (1, 1)$ does not belong to $E_1 \cup E_2$, because:

- $(1, 1) \notin E_1$ (second component is not zero).
- $(1, 1) \notin E_2$ (first component is not zero).

Therefore, $E_1 \cup E_2$ is not closed under vector addition, and hence not a subspace.

Definition 1.4: Sum of Subspaces

Let E be a vector space over a field $\mathbb{K}$, and let F and G be two subspaces of E. The *sum* of F and G, denoted $F + G$, is defined as:

$$F + G = \{u + v \mid u \in F,\ v \in G\}.$$

Proposition 1.3: Properties of the Sum

Let F and G be subspaces of a vector space E over $\mathbb{K}$. The sum $F + G$ has the following properties:

1. $F + G$ is a vector subspace of E.
2. $F + G$ is the smallest subspace of E containing both F and G.
3. $F \cup G \subseteq F + G$.

Example 1.6: Sum and Intersection of Subspaces in $\mathbb{R}^3$

In $\mathbb{R}^3$, consider the following subspaces:

- $F = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$ (the xy-plane).
- $G = \{(0, y, z) \mid y, z \in \mathbb{R}\}$ (the yz-plane).

Then their sum is:

$$F + G = \{(x, y, z) \mid x, y, z \in \mathbb{R}\} = \mathbb{R}^3,$$

while their intersection is:

$$F \cap G = \{(0, y, 0) \mid y \in \mathbb{R}\} \quad \text{(the } y\text{-axis)}.$$

Remark 1.2: Properties of Subspace Sum

Let F, G, and H be three vector subspaces of E. The following properties hold:

1. **Algebraic properties:**
    - $F + G = G + F$ (Commutativity).
    - $F + (G + H) = (F + G) + H$ (Associativity).
    - $F + \{\mathbf{0}_E\} = F$ (Identity element).
    - $F + E = E$.
    - $F + F = F$.
2. **Non-invertibility:** If $H = F + G$, the expression $F = H - G$ does not make sense. Subtraction of subspaces is not defined in vector spaces.
3. **Non-cancellation property:** If $F + G = F + H$, we cannot immediately conclude that $G = H$.

Example 1.7: Non-Cancellation Property of Subspace Sum

In $\mathbb{R}^2$, take:

- $F = \{(x, 0) \mid x \in \mathbb{R}\}$ (x-axis),
- $G = \{(0, y) \mid y \in \mathbb{R}\}$ (y-axis),

- $H = \{(y, y) \mid y \in \mathbb{R}\}$ (diagonal line).

Then $F + G = F + H = \mathbb{R}^2$, but clearly $G \neq H$.

Definition 1.5: Direct Sum of Subspaces

Let E_1 and E_2 be two vector subspaces of a vector space E over $\mathbb{K}$. We say that E_1 and E_2 are in *direct sum* in E (or that E_1 is *complementary* to E_2 in E) if one of the following equivalent conditions is satisfied:

1. **Trivial intersection and spanning:** $E_1 \cap E_2 = \{\mathbf{0}_E\}$ and $E_1 + E_2 = E$.
2. **Unique decomposition:** For every vector $w \in E$, there exists a unique pair $(u, v) \in E_1 \times E_2$ such that $w = u + v$.

In this case, we write $E = E_1 \oplus E_2$.

Remark 1.3

The two conditions are equivalent because:

1. Condition (1) ensures the decomposition exists $(E_1 + E_2 = E)$ and is unique (since $E_1 \cap E_2 = \{0_E\}$ prevents multiple representations).
2. Condition (2) implies that the intersection must be trivial, otherwise some vectors would have multiple decompositions.

Example 1.8: Direct Sum Decomposition of $\mathbb{R}^3$

Consider the following subspaces of $\mathbb{R}^3$:

$$E_1 = \left\{ \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} \middle| x, z \in \mathbb{R} \right\} \quad \text{(the } xz\text{-plane).}$$

$$E_2 = \left\{ \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} \middle| y \in \mathbb{R} \right\} \quad \text{(the } y\text{-axis).}$$

1. **Intersection Verification:**

$$E_1 \cap E_2 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

   The only vector common to both subspaces is the zero vector.

2. **Sum Verification:**

$$E_1 + E_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\} = \mathbb{R}^3.$$

   Any vector in $\mathbb{R}^3$ can be expressed as a sum of vectors from E_1 and E_2.

3. **Unique Decomposition:** For any $\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$, there exists a unique decomposition:

$$\mathbf{v} = \underbrace{\begin{pmatrix} a \\ 0 \\ c \end{pmatrix}}_{\in E_1} + \underbrace{\begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}}_{\in E_2}.$$

Conclusion:

$$\mathbb{R}^3 = E_1 \oplus E_2.$$

Example 1.9

The vector subspaces of $\mathbb{R}^3$:

$$F = \{(a, a, a) \in \mathbb{R}^3 \mid a \in \mathbb{R}\} \quad \text{and} \quad G = \{(x, y, z) \in \mathbb{R}^3 \mid x = y - 2z\}$$

are in direct sum, since if $(a, a, a) \in G$, then

$$a = a - 2a \quad \Rightarrow \quad a = 0,$$

and therefore

$$F \cap G = \{(0,0,0)\}.$$

Proposition 1.4: 1.12

Let E be a vector space over the field $\mathbb{K}$. Every vector subspace of E has at least one complementary vector subspace.

1.3 Generated subspace

Theorem 1.1

Let $v_1, \ldots, v_n$ be a finite set of vectors in a vector space E over a field $\mathbb{K}$. Then:

1. The set of all linear combinations of $v_1, \ldots, v_n$ is a subspace of E.
2. It is the smallest subspace of E (with respect to inclusion) containing $v_1, \ldots, v_n$.

Proof 1.2

1. Let

   $$W = \{\lambda_1 v_1 + \cdots + \lambda_n v_n \mid \lambda_i \in \mathbb{K}\}.$$

   To show that W is a subspace:

   - **Closed under addition:** If $u, w \in W$, then $u + w$ is also a linear combination of $v_1, \ldots, v_n$.
   - **Closed under scalar multiplication:** If $\alpha \in \mathbb{K}$ and $w \in W$, then $\alpha w \in W$.
   - **Contains zero:** $0 = 0v_1 + \cdots + 0v_n \in W$.

2. Let U be any subspace of E containing $v_1, \ldots, v_n$. Since U is closed under linear combinations, we have $W \subseteq U$. Thus, W is the smallest such subspace.

Notation 1.1

This subspace is called the subspace generated by $v_1, \dots, v_n$, and is denoted $\mathrm{Span}(v_1, \dots, v_n)$. Therefore, we have:

$$u \in \mathrm{Span}(v_1, \dots, v_n) \iff \exists \lambda_1, \dots, \lambda_n \in \mathbb{K} \text{ such that } u = \lambda_1 v_1 + \cdots + \lambda_n v_n.$$

Remark 1.4

1. Saying that $\mathrm{Span}(v_1, \dots, v_n)$ is the smallest subspace of E containing the vectors $v_1, \dots, v_n$ means that if F is a subspace of E containing $v_1, \dots, v_n$, then
$$\mathrm{Span}(v_1, \dots, v_n) \subseteq F.$$
2. More generally, we can define the subspace spanned by any subset V (not necessarily finite) of a vector space: $\mathrm{Span}(V)$ is the smallest subspace containing V.

Example 1.10

1. The vector subspace of $\mathbb{R}^3$ generated by the set
$$A = \{(-2, 0, 1), (3, 1, 1)\}$$
is
$$\begin{aligned}\mathrm{Span}(A) &= \{a(-2,0,1) + b(3,1,1) \mid a, b \in \mathbb{R}\} \\ &= \{(-2a + 3b,\ b,\ a + b) \mid a, b \in \mathbb{R}\}.\end{aligned}$$
2. In $\mathbb{C}_2[X]$, let the vectors $u = iX - X^2$ and $v = 1 + i + X$. Then
$$\begin{aligned}\mathrm{Span}(u, v) &= \{\beta(iX - X^2) + \gamma(1 + i + X) \mid \beta, \gamma \in \mathbb{C}\} \\ &= \{\gamma(1 + i) + (\beta i + \gamma)X - \beta X^2 \mid \beta, \gamma \in \mathbb{C}\}.\end{aligned}$$

Theorem 1.2

1. If a vector w is a linear combination of the vectors $(v_1, \dots, v_n)$, then
$$\mathrm{Span}(v_1, \dots, v_n, w) = \mathrm{Span}(v_1, \dots, v_n).$$

2. If a vector w is a linear combination of the vectors $(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n)$, then

$$\text{Span}(v_1, \dots, v_{k-1}, v_k + w, v_{k+1}, \dots, v_n) = \text{Span}(v_1, \dots, v_n).$$

Proof 1.3

1. For part (1):

   - ($\subseteq$) Any vector in $\text{Span}(v_1, \dots, v_n, w)$ can be written as

     $$\sum_{i=1}^{n} \lambda_i v_i + \mu w.$$

     Since w is a linear combination of $v_1, \dots, v_n$, this reduces to a linear combination of $v_1, \dots, v_n$.
   - ($\supseteq$) Trivial, since $\{v_1, \dots, v_n\} \subseteq \{v_1, \dots, v_n, w\}$.

2. For part (2): Let

   $$S_1 = \text{Span}(v_1, \dots, v_{k-1}, v_k + w, v_{k+1}, \dots, v_n), \quad S_2 = \text{Span}(v_1, \dots, v_n).$$

   - ($S_1 \subseteq S_2$): Since
     $$v_k + w = v_k + \sum_{i \neq k} \alpha_i v_i,$$
     it follows that $v_k + w \in S_2$. Hence $S_1 \subseteq S_2$.
   - ($S_2 \subseteq S_1$): We can write

     $$v_k = (v_k + w) - w,$$

     where $w \in S_1$ by construction. Therefore, $v_k \in S_1$, and all other $v_i \in S_1$. Thus $S_2 \subseteq S_1$.

Theorem 1.3

Let A and B be two subsets of a $\mathbb{K}$-vector space E. Then:

1. $A \subseteq B \implies \text{Span}(A) \subseteq \text{Span}(B)$.

2. $\text{Span}(A \cup B) = \text{Span}(A) + \text{Span}(B)$.

Proof 1.4

1. If $A \subseteq B$, then any linear combination of elements from A is also a linear combination of elements from B. Thus $\text{Span}(A) \subseteq \text{Span}(B)$.

2. For the equality:

   - ($\subseteq$) Any vector in $\text{Span}(A \cup B)$ can be written as a linear combination of vectors from both A and B, hence it belongs to $\text{Span}(A) + \text{Span}(B)$.
   - ($\supseteq$) Since $A, B \subseteq A \cup B$, by part (1) we have $\text{Span}(A), \text{Span}(B) \subseteq \text{Span}(A \cup B)$, and thus their sum is also contained in $\text{Span}(A \cup B)$.

Definition 1.6

A family of vectors $S = \{v_i,\ i \in I\}$ in a vector space E is called a **generating family** (or **spanning set**) if every vector in E can be expressed as a linear combination of the vectors in S, i.e.,

$$\text{Span}(S) = E.$$

Mathematically, S generates E if

$$\forall v \in E,\ \exists\, \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{K}, \quad \text{such that} \quad v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n.$$

Example 1.11

In $\mathbb{R}^3$, the set

$$\{(1,0,0),\ (0,1,0),\ (0,0,1)\}$$

is a generating family of $\mathbb{R}^3$.
Also, the set

$$\{(1,1,0),\ (0,1,1),\ (1,0,1)\}$$

is a generating family as well.

Example 1.12

Consider the vector space of polynomials of degree at most 4, denoted by $\mathbb{R}_4[X]$, which consists of all polynomials of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4.$$

A generating set for this space is the set

$$\{1, x, x^2, x^3, x^4\}.$$

Remark 1.5

The generating set of a $\mathbb{K}$-vector space is not unique.

Definition 1.7: Free Family (Linearly Independent)

Let E be a $\mathbb{K}$-vector space and let $\{v_1, \dots, v_n\}$ be a family of vectors in E. We say that the family $\{v_1, \dots, v_n\}$ is **free** (or **linearly independent**) in E if

$$\forall \lambda_1, \dots, \lambda_n \in \mathbb{K}, \quad \lambda_1 v_1 + \cdots + \lambda_n v_n = 0_E \ \Rightarrow\ \lambda_1 = \cdots = \lambda_n = 0_{\mathbb{K}}.$$

Example 1.13: Standard basis in $\mathbb{R}^n$

The canonical basis vectors form a free family in $\mathbb{R}^n$:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}.$$

This family is free because the equation

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \lambda_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = 0$$

implies $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.

Example 1.14: Polynomials in $\mathbb{R}[x]$

The family $\{1, x, x^2, \dots, x^n\}$ is free in $\mathbb{R}[x]$.
Indeed, for any linear combination

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0,$$

we must have

$$a_0 = a_1 = \cdots = a_n = 0,$$

since a nonzero polynomial of degree at most n cannot vanish identically.

Definition 1.8: Dependent Family

Let E be a $\mathbb{K}$-vector space and let $\{v_1, \dots, v_n\}$ be a family of vectors in E.
We say that the family $\{v_1, \dots, v_n\}$ is **dependent** (or **linearly dependent**) if it is not free. This means that there exist scalars $\lambda_1, \dots, \lambda_n \in \mathbb{K}$, not all zero, such that

$$\lambda_1 v_1 + \cdots + \lambda_n v_n = 0_E.$$

Example 1.15: Linearly dependent vectors in $\mathbb{R}^2$

Consider the following two vectors in $\mathbb{R}^2$:

$$v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

These vectors are linearly dependent because

$$v_2 - 2v_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Theorem 1.4: Characterization of Linear Dependence

Let $n \geq 2$. The family $\{v_1, v_2, \dots, v_n\}$ is linearly dependent if and only if one of the vectors $v_1, v_2, \dots, v_n$ is a linear combination of the others.

1.4 Basis

Definition 1.9: Basis

A family of vectors $\{x_i\}_{i\in I}$ in a vector space E over a field $\mathbb{K}$ is called a **basis** of E if the following two conditions are satisfied:

1. The family is linearly independent.
2. The family spans E; i.e., every vector in E can be expressed as a linear combination of the vectors x_i.

Example 1.16: Examples of bases

1. The set $\{(1,0),(0,1)\}$ is the standard basis for $\mathbb{R}^2$.
2. The polynomials $\{1,t,t^2\}$ form a basis for the space of quadratic polynomials $\mathbb{R}_2[X]$.

Remark 1.6

Observations:

1. The empty set $\varnothing$ is a basis for the zero vector space $\{0\}$.
2. The basis of a given vector space is not unique.

Theorem 1.5: Unique Representation

Let $B=\{e_1,e_2,\ldots,e_n\}$ be a basis for a vector space E over a field $\mathbb{K}$. Then every vector $v\in E$ can be written uniquely as

$$v=\sum_{i=1}^{n}\lambda_i e_i,\qquad \lambda_i\in\mathbb{K}.$$

The scalars λ_i are called the **coordinates** of v with respect to the basis B.

Proposition 1.5: Properties of Linear Independence and Span

1. A singleton $\{x\}$ is linearly independent if and only if $x \neq 0$.
2. Any family that contains a spanning set is itself a spanning set.
3. A subset of a linearly independent set is linearly independent.
4. Any set containing a linearly dependent subset is itself linearly dependent.
5. If a set $\{v_1, \ldots, v_p\}$ contains the zero vector, it is linearly dependent.

Theorem 1.6: Basis of a Direct Sum

Let E be a vector space over $\mathbb{K}$, and let F and G be nontrivial subspaces of E (i.e., $F, G \neq \{0\}$).

1. If B_F is a basis for F and B_G is a basis for G, then $B_F \cup B_G$ spans $F + G$.
2. If $F \cap G = \{0\}$ (direct sum), then $B_F \cup B_G$ is a basis for $F \oplus G$.

Example 1.17: Direct Sum in $\mathbb{R}^4$

Consider the following vector subspaces in $\mathbb{R}^4$:

$$F = \text{Span}\{(1, -1, 0, 2)\}, \quad G = \text{Span}\{(-2, 5, 3, 1), (1, 1, -2, -2)\}.$$

Since $(1, -1, 0, 2) \neq 0$ in $\mathbb{R}^4$, and the vectors $(-2, 5, 3, 1)$ and $(1, 1, -2, -2)$ are linearly independent, it follows that:

- $\{(1, -1, 0, 2)\}$ is a basis for F,
- $\{(-2, 5, 3, 1), (1, 1, -2, -2)\}$ is a basis for G.

Therefore, the set

$$\{(1, -1, 0, 2),\ (-2, 5, 3, 1),\ (1, 1, -2, -2)\}$$

is a generating set for $F + G$.

1.5 Finite-Dimensional Vector Spaces

Definition 1.10: Finite and Infinite Dimensional Spaces

1. A vector space E over a field $\mathbb{K}$ is called **finite-dimensional** if it has a finite generating set; that is, there exists a finite family of vectors $\{v_1, v_2, \dots, v_n\}$ in E such that $\text{Span}\{v_1, v_2, \dots, v_n\} = E$.
2. A vector space E is called **infinite-dimensional** if it is not finite-dimensional; this means that every generating set of E is infinite.

Example 1.18

1. The vector spaces $\mathbb{R}^n$ for $n \in \mathbb{N}$ and $\mathbb{R}_n[X]$ for $n \in \mathbb{N}$ are finite-dimensional.
2. The vector space $\mathbb{R}[X]$ is infinite-dimensional.

Theorem 1.7: Dimension of a Vector Space

Let E be a finite-dimensional vector space over a field $\mathbb{K}$. Then all bases of E have the same cardinality. This common number is called the **dimension** of E, and is denoted by $\dim(E)$ or $\dim_{\mathbb{K}}(E)$.

Theorem 1.8: Existence of a Basis

Every non-trivial finite-dimensional vector space $E \neq \{0\}$ over a field $\mathbb{K}$ admits a basis.

Corollary 1.1: Properties of Dimension

Let E be a finite-dimensional vector space over a field $\mathbb{K}$ with $\dim(E) = n$. Then:

1. Every linearly independent set in E has at most n elements.
2. Every generating set of E has at least n elements.

Remark 1.7: Basis Verification in Finite-Dimensional Spaces

Let E be a finite-dimensional vector space with $\dim(E) = n$. To show that a set of n vectors is a basis of E, it is sufficient to prove that the set is either linearly independent or a generating set for E.

Theorem 1.9: Incomplete Basis Theorem

Let E be a finite-dimensional vector space over a field $\mathbb{K}$, and let L be a linearly independent subset of E. Then there exists a basis B of E such that $L \subseteq B$ and B has finite cardinality.

Theorem 1.10: Grassmann's Formula

Let E be a finite-dimensional vector space over a field $\mathbb{K}$, and let F and G be two subspaces of E. Then

$$\dim(F+G) = \dim(F) + \dim(G) - \dim(F \cap G).$$

In particular, F and G are in direct sum (i.e., $F+G = F \oplus G$) if and only if

$$\dim(F+G) = \dim(F) + \dim(G).$$

Theorem 1.11: Subspace Dimension Theorem

Let E be a finite-dimensional vector space. If F is a subspace of E, then:

1. F is finite-dimensional.
2. $\dim(F) \leq \dim(E)$.
3. $\dim(F) = \dim(E)$ if and only if $F = E$.

Remark 1.8

The dimension of the zero vector space is 0.

Example 1.19: Finite and Infinite Dimensional Spaces

1. $\dim(\mathbb{K}^n) = n$.
2. $\dim(\mathbb{K}_n[X]) = n+1$.
3. $\dim(\mathbb{K}[X])$ is infinite.
4. $\dim_{\mathbb{C}} \mathbb{C} = 1$ and $\dim_{\mathbb{R}} \mathbb{C} = 2$.

Theorem 1.12: Characterization of Supplementary Subspaces

Let F and G be two subspaces of a finite-dimensional vector space E. Then F and G are supplementary in E (i.e., $E = F \oplus G$) if and only if any two of the following three conditions hold:

1. $\dim F + \dim G = \dim E$,
2. $F \cap G = \{0\}$,
3. $F + G = E$.

1.6 Exercises

Exercise 1.1: Vector Space Verification

Determine whether $\mathbb{R}^2$, equipped with the following operations, forms an $\mathbb{R}$-vector space.

(a) For all $(a,b),(c,d) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$:

$$(a,b)+(c,d)=(a+c,b+d), \quad \lambda \cdot (a,b)=(\lambda a,b).$$

(b) For all $(a,b),(c,d) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$:

$$(a,b)+(c,d)=(a+c,b+d), \quad \lambda \cdot (a,b)=(\lambda^2 a,\lambda^2 b).$$

(c) For all $(a,b),(c,d) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$:

$$(a,b)+(c,d)=(a+c,b+d), \quad \lambda \cdot (a,b)=(\lambda a,\lambda b).$$

Exercise 1.2: Subspaces Verification

Among the following sets, determine which are, or are not, vector subspaces.

1. $E_1 = \{(x,y,z) \in \mathbb{R}^3 \mid x+2y+z=0\}$.
2. $E_2 = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=5\}$.
3. $E_3 = \{(x,y,z,t) \in \mathbb{R}^4 \mid x=2y=3z=4t\}$.
4. $E_4 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$.
5. $E_5 = \{(x,y) \in \mathbb{R}^2 \mid y=x^3\}$.
6. $E_6 = \{(x,y,z) \in \mathbb{R}^3 \mid x+2y+3z=0\} \setminus \{(x,y,z) \in \mathbb{R}^3 \mid x-y+z=0\}$.
7. $E_7 = \{(x,y,z) \in \mathbb{R}^3 \mid x+2y+3z=0\} \cup \{(x,y,z) \in \mathbb{R}^3 \mid x-y+z=0\}$.
8. $E_8 = \{P \in \mathbb{R}[X] \mid P(0)=P(1)\}$.
9. $E_9 = \{P \in \mathbb{R}[X] \mid \deg(P) \leq 2\}$.

10. $E_{10} = \{P \in \mathbb{R}[X] \mid P'(X) \text{ divides } P(X)\}$.
11. $E_{11} = \{f \in F(\mathbb{R}, \mathbb{R}) \mid f \text{ is bounded}\}$.
12. $E_{12} = \{f \in F(\mathbb{R}, \mathbb{R}) \mid f \text{ is bounded below}\}$.
13. $E_{13} = \{f \in C^1(\mathbb{R}, \mathbb{R}) \mid f' + 3f = 0\}$.
14. $E_{14} = \{f \in C^1([a, b], \mathbb{R}) \mid \int_a^b f(t)\, dt = 0\}$.

Exercise 1.3

1. Let the vector $t = (2, 1, 0, -3)$. Does this vector belong to the subspace of $\mathbb{R}^4$ spanned by the vectors:

$$u = (2, 3, 1, 0), \quad v = (1, -1, 2, 3), \quad w = (0, 1, 3, -1)?$$

2. Let G be the subspace of $\mathbb{R}^3$ spanned by the vectors $(3, 1, -2)$ and $(1, -1, 4)$. Prove that there exist real numbers a, b, c such that

$$G = \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = 0\}.$$

Exercise 1.4

Let E be the set of all convergent real sequences, F the set of all real sequences converging to zero, and G the set of all constant real sequences.

1. Prove that E, F, and G are subspaces of $\mathbb{R}^{\mathbb{N}}$.
2. Prove that $E = F \oplus G$.

Exercise 1.5

Are the following families linearly independent?

1. $\{(1, -2, 3),\ (0, 1, -1),\ (2, 3, 5)\}$ in $\mathbb{R}^3$.
2. $\{(3, -1, 4),\ (1, 0, -2),\ (5, -3, 10)\}$ in $\mathbb{R}^3$.
3. $\{(1, 0, 0),\ (0, 2, 0),\ (1, 2, 3),\ (2, -1, -1)\}$ in $\mathbb{R}^3$.
4. $\{(2, -1, 0, 3),\ (0, 1, -2, -1),\ (5, -2, 4, -3)\}$ in $\mathbb{R}^4$.

5. $\{1,\ X,\ 1+X^2\}$ in $\mathbb{R}_2[X]$.
6. $\{1,\ X^2-X,\ X^3+2X\}$ in $\mathbb{R}_3[X]$.

Exercise 1.6

Let F, G, and H be three subspaces of a $\mathbb{K}$-vector space E such that

$$F+G=F+H, \quad F\cap G=F\cap H, \quad G\subseteq H.$$

Prove that $G=H$.

Exercise 1.7

Show that the vectors

$$v_1=(0,1,1), \quad v_2=(1,0,1), \quad v_3=(1,1,0)$$

form a basis of $\mathbb{R}^3$.
Find the components of the vector

$$w=(1,1,1)$$

in this basis (v_1, v_2, v_3).

Exercise 1.8: Subspaces and Bases in $\mathbb{R}^3$

Let

$$E=\{(x,y,z)\in\mathbb{R}^3 \mid x+y-z=0 \text{ and } x-y-z=0\},$$

$$F=\{(x,y,z)\in\mathbb{R}^3 \mid x+y-2z=0\}.$$

be two subsets of $\mathbb{R}^3$. Assume that F is a subspace of $\mathbb{R}^3$. Let

$$a=(1,0,1), \quad b=(1,1,1), \quad c=(0,2,1).$$

1. Show that E is a subspace of $\mathbb{R}^3$.
2. Determine a generating family of E and show that this family is a basis.

3. Show that $\{b, c\}$ is a basis of F.
4. Show that $\{a, b, c\}$ is a linearly independent family in $\mathbb{R}^3$.
5. Prove that $E \oplus F = \mathbb{R}^3$.
6. Let $u = (x, y, z)$. Express u in the basis $\{a, b, c\}$.

Exercise 1.9: Spanning and Completing Bases in $\mathbb{R}^3$

Let

$$E = \text{span}(a, b, c, d)$$

be a subspace of $\mathbb{R}^3$, where

$$a = (2, -1, -1), \quad b = (-1, 2, 3), \quad c = (1, 4, 7), \quad d = (1, 1, 2).$$

1. Is (a, b, c, d) a basis of $\mathbb{R}^3$?
2. Show that (a, b) is a basis of E.
3. Determine one or more equations characterizing E.
4. Complete a basis of E to form a basis of $\mathbb{R}^3$.

1.7 Additional exercises

Exercise 1.10: Subspaces and Direct Sum in $\mathbb{R}^4$

We consider the subset

$$F = \{(x, y, z, t) \in \mathbb{R}^4 \mid x + y = 0 \text{ and } x + z = 0\}.$$

1. Show that F is a subspace of $\mathbb{R}^4$ and provide a basis for F.
2. Complete the basis found in part (1) to form a basis of $\mathbb{R}^4$.
3. Let
$$u_1 = (1, 1, 1, 1), \quad u_2 = (1, 2, 3, 4), \quad u_3 = (-1, 0, -1, 0).$$

Show whether the family $\{u_1, u_2, u_3\}$ is linearly independent.

4. Let G be the vector space spanned by u_1, u_2, u_3. Determine $\dim G$.
5. Provide a basis for $F \cap G$. Deduce that $F + G = \mathbb{R}^4$.
6. Can every vector in $\mathbb{R}^4$ be written uniquely as the sum of a vector from F and a vector from G?

Exercise 1.11: Subspaces and Bases in $\mathbb{R}^4$

Let

$$E = \{(x, y, z, t) \in \mathbb{R}^4 \mid x + y + z - t = 0,\ x - 2y + 2z + t = 0,\ x - y + z = 0\}$$

and assume that E is a vector space. Let

$$F = \{(x, y, z, t) \in \mathbb{R}^4 \mid 2x + 6y + 7z - t = 0\}.$$

Let

$$a = (2, 1, -1, 2), \quad b = (1, 1, -1, 1), \quad c = (-1, -2, 3, 7), \quad d = (4, 4, -5, -3)$$

be vectors in $\mathbb{R}^4$.

1. (a) Determine a basis for E and deduce its dimension.
   (b) Complete this basis to form a basis of $\mathbb{R}^4$.
2. (a) Show that F is a subspace of $\mathbb{R}^4$.
   (b) Determine a basis for F.
   (c) Determine whether $E + F = \mathbb{R}^4$.
3. (a) Show that $F = \text{Span}\{b, c, d\}$.
   (b) Let $u = (x, y, z, t) \in F$. Express u as a linear combination of b, c, and d.

Exercise 1.12: Subspaces, Bases, and Direct Sum in $\mathbb{R}^3$

Let

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid x + y - z = 0 \text{ and } x - y - z = 0\}$$

and

$$F = \{(x, y, z) \in \mathbb{R}^3 \mid x + y - 2z = 0\}$$

be two subsets of $\mathbb{R}^3$. Assume that F is a subspace of $\mathbb{R}^3$. Let

$$a = (1, 0, 1), \quad b = (1, 1, 1), \quad c = (0, 2, 1).$$

1. Show that E is a subspace of $\mathbb{R}^3$.
2. Determine a generating family for E and show that this family forms a basis.
3. Show that $\{b, c\}$ is a basis for F.
4. Show that $\{a, b, c\}$ is a linearly independent set in $\mathbb{R}^3$.
5. Prove that $E + F = \mathbb{R}^3$.
6. Let $u = (x, y, z)$. Express u in the basis $\{a, b, c\}$.

Exercise 1.13: Bases and Subspaces in $\mathbb{R}^3$

Let

$$u_1 = (1, -1, 2), \quad u_2 = (1, 1, -1), \quad u_3 = (-1, -5, -7), \quad E = \text{Span}\{u_1, u_2, u_3\}.$$

Let

$$F = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}.$$

1. Provide a basis for E.
2. Show that F is a subspace of $\mathbb{R}^3$.
3. Provide a basis for F.
4. Provide a basis for $E \cap F$.

1.8 Exercise Solutions

Solution 1.1

Let us analyze each case separately to determine if $(\mathbb{R}^2, +, \cdot)$ forms a vector space over $\mathbb{R}$.

(a)

The internal operation (vector addition) is the standard one, which satisfies all the required axioms. However, the external operation (scalar multiplication) is defined as:

$$\lambda \cdot (a, b) = (\lambda a, b).$$

This structure is **not** a vector space, because it violates the following vector space axiom:

- Distributivity of scalar multiplication with respect to field addition:

  $$(\lambda + \mu) \cdot (a, b) = ((\lambda + \mu)a, b) \neq \lambda \cdot (a, b) + \mu \cdot (a, b) = ((\lambda + \mu)a, 2b),$$

  unless $b = 0$.

(b)

The scalar multiplication is defined as:

$$\lambda \cdot (a, b) = (\lambda^2 a, \lambda^2 b).$$

This structure is **not** a vector space, because the scalar multiplication is not linear with respect to field addition:

$$(\lambda+\mu)\cdot(a, b) = ((\lambda+\mu)^2 a, (\lambda+\mu)^2 b) \neq \lambda\cdot(a, b)+\mu\cdot(a, b) = ((\lambda^2+\mu^2)a, (\lambda^2+\mu^2)b).$$

Thus, it is not a vector space.

(c)

The operations are defined as:

$$(a, b) + (c, d) = (a + c, b + d), \quad \lambda \cdot (a, b) = (\lambda a, \lambda b).$$

This is the standard vector space structure on $\mathbb{R}^2$. All vector space axioms are satisfied:

- Associativity and commutativity of addition.
- Existence of additive identity $(0, 0)$ and additive inverses $(-a, -b)$.
- Distributivity and compatibility of scalar multiplication.
- Scalar identity $1 \cdot (a, b) = (a, b)$ holds.

Thus, this is a vector space over $\mathbb{R}$.

Solution 1.2

Determine which of the given sets are vector subspaces.

1. E_1: **Subspace.** Closed under addition and scalar multiplication (solution set of a homogeneous linear equation).
2. E_2: **Not a subspace.** Does not contain the zero vector, since $0 + 0 + 0 \neq 5$.
3. E_3: **Subspace.** The conditions $x = 2y = 3z = 4t$ define a line through the origin in $\mathbb{R}^4$.
4. E_4: **Not a subspace.** Not closed under addition; for example, $(1, 0)+(0, 1) = (1, 1)$ is not on the unit circle.
5. E_5: **Not a subspace.** Contains the zero vector, but not closed under addition. For example, $(1, 1) \in E_5$ and $(2, 8) \in E_5$, but $(1 + 2, 1 + 8) = (3, 9) \notin E_5$, because $9 \neq 3^3 = 27$.
6. E_6: **Subspace.** Intersection of two planes through the origin (solution space of a homogeneous system).
7. E_7: **Not a subspace.** The union of two planes is not closed under addition (take vectors from different planes).
8. E_8: **Subspace.** The condition $P(0) = P(1)$ is linear and preserved under polynomial operations.

9. E_9: **Subspace.** Polynomials of degree ≤ 2 are closed under addition and scalar multiplication.
10. E_{10}: **Not a subspace.** The divisibility condition $P' \mid P$ is not preserved under addition; for example, X^2 and X^3 satisfy it, but their sum does not.
11. E_{11}: **Subspace.** Bounded functions form a subspace of $F(\mathbb{R}, \mathbb{R})$.
12. E_{12}: **Not a subspace.** Functions bounded below are not closed under scalar multiplication, since multiplying by -1 gives functions unbounded above.
13. E_{13}: **Subspace.** Solutions of the linear differential equation $f' + 3f = 0$ form a vector space.
14. E_{14}: **Subspace.** The integral condition $\int_a^b f(t)\,dt = 0$ is linear and preserved under addition and scalar multiplication.

Solution 1.3

1. To determine if $t = (2, 1, 0, -3)$ belongs to the span of $\{u, v, w\}$, we solve:

$$t = \alpha u + \beta v + \gamma w.$$

This gives the system:

$$\begin{cases} 2\alpha + \beta = 2, \\ 3\alpha - \beta + \gamma = 1, \\ \alpha + 2\beta + 3\gamma = 0, \\ 3\beta - \gamma = -3. \end{cases}$$

Solving: from the first equation, $\beta = 2 - 2\alpha$. Substituting into the fourth equation gives $\gamma = 9 - 6\alpha$. Plugging these into the third equation yields $\alpha = 1$, leading to $\beta = 0$ and $\gamma = 3$. These values satisfy all equations; therefore, t is in the span.

2. To find the equation of the plane G spanned by $(3, 1, -2)$ and $(1, -1, 4)$:

(a) **Parametric approach:** any vector in G can be written as

$$(x, y, z) = \lambda(3, 1, -2) + \mu(1, -1, 4) = (3\lambda + \mu, \lambda - \mu, -2\lambda + 4\mu).$$

(b) **Eliminate parameters:** from the first two components,

$$x = 3\lambda + \mu, \quad y = \lambda - \mu.$$

Solving for λ and μ,

$$\lambda = \frac{x+y}{4}, \quad \mu = \frac{x-3y}{4}.$$

(c) **Substitute into the third component:**

$$z = -2\lambda + 4\mu = -2\frac{x+y}{4} + 4\frac{x-3y}{4} = \frac{2x-14y}{4}.$$

Simplifying gives

$$2x - 14y - 4z = 0, \quad \text{or equivalently,} \quad x - 7y - 2z = 0.$$

Thus, G is the solution space of $x - 7y - 2z = 0$.

Solution 1.4

1. **Subspace verification:**
All three sets are subsets of the vector space $\mathbb{R}^{\mathbb{N}}$ (the space of all real sequences). We verify the subspace conditions:

- For E (convergent sequences):
  - Contains the zero sequence, which converges to 0.
  - If $(x_n), (y_n) \in E$ with limits L_x, L_y, then $(x_n + y_n)$ converges to $L_x + L_y$.
  - For $\alpha \in \mathbb{R}$, (αx_n) converges to αL_x.
- For F (sequences converging to 0):
  - Contains the zero sequence.
  - If $(x_n), (y_n) \in F$, then $(x_n + y_n)$ converges to $0 + 0 = 0$.
  - For $\alpha \in \mathbb{R}$, (αx_n) converges to $\alpha \cdot 0 = 0$.
- For G (constant sequences):
  - Contains the zero sequence.

- If $(x_n) = (c_1)$ and $(y_n) = (c_2)$ are constant, then $(x_n + y_n) = (c_1 + c_2)$ is constant.
- For $\alpha \in \mathbb{R}$, $(\alpha x_n) = (\alpha c_1)$ is constant.

2. **Direct sum decomposition $E = F \oplus G$:**
We must show:

- $E = F + G$; every convergent sequence can be written as the sum of a sequence converging to 0 and a constant sequence.
- $F \cap G = \{0\}$; the only constant sequence converging to 0 is the zero sequence.

Proof:
For any $(x_n) \in E$ with limit L, write:

$$x_n = (x_n - L) + L,$$

where $(x_n - L) \in F$ (converges to 0), and $L \in G$ (constant sequence).
If $(z_n) \in F \cap G$, then it is constant and converges to 0, so $z_n = 0$ for all n.
Thus, E is the direct sum of F and G.

Solution 1.5

1. Consider the linear combination:

$$\lambda_1(1, -2, 3) + \lambda_2(0, 1, -1) + \lambda_3(2, 3, 5) = (0, 0, 0).$$

This gives the system:

$$\begin{cases} \lambda_1 + 2\lambda_3 = 0, \\ -2\lambda_1 + \lambda_2 + 3\lambda_3 = 0, \\ 3\lambda_1 - \lambda_2 + 5\lambda_3 = 0. \end{cases}$$

The only solution is $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Thus, the family is linearly independent.
2. For the vectors $(3, -1, 4), (1, 0, -2), (5, -3, 10)$:

$$(5, -3, 10) = 2(3, -1, 4) - (1, 0, -2).$$

This non-trivial linear combination equals zero; therefore, the family is linearly dependent.
3. Any four vectors in $\mathbb{R}^3$ must be linearly dependent by dimension considerations. Thus, the family is linearly dependent.
4. For the vectors in $\mathbb{R}^4$:

$$\lambda_1(2,-1,0,3)+\lambda_2(0,1,-2,-1)+\lambda_3(5,-2,4,-3)=(0,0,0,0).$$

Solving the resulting system shows that the only solution is $\lambda_1=\lambda_2=\lambda_3=0$. Thus, the family is linearly independent.
5. In $\mathbb{R}_2[X]$, consider:

$$a\cdot 1+bX+c(1+X^2)=0.$$

This implies $a+c=0$, $b=0$, and $c=0$. The only solution is $a=b=c=0$; so the family is linearly independent.
6. In $\mathbb{R}_3[X]$, the polynomials $1, X^2-X, X^3+2X$ have distinct degrees. Any linear combination:

$$a\cdot 1+b(X^2-X)+c(X^3+2X)=0$$

must have $a=b=c=0$, by comparing coefficients. Thus, the family is linearly independent.

Solution 1.6

We will prove that $G=H$ under the given conditions.
Given:

1. $F+G=F+H$;
2. $F\cap G=F\cap H$;
3. $G\subset H$.

To show: $G=H$.
Proof:
Since $G\subset H$, by condition (3), we only need to prove that $H\subset G$.
Let $h\in H$ be arbitrary. From condition (1), since $h\in H\subset F+H=F+G$, there

exist $f \in F$ and $g \in G$ such that

$$h = f + g.$$

Now, since $G \subset H$ and $h \in H$, we have $f = h - g \in H$ (because H is a subspace). Thus,

$$f \in F \cap H.$$

By condition (2), $F \cap H = F \cap G$, so $f \in F \cap G$. This implies $f \in G$ (since $F \cap G \subset G$).

Therefore,

$$h = f + g \in G + G = G,$$

because G is a subspace (closed under addition).

Since $h \in H$ was arbitrary, we conclude that $H \subset G$.

Together with condition (3), $G \subset H$, this proves that $G = H$.

Solution 1.7

Part 1: Proving that $\{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$.

To show that the vectors $v_1 = (0, 1, 1)$, $v_2 = (1, 0, 1)$, and $v_3 = (1, 1, 0)$ form a basis for $\mathbb{R}^3$, it is sufficient to verify either linear independence or that they span $\mathbb{R}^3$, since we have three vectors in a three-dimensional space.

Method: Checking linear independence.

Consider the linear combination:

$$\alpha v_1 + \beta v_2 + \gamma v_3 = 0,$$

which gives the system:

$$\begin{cases} 0\alpha + 1\beta + 1\gamma = 0, \\ 1\alpha + 0\beta + 1\gamma = 0, \\ 1\alpha + 1\beta + 0\gamma = 0. \end{cases}$$

Solving this system:

From the first equation: $\beta + \gamma = 0 \Rightarrow \beta = -\gamma$;

from the second equation: $\alpha + \gamma = 0 \Rightarrow \alpha = -\gamma$;

substitute into the third equation: $(-\gamma) + (-\gamma) = -2\gamma = 0 \Rightarrow \gamma = 0$.

Thus, $\alpha = \beta = \gamma = 0$. The only solution is the trivial one, proving that the vectors are linearly independent.
Since we have three linearly independent vectors in $\mathbb{R}^3$, they form a basis.
Part 2: Finding the components of $w = (1, 1, 1)$.
We need to find scalars x, y, z such that:

$$w = xv_1 + yv_2 + zv_3,$$

which gives the system:

$$\begin{cases} 0x + 1y + 1z = 1, \\ 1x + 0y + 1z = 1, \\ 1x + 1y + 0z = 1. \end{cases}$$

Solving:
From the first equation: $y + z = 1$;
from the second equation: $x + z = 1$;
from the third equation: $x + y = 1$.
Subtract (2) from (1): $y - x = 0 \Rightarrow y = x$;
substitute into (3): $x + x = 1 \Rightarrow x = \frac{1}{2}$;
thus, $y = \frac{1}{2}$;
from (2): $\frac{1}{2} + z = 1 \Rightarrow z = \frac{1}{2}$.
Therefore, the coordinates of w in the basis $\{v_1, v_2, v_3\}$ are:

$$w = \frac{1}{2}v_1 + \frac{1}{2}v_2 + \frac{1}{2}v_3.$$

Solution 1.8

1. Showing that E is a subspace.

- **Contains zero vector:** $(0, 0, 0)$ satisfies both equations $x + y - z = 0$ and $x - y - z = 0$.
- **Closed under addition:** For $(x_1, y_1, z_1), (x_2, y_2, z_2) \in E$:

$$(x_1+x_2)+(y_1+y_2)-(z_1+z_2) = 0, \quad (x_1+x_2)-(y_1+y_2)-(z_1+z_2) = 0.$$

- **Closed under scalar multiplication:** For $\alpha \in \mathbb{R}$ and $(x, y, z) \in E$:

$$\alpha x + \alpha y - \alpha z = 0, \quad \alpha x - \alpha y - \alpha z = 0.$$

Thus, E is a subspace.
2. Basis for E: Solve the system:

$$\begin{cases} x + y - z = 0, \\ x - y - z = 0. \end{cases}$$

Adding equations: $2x - 2z = 0 \Rightarrow x = z$. Substituting: $y = 0$. Therefore,

$$E = \{(z, 0, z) \mid z \in \mathbb{R}\}.$$

A basis is $\{(1, 0, 1)\}$, which is linearly independent and generates E.
3. Basis for F: The plane $x + y - 2z = 0$ has dimension 2. Verify:

$$b = (1, 1, 1) \in F : \ 1 + 1 - 2(1) = 0;$$

$$c = (0, 2, 1) \in F : \ 0 + 2 - 2(1) = 0.$$

Linear independence: $\alpha(1, 1, 1) + \beta(0, 2, 1) = 0$ has only $\alpha = \beta = 0$. Hence, $\{b, c\}$ is a basis of F.
4. Linear independence of $\{a, b, c\}$: Solve $\alpha(1, 0, 1) + \beta(1, 1, 1) + \gamma(0, 2, 1) = 0$:

$$\begin{cases} \alpha + \beta = 0, \\ \beta + 2\gamma = 0, \\ \alpha + \beta + \gamma = 0. \end{cases}$$

The only solution is $\alpha = \beta = \gamma = 0$. So the family is linearly independent.
5. Direct sum $E \oplus F = \mathbb{R}^3$:

- $E \cap F = \{0\}$: The only solution to $x = z, \ y = 0, \ x + y - 2z = 0$ is $(0, 0, 0)$.
- $\dim E + \dim F = 1 + 2 = 3 = \dim \mathbb{R}^3$.

Therefore, $E \oplus F = \mathbb{R}^3$.

6. Expressing $u = (x, y, z)$ in the basis $\{a, b, c\}$: Solve

$$\alpha(1, 0, 1) + \beta(1, 1, 1) + \gamma(0, 2, 1) = (x, y, z),$$

which gives the system:

$$\begin{cases} \alpha + \beta = x, \\ \beta + 2\gamma = y, \\ \alpha + \beta + \gamma = z. \end{cases}$$

Solution:

$$\gamma = z - x, \quad \beta = y - 2(z - x), \quad \alpha = x - (y - 2(z - x)).$$

Thus,

$$u = (3x - y - 2z)a + (y - 2z + 2x)b + (z - x)c.$$

Solution 1.9

1. Is (a, b, c, d) a basis of $\mathbb{R}^3$?

No. $\mathbb{R}^3$ has dimension 3, so any basis must consist of exactly three linearly independent vectors. Here we have four vectors, which are necessarily linearly dependent; therefore, they cannot form a basis.

2. Show that (a, b) is a basis of E:

Step 1: Check linear independence. Solve $\alpha(2, -1, -1) + \beta(-1, 2, 3) = (0, 0, 0)$:

$$\begin{cases} 2\alpha - \beta = 0, \\ -\alpha + 2\beta = 0, \\ -\alpha + 3\beta = 0. \end{cases}$$

The only solution is $\alpha = \beta = 0$. Hence, a and b are linearly independent.

Step 2: Express c and d as linear combinations of a and b:

$$c = 2a + 3b = (1, 4, 7), \quad d = a + b = (1, 1, 2).$$

Hence, $E = \text{span}(a, b)$, and (a, b) is a basis.

3. Determine equations characterizing E:

For $(x, y, z) \in E:$

$$(x, y, z) = \alpha(2, -1, -1) + \beta(-1, 2, 3),$$

which gives

$$x = 2\alpha - \beta, \quad y = -\alpha + 2\beta, \quad z = -\alpha + 3\beta.$$

Solving for α and $\beta:$

$$\alpha = \frac{2x + y}{3}, \quad \beta = \frac{x + 2y}{3}.$$

Substitute into $z:$

$$z = \frac{x + 5y}{3} \implies x + 5y - 3z = 0.$$

Thus,

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid x + 5y - 3z = 0\}.$$

4. Complete to a basis of $\mathbb{R}^3$:

Choose a vector not in E, e.g., $v = (1, 0, 0):$

$$1 + 5 \cdot 0 - 3 \cdot 0 = 1 \neq 0.$$

Check linear independence of $(a, b, v):$

$$\det \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ -1 & 3 & 0 \end{pmatrix} = -1 \neq 0.$$

Hence, (a, b, v) is a basis of $\mathbb{R}^3$. One possible completion: $(a, b, (1, 0, 0))$.

Chapter 2

Linear Mappings

Linear Mappings

Linear mappings are the cornerstone of linear algebra, providing a systematic framework to study structure-preserving transformations between vector spaces. This chapter explores their fundamental properties, classifications, and applications across mathematics.

A linear map $f : E \to F$ is characterized by its ability to commute with vector space operations:

$$f(\alpha u + v) = \alpha f(u) + f(v) \quad \forall u, v \in E,\ \alpha \in \mathbb{K}.$$

We will examine key examples—from differentiation and integration to matrix transformations—and establish tools to analyze their behavior through kernels, images, and the rank-nullity theorem.

Key Objectives

- Characterize linearity via additivity and homogeneity.
- Compute kernels and images, and relate their dimensions.
- Classify maps by injectivity, surjectivity, and isomorphisms.
- Construct linear maps from basis assignments.
- Apply rank inequalities to solve dimension constraints.

Notation 2.1

- $\mathcal{L}(E, F)$: space of linear maps $E \to F$;
- $\ker f$: kernel (null space) of f;
- $\operatorname{Im} f$: image of f;
- $\operatorname{rg}(f)$: rank of f.

2.1 Definitions

Definition 2.1: Linear Mapping

Let E, F be two vector spaces over a field $\mathbb{K}$. A linear mapping (or linear transformation) $f : E \to F$ is a mapping that satisfies the following two properties:

1. **Additivity:** For any vectors $u, v \in E$,

$$f(u + v) = f(u) + f(v).$$

2. **Homogeneity:** For any vector $u \in E$ and any scalar $c \in \mathbb{K}$,

$$f(cu) = cf(u).$$

Let's look at some examples of linear maps.

Example 2.1: Linear Maps

1. **Zero linear map:** For vector spaces E and F over the same field $\mathbb{K}$, the zero linear map is defined by

$$0 : E \to F, \quad 0(x) = 0_F \text{ for all } x \in E,$$

where 0_F denotes the zero vector in F.

2. **Identity map:** On a vector space E, the identity map is given by

$$\mathrm{Id}_E : E \to E, \quad \mathrm{Id}_E(x) = x \text{ for all } x \in E.$$

This map is linear and is sometimes denoted simply by I when the space is clear from context.

3. **Differentiation:** Let $P(\mathbb{R})$ be the space of real polynomials. Define

$$D \in \mathcal{L}(P(\mathbb{R})) \text{ by } D(p) = p'.$$

Differentiation is a linear map.

4. **Integration:** Define

$$T \in \mathcal{L}(P(\mathbb{R}), \mathbb{R}) \text{ by } T(p) = \int_0^1 p(x)\,dx.$$

Integration is also a linear map.

5. **Matrix-type linear map:** Define

$$T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2) \text{ by } T(x, y, z) = (2x - y + 3z,\ 7x + 5y - 6z).$$

This corresponds to multiplication by the matrix

$$\begin{pmatrix} 2 & -1 & 3 \\ 7 & 5 & -6 \end{pmatrix}.$$

Theorem 2.1: Linear Map Characterization

Let E and F be two $\mathbb{K}$-vector spaces, and let $f : E \to F$ be a function. Then f is a linear map if and only if

$$\forall x, y \in E,\ \forall \lambda \in \mathbb{K}, \quad f(\lambda x + y) = \lambda f(x) + f(y).$$

Remark 2.1: Linear Map Terminology

1. If a linear map f is injective, we say that f is a *monomorphism*.
2. If a linear map f is surjective, we say that f is an *epimorphism*.
3. If a linear map f is bijective, we say that f is an *isomorphism* of vector spaces, and that E and F are *isomorphic*.

4. A bijective endomorphism is called an *automorphism*.
5. The set of all linear mappings from E to F is denoted by $\mathcal{L}(E,F)$.
6. The set of linear mappings from E to itself is denoted by $\mathcal{L}(E)$; in other words, $\mathcal{L}(E)=\mathcal{L}(E,E)$.

Proposition 2.1: Vector Space of Linear Maps

Let E and F be vector spaces over $\mathbb{K}$. The set $\mathcal{L}(E,F)$ of all linear maps from E to F, equipped with:

- Addition: $(f+g)(x)=f(x)+g(x)$;
- Scalar multiplication: $(\lambda f)(x)=\lambda f(x)$,

forms a vector space over $\mathbb{K}$.

Proposition 2.2: Composition of Linear Maps

Let E, F, G be vector spaces over $\mathbb{K}$. If $f\in\mathcal{L}(E,F)$ and $g\in\mathcal{L}(F,G)$, then their composition $g\circ f$ belongs to $\mathcal{L}(E,G)$.

Theorem 2.2

Let f be a linear map. Then:

1. $f(0_E)=0_F$;
2. If A is a subspace of E, then the restriction $f|_A$ is a linear map on A;
3. For all $x\in E$, $f(-x)=-f(x)$;
4. For any scalars $\alpha_1,\dots,\alpha_n$ and vectors $x_1,\dots,x_n\in E$,

$$f\Big(\sum_{i=1}^{n}\alpha_i x_i\Big)=\sum_{i=1}^{n}\alpha_i f(x_i).$$

Theorem 2.3: Existence and Uniqueness of a Linear Map from Basis Assignment

Let E and F be two vector spaces over the same field $\mathbb{K}$. Suppose E is finite-dimensional with dimension n, and let $(e_1, \dots, e_n)$ be a basis of E. Then for any choice of n vectors $(v_1, \dots, v_n)$ in F, there exists a unique linear map $f : E \to F$ such that, for all $i = 1, \dots, n$,

$$f(e_i) = v_i.$$

This theorem makes no assumption about the dimension of the target vector space F.

Proof 2.1

Uniqueness. Suppose there exists a linear map $f : E \to F$ such that $f(e_i) = v_i$ for all $i = 1, \dots, n$. For any $x \in E$, there exist unique scalars $x_1, \dots, x_n$ such that

$$x = \sum_{i=1}^{n} x_i e_i.$$

Since f is linear, we have

$$f(x) = f\Big(\sum_{i=1}^{n} x_i e_i\Big) = \sum_{i=1}^{n} x_i f(e_i) = \sum_{i=1}^{n} x_i v_i. \qquad (*)$$

Therefore, if such f exists, it must be unique.

Existence. Define the map $f : E \to F$ by equation (*) as

$$f\Big(\sum_{i=1}^{n} x_i e_i\Big) = \sum_{i=1}^{n} x_i v_i.$$

We show that this map is linear and satisfies $f(e_i) = v_i$.

Let $(x_1, \dots, x_n)$ (respectively, $(y_1, \dots, y_n)$) be the coordinates of x (respectively,

y) in the basis $(e_1, \dots, e_n)$. Then, for any scalars $\alpha, \beta \in \mathbb{K}$,

$$\begin{aligned} f(\alpha x + \beta y) &= f\Big(\sum_{i=1}^{n} (\alpha x_i + \beta y_i) e_i\Big) \\ &= \sum_{i=1}^{n} (\alpha x_i + \beta y_i) v_i \\ &= \alpha \sum_{i=1}^{n} x_i v_i + \beta \sum_{i=1}^{n} y_i v_i \\ &= \alpha f(x) + \beta f(y). \end{aligned}$$

Finally, since the coordinates of e_i are $(0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the i-th position, we have

$$f(e_i) = 1 \cdot v_i = v_i.$$

This completes the proof of the theorem.

Example 2.2

There exists a unique linear map $f : \mathbb{R}^n \to \mathbb{R}[X]$ such that

$$f(e_i) = (X+1)^i, \quad \text{for } i = 1, \dots, n,$$

where $(e_1, \dots, e_n)$ is the standard basis of $\mathbb{R}^n$.
For a vector $x = (x_1, \dots, x_n)$, we have

$$f(x_1, \dots, x_n) = f\Big(\sum_{i=1}^{n} x_i e_i\Big) = \sum_{i=1}^{n} x_i f(e_i) = \sum_{i=1}^{n} x_i (X+1)^i.$$

Theorem 2.4: Characterization of Injectivity and Surjectivity via Basis Image

Let $f : E \to F$ be a linear map, where E has a basis $(e_i)_{i \in I}$. Then:

1. f is surjective if and only if $F = \text{span}\{f(e_i)\}_{i \in I}$;
2. f is injective if and only if $(f(e_i))_{i \in I}$ is linearly independent;
3. f is bijective if and only if $(f(e_i))_{i \in I}$ is a basis of F.

Example 2.3: Surjective Map

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x + y$. Consider the basis $(e_1, e_2) = ((1, 0), (0, 1))$ of $\mathbb{R}^2$:

- $f(e_1) = 1$, $f(e_2) = 1$;
- $\text{Vect}\{f(e_1), f(e_2)\} = \text{Vect}\{1\} = \mathbb{R}$.

Thus, f is surjective.

Example 2.4: Injective Map

Consider $g : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $g(x, y) = (x, y, 0)$:

- $g(e_1) = (1, 0, 0)$, $g(e_2) = (0, 1, 0)$;
- The set $\{g(e_1), g(e_2)\}$ is linearly independent.

Hence, g is injective.

Example 2.5: Bijective Map

Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $h(x, y) = (y, x)$:

- $h(e_1) = e_2$, $h(e_2) = e_1$;
- The set $\{h(e_1), h(e_2)\} = \{e_2, e_1\}$ forms a basis of $\mathbb{R}^2$.

Therefore, h is bijective by Theorem 2.9(iii).

Theorem 2.5: Equivalence of Injectivity, Surjectivity, and Bijectivity

Let E and F be finite-dimensional $\mathbb{K}$-vector spaces of the same dimension, and let $f : E \to F$ be a linear transformation. Then:

$$f \text{ is bijective} \iff f \text{ is injective} \iff f \text{ is surjective.}$$

2.2 Image and Kernel of a Linear map

Definition 2.2: Kernel of a Linear Map

Let $f : E \to F$ be a linear map. The **kernel** of f, denoted by $\ker f$, is the set defined by

$$\ker f = \{u \in E \mid f(u) = 0_F\}.$$

It is sometimes also denoted by $f^{-1}(0)$.

Example 2.6

Determine the kernel of the linear map f_1, defined by

$$f_1 : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto x + 2y.$$

We are looking for

$$\ker f_1 = \{(x, y) \in \mathbb{R}^2 \mid f_1(x, y) = 0\} = \{(x, y) \in \mathbb{R}^2 \mid x + 2y = 0\}.$$

Solving the equation $x + 2y = 0$, we get

$$x = -2y.$$

Thus, every element in the kernel has the form

$$(x, y) = (-2y, y) = y(-2, 1), \quad y \in \mathbb{R}.$$

Therefore,

$$\ker f_1 = \text{Span}\{(-2, 1)\}.$$

This means that the kernel is a vector subspace of $\mathbb{R}^2$ of dimension 1, and a basis of $\ker f_1$ is $\{(-2, 1)\}$.

Proposition 2.3: Preimage of a Subspace

Let $f : E \to F$ be a linear map. Then:

1. More generally, if $B \subset F$ is a subspace, then the preimage

$$f^{-1}(B) = \{x \in E \mid f(x) \in B\}$$

is a subspace of E. In particular, since $\{0\} \subset F$ is a subspace, it follows that

$$\ker f = f^{-1}(\{0\})$$

is a subspace of E.

2. The map f is injective if and only if

$$\ker f = \{0\}.$$

Example 2.7: Kernel as a Special Case

For the linear map $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$f(x, y, z) = (x + y,\ y - z),$$

the kernel is

$$\ker f = \{(x, y, z) \mid x + y = 0,\ y - z = 0\} = \{(-y, y, y) \mid y \in \mathbb{R}\}.$$

This line through the origin shows that $\ker f$ is indeed a subspace, as it is the preimage of $\{0\}$.

Example 2.8: Injectivity Criterion

Define $g : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$g(x, y) = (x, 2y, x + y).$$

- $\ker g = \{(0, 0)\}$, since $g(x, y) = (0, 0, 0)$ implies $x = y = 0$;
- By Proposition 2.12(ii), g is injective;
- Visually, no two distinct input vectors map to the same output.

Contrast with $h : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$h(x, y) = (x + y,\, x + y,\, 0),$$

where

$$\ker h = \{(x, -x) \mid x \in \mathbb{R}\} \neq \{(0, 0)\}.$$

Thus, h is not injective, showing that the kernel criterion works.

Definition 2.3: Image of a Linear Map

Let $f : E \to F$ be a linear map between $\mathbb{K}$-vector spaces. The **image** of f, denoted $\mathrm{Im}(f)$, is the subset of F defined by

$$\mathrm{Im}(f) := \{f(x) \mid x \in E\} = f(E).$$

Example 2.9: (Finding the Image of a Linear Transformation)

Consider the linear transformation

$$f_2 : \mathbb{R}^3 \to \mathbb{R}^3$$

defined by

$$f_2(x, y, z) = (-x + y,\ x - z,\ y).$$

Image Calculation. The image of f_2 is

$$\begin{aligned} \mathrm{Im}\, f_2 &= \{\, f_2(x, y, z) \mid (x, y, z) \in \mathbb{R}^3 \,\} \\ &= \{\, (-x + y, x - z, y) \mid (x, y, z) \in \mathbb{R}^3 \,\} \\ &= \{\, x(-1, 1, 0) + y(1, 0, 1) + z(0, -1, 0) \mid (x, y, z) \in \mathbb{R}^3 \,\}. \end{aligned}$$

Thus, $\mathrm{Im} f_2$ is a subspace of $\mathbb{R}^3$ generated by the vectors

$$\{(-1, 1, 0), (1, 0, 1), (0, -1, 0)\}.$$

Basis and Dimension. It is easy to show that this family of vectors is linearly independent and forms a basis for $\mathbb{R}^3$. Therefore,

$$\dim \operatorname{Im} f_2 = 3 \quad \text{and} \quad \operatorname{Im} f_2 = \mathbb{R}^3.$$

Theorem 2.6: (Image of a Vector Subspace under a Linear Map)

Let f be a linear map from E to F.

1. If A is a vector subspace of E, then $f(A)$ is a vector subspace of F. In particular, $\operatorname{Im}(f) = f(E)$ is a vector subspace of F.
2. f is surjective if and only if $\operatorname{Im}(f) = F$.

Example 2.10

For $n \geq 1$, consider the linear map

$$f : \mathbb{R}_n[X] \to \mathbb{R}_{n-1}[X], \quad P \mapsto P',$$

where P' denotes the derivative of P.
The image of f is

$$\begin{aligned}\operatorname{Im}(f) &= \{P' \mid P \in \mathbb{R}_n[X]\} \\ &= \{a_1 + 2a_2 X + \cdots + n a_n X^{n-1} \mid a_1, \ldots, a_n \in \mathbb{R}\} \\ &= \mathbb{R}_{n-1}[X].\end{aligned}$$

which shows that f is surjective.
Moreover,

$$\ker(f) = \{P \in \mathbb{R}_n[X] \mid P' = 0\} = \mathbb{R}_0[X],$$

the space of constant polynomials. Thus, for $n \geq 1$, f is not injective.

Theorem 2.7: (Image of a Span under a Linear Map)

Let f be a linear map from E to F. For any subset X of E:

$$f(\operatorname{Span}(X)) = \operatorname{Span}(f(X)).$$

In particular, if E has a basis $(e_i)_{i\in I}$:

$$\mathrm{Im}(f) = \mathrm{Span}\{f(e_i) \mid i \in I\}.$$

Proof 2.2

(1) To prove the equality $f(\mathrm{Span}(X)) = \mathrm{Span}(f(X))$:

(i) Let $y \in f(\mathrm{Span}(X))$. Then there exist $x_1, \ldots, x_n \in X$ and scalars $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ such that

$$y = f\Big(\sum_{i=1}^{n} \lambda_i x_i\Big).$$

By linearity of f,

$$y = \sum_{i=1}^{n} \lambda_i f(x_i) \in \mathrm{Span}(f(X)).$$

Hence, $f(\mathrm{Span}(X)) \subseteq \mathrm{Span}(f(X))$.

(ii) Conversely, let $y \in \mathrm{Span}(f(X))$. Then there exist $x_1, \ldots, x_n \in X$ and scalars $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ such that

$$y = \sum_{i=1}^{n} \lambda_i f(x_i) = f\Big(\sum_{i=1}^{n} \lambda_i x_i\Big).$$

Since $\sum_{i=1}^{n} \lambda_i x_i \in \mathrm{Span}(X)$, we have $y \in f(\mathrm{Span}(X))$. Thus, $\mathrm{Span}(f(X)) \subseteq f(\mathrm{Span}(X))$.

Combining (i) and (ii), we get $f(\mathrm{Span}(X)) = \mathrm{Span}(f(X))$.

(2) For the particular case $X = \{e_i\}_{i\in I}$, a basis of E, we have

$$\mathrm{Span}(\{e_i\}_{i\in I}) = E, \quad \text{hence} \quad f(E) = \mathrm{Im}(f) = \mathrm{Span}\{f(e_i) \mid i \in I\}.$$

Example 2.11

Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $f(x, y, z) = (2x - y,\ y + z)$.
Take $X = \{(1, 0, 0), (0, 1, 0)\}$.

1. **Computing both sides:**

$$\begin{aligned} f(\text{Span}(X)) &= f(\{\, a(1,0,0) + b(0,1,0) \mid a, b \in \mathbb{R} \,\}) \\ &= f(\{\, (a, b, 0) \mid a, b \in \mathbb{R} \,\}) \\ &= \mathbb{R}^2. \end{aligned}$$

$$\text{Span}(f(X)) = \text{Span}\{f(1,0,0), f(0,1,0)\} = \text{Span}\{(2,-1), (0,1)\} = \mathbb{R}^2.$$

2. **Using the standard basis** $\{e_1, e_2, e_3\}$:

$$\text{Im}(f) = \text{Span}\{f(e_1), f(e_2), f(e_3)\} = \text{Span}\{(2,0), (-1,1), (0,1)\} = \mathbb{R}^2.$$

2.3 Rank-Nullity Theorem

Definition 2.4: (Rank of a Linear Map)

Let f be a linear map from E to F. We say that f has **finite rank** if $\text{Im}(f)$ is finite-dimensional, and **infinite rank** otherwise. When f has finite rank, the **rank** of f, denoted by $\text{rg}(f)$, is defined as

$$\text{rg}(f) = \dim(\text{Im}(f)).$$

Theorem 2.8: (Rank–Nullity Theorem)

Let $f : E \to F$ be a linear map between $\mathbb{K}$-vector spaces. If E is finite-dimensional, then

$$\dim E = \dim(\ker f) + \text{rg}(f).$$

Example 2.12

Consider the differentiation map:

$$D : \mathbb{R}_3[X] \to \mathbb{R}_2[X], \quad P \mapsto P'.$$

- $\dim \mathbb{R}_3[X] = 4$ (basis: $\{1, X, X^2, X^3\}$);
- $\ker D = \{P \mid P' = 0\} = \mathbb{R}_0[X]$, with $\dim \ker D = 1$;
- $\mathrm{Im}(D) = \mathbb{R}_2[X]$, with $\mathrm{rg}(D) = 3$.

Verification: $4 = 1 + 3$ confirms the theorem.

Theorem 2.9: (Rank Inequalities)

Let $f : E \to F$ be a linear map between $\mathbb{K}$-vector spaces.

1. If F is finite-dimensional, then f has finite rank and

$$\mathrm{rg}(f) \leq \dim F.$$

   Moreover, f is surjective if and only if $\mathrm{rg}(f) = \dim F$.

2. If E is finite-dimensional, then f has finite rank and

$$\mathrm{rg}(f) \leq \dim E.$$

   Moreover, f is injective if and only if $\mathrm{rg}(f) = \dim E$.

Proof 2.3

1. Since $\mathrm{Im}(f) \subset F$ is a subspace, if $\dim F < \infty$, then

$$\mathrm{rg}(f) = \dim(\mathrm{Im}(f)) \leq \dim F.$$

   Surjectivity is equivalent to $\mathrm{Im}(f) = F$, which implies equality of dimensions.

2. When $\dim E < \infty$, the Rank–Nullity Theorem gives

$$\operatorname{rg}(f) = \dim E - \dim(\ker f) \leq \dim E.$$

Injectivity ($\ker f = \{0\}$) occurs precisely when $\dim(\ker f) = 0$, yielding $\operatorname{rg}(f) = \dim E$.

Example 2.13

Consider the projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$\pi(x, y, z) = (x, y).$$

- $\dim F = 2$, $\operatorname{rg}(\pi) = 2 \leq 2$ (surjective);
- $\dim E = 3$, $\operatorname{rg}(\pi) = 2 < 3$ (not injective).

2.4 Exercises

Exercise 2.1

Are the following applications from E to F linear? If so, determine a basis of the kernel and a basis of the image.

1. $E = F = \mathbb{R}^2$; for all $(x, y) \in \mathbb{R}^2$:
$$f(x, y) = (2x + 3y, x)$$
2. $E = F = \mathbb{R}^2$; for all $(x, y) \in \mathbb{R}^2$:
$$f(x, y) = (y, x + y + 1)$$
3. $E = \mathbb{R}^3$, $F = \mathbb{R}$; for all $(x, y, z) \in \mathbb{R}^3$:
$$f(x, y, z) = x + 2y + z$$
4. $E = F = \mathbb{R}^2$; for all $(x, y) \in \mathbb{R}^2$:
$$f(x, y) = (x + y, xy)$$
5. $E = F = \mathbb{R}$; for all $x \in \mathbb{R}$:
$$f(x) = x^2$$

Exercise 2.2

Consider the linear map:

$$f : \mathbb{R}[X] \to \mathbb{R}[X]$$
$$P \mapsto P + XP',$$

where P' denotes the derivative of P.

1. Verify that f is indeed a linear transformation.
2. Determine the kernel $\ker f$ of this mapping.

3. Characterize the image $\operatorname{Im}(f)$ of this transformation.
4. Is f injective? Surjective? Justify your answers.

Exercise 2.3

Let f be a function defined from $\mathbb{R}^3$ to $\mathbb{R}^3$ by:

$$f(x, y, z) = (2x + y,\ y - z,\ x - y).$$

1. Show that f is a linear map.
2. Find $\ker f$ and $\operatorname{Im} f$, and determine their dimensions. Is f bijective?
3. Determine $f \circ f$.

Exercise 2.4

We denote $B = \{e_1, e_2, e_3\}$ the canonical basis of $\mathbb{R}^3$, and f the endomorphism in $\mathbb{R}^3$ defined by:

$$f(e_1) = -2e_1 + 2e_3, \quad f(e_2) = 3e_2, \quad f(e_3) = -4e_1 + 4e_3.$$

1. Let $u = (x, y, z) \in \mathbb{R}^3$. Calculate $f(u)$.
2. Determine a basis of $\ker f$.
3. Is f injective? Can it be surjective? Why?
4. Determine a basis of $\operatorname{Im} f$. Deduce the rank of f.
5. Show that $\mathbb{R}^3 = \ker f \oplus \operatorname{Im} f$.

Exercise 2.5

Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear map such that:

$$(1, 2, 0) \in \ker(f), \quad f(0, 0, 1) = (1, 0), \quad f(0, t, 0) = (t, t), \quad \forall t \in \mathbb{R}.$$

1. Determine the expression of $f(x, y, z)$.

2. Find a basis and the dimension of $\ker(f)$ and $\operatorname{Im}(f)$.
3. Find all vectors whose image is the vector $(0, 1)$. Is this set a vector subspace of $\mathbb{R}^3$?

Exercise 2.6

Let E, F, and G be three vector spaces over a field $\mathbb{K}$, and let $f \in \mathcal{L}(E, F)$ and $g \in \mathcal{L}(F, G)$ be linear maps. Prove that:

$$g \circ f = 0 \iff \operatorname{Im}(f) \subset \ker(g).$$

Exercise 2.7

Let E be a real vector space of dimension 3, and let $f \in \mathcal{L}(E)$ be an endomorphism such that:

$$f^2 \neq 0 \quad \text{and} \quad f^3 = 0.$$

Let $x_0 \in E$ such that $f^2(x_0) \neq 0$.

1. Show that the set $B = \{x_0, f(x_0), f^2(x_0)\}$ is a basis of E.
2. Let $g \in \mathcal{L}(E)$ such that $g \circ f = f \circ g$.
   (a) Show that there exist scalars $\alpha, \beta, \gamma \in \mathbb{R}$ such that:
   $$g(x_0) = \alpha x_0 + \beta f(x_0) + \gamma f^2(x_0).$$
   (b) Show that:
   $$g = \alpha \mathrm{Id}_E + \beta f + \gamma f^2.$$
   (c) Deduce the set of endomorphisms of E that commute with f.

2.5 Exercise Solutions

Solution 2.1

1. Let $f(x, y) = (2x + 3y, x)$. We check linearity:

$$f(x_1 + x_2, y_1 + y_2) = (2(x_1 + x_2) + 3(y_1 + y_2), x_1 + x_2)$$

$$= (2x_1 + 2x_2 + 3y_1 + 3y_2, x_1 + x_2) = f(x_1, y_1) + f(x_2, y_2).$$

$$f(\lambda x, \lambda y) = (2\lambda x + 3\lambda y, \lambda x) = \lambda f(x, y).$$

Thus, f is linear.

To find the kernel:

$$f(x, y) = (0, 0) \Rightarrow \begin{cases} 2x + 3y = 0, \\ x = 0 \end{cases} \Rightarrow x = 0 \Rightarrow y = 0.$$

So, ker $f = \{(0, 0)\}$, and a basis is $\emptyset$ (or $\{0\}$).

To find the image, we compute the image of a basis of $\mathbb{R}^2$.

Let $B_E = \{e_1 = (1, 0), e_2 = (0, 1)\}$.

Then:

$$f(e_1) = f(1, 0) = (2, 1), \quad f(e_2) = f(0, 1) = (3, 0).$$

Let us check whether these two vectors are linearly independent. Assume $\alpha(2, 1) + \beta(3, 0) = (0, 0)$:

$$(2\alpha + 3\beta, \alpha) = (0, 0) \Rightarrow \begin{cases} 2\alpha + 3\beta = 0, \\ \alpha = 0 \end{cases} \Rightarrow \beta = 0.$$

So, the vectors $(2, 1)$ and $(3, 0)$ are linearly independent. Therefore, they form a basis of Im(f).

Since there are two linearly independent vectors in $\mathbb{R}^2$, we have:

$$\dim(\text{Im}(f)) = 2 \Rightarrow \text{Im}(f) = \mathbb{R}^2.$$

So, a basis of the image is $\{(2,1),(3,0)\}$.

2. $f(x,y) = (y, x+y+1)$. Check linearity:

   $$f(0,0) = (0,1) \neq (0,0).$$

   So, f is not linear.

3. $f(x,y,z) = x+2y+z$. This is a linear form, since it is a linear combination of the inputs.

   To find the kernel of f, we solve:

   $$f(x,y,z) = 0 \Rightarrow x+2y+z = 0.$$

   Let $y = s$ and $z = t$, where $s,t \in \mathbb{R}$ are free parameters. Then:

   $$x = -2s - t.$$

   So, the general solution of the kernel is:

   $$(x,y,z) = (-2s-t, s, t) = s(-2,1,0) + t(-1,0,1).$$

   Hence, the kernel is the span of two linearly independent vectors:

   $$\ker f = \text{Span}\{(-2,1,0),(-1,0,1)\}.$$

   Therefore, a basis of the kernel is:

   $$\{(-2,1,0),(-1,0,1)\}.$$

   The image is $\mathbb{R}$, since any real number can be written as a linear combination of $x+2y+z$. A basis of the image is $\{1\}$.

4. $f(x, y) = (x + y, xy)$. Check linearity:

$$f(1,1) = (2,1), \quad f(2,0) = (2,0), \quad f(1,1) + f(2,0) = (4,1),$$

while

$$f((1,1) + (2,0)) = f(3,1) = (4,3) \neq (4,1).$$

So, f is not linear.

5. $f(x) = x^2$. This is not linear, since:

$$f(x + y) = (x + y)^2 = x^2 + 2xy + y^2 \neq f(x) + f(y).$$

So, f is not linear.

Solution 2.2

Consider the linear map:

$$f : \mathbb{R}[X] \to \mathbb{R}[X], \quad P \mapsto P + XP',$$

where P' denotes the derivative of P.

1. **Linearity of f:**

Let $P(X), Q(X) \in \mathbb{R}[X]$ and let $\lambda \in \mathbb{R}$. We have:

$$\begin{aligned} f(P + Q) &= (P + Q) + X(P + Q)' \\ &= P + Q + X(P' + Q') \\ &= (P + XP') + (Q + XQ') \\ &= f(P) + f(Q), \end{aligned}$$

and

$$f(\lambda P) = \lambda P + X(\lambda P)' = \lambda P + \lambda XP' = \lambda(P + XP') = \lambda f(P).$$

Hence, f is linear.

2. **Kernel of f:**

We want to find all polynomials P such that:

$$f(P) = P + XP' = 0.$$

Write $P(X) = a_0 + a_1X + a_2X^2 + \cdots + a_nX^n$. Then:

$$P'(X) = a_1 + 2a_2X + \cdots + na_nX^{n-1},$$

and

$$XP'(X) = a_1X + 2a_2X^2 + \cdots + na_nX^n.$$

So:

$$\begin{aligned} P + XP' &= a_0 + (a_1 + a_1)X + (a_2 + 2a_2)X^2 + \cdots + (a_n + na_n)X^n \\ &= a_0 + 2a_1X + 3a_2X^2 + \cdots + (n+1)a_nX^n. \end{aligned}$$

Set this equal to the zero polynomial:

$$a_0 = 0, \quad 2a_1 = 0, \quad 3a_2 = 0, \quad \ldots, \quad (n+1)a_n = 0.$$

This implies:

$$a_0 = a_1 = a_2 = \cdots = a_n = 0.$$

Therefore, the only solution is the zero polynomial:

$$\ker f = \{0\}.$$

3. **Image of f:**

   Let us compute f on each basis element:

$$f(1) = 1 + X \cdot 0 = 1, \quad f(X) = X + X \cdot 1 = 2X,$$

$$f(X^2) = X^2 + X \cdot 2X = 3X^2, \quad f(X^3) = X^3 + X \cdot 3X^2 = 4X^3.$$

$$\vdots$$

$$f(X^n) = X^n + X \cdot nX^{n-1} = (n+1)X^n.$$

So:

$$f(1)=1,\quad f(X)=2X,\quad f(X^2)=3X^2,\quad \dots,\quad f(X^n)=(n+1)X^n.$$

We see that f sends each basis monomial X^k to a scalar multiple of X^k. Hence:

$$\mathrm{Im}(f)=\mathrm{Span}\{1,X,X^2,\dots,X^n\}=\mathbb{R}_n[X].$$

Since this is true for arbitrary n, we conclude:

$$\mathrm{Im}(f)=\mathbb{R}[X].$$

Thus, the image of f is the entire space of real polynomials.

4. **Injectivity and Surjectivity:**

   - f is injective since $\ker f=\{0\}$. - f is surjective since $\mathrm{Im}(f)=\mathbb{R}[X]$.

   Therefore, f is an automorphism of $\mathbb{R}[X]$ (a bijective linear operator).

Solution 2.3

Let f be a function defined from $\mathbb{R}^3$ to $\mathbb{R}^3$ by:

$$f(x,y,z)=(2x+y,\ y-z,\ x-y).$$

1. **Linearity of f:**

   Let $u=(x_1,y_1,z_1),\ v=(x_2,y_2,z_2)\in\mathbb{R}^3$, and $\lambda\in\mathbb{R}$.

   Additivity:

$$\begin{aligned}f(u+v)&=f(x_1+x_2,\ y_1+y_2,\ z_1+z_2)\\&=(2(x_1+x_2)+(y_1+y_2),\ (y_1+y_2)-(z_1+z_2),\ (x_1+x_2)-(y_1+y_2))\\&=(2x_1+2x_2+y_1+y_2,\ y_1-z_1+y_2-z_2,\ x_1-y_1+x_2-y_2)\\&=f(u)+f(v).\end{aligned}$$

   Homogeneity:

$$f(\lambda u)=f(\lambda x_1,\ \lambda y_1,\ \lambda z_1)=(2\lambda x_1+\lambda y_1,\ \lambda y_1-\lambda z_1,\ \lambda x_1-\lambda y_1)=\lambda f(u).$$

Therefore, f is linear.

2. **Kernel and image of f:**

   We solve:

$$f(x,y,z) = (2x+y,\ y-z,\ x-y) = (0,0,0) \Rightarrow \begin{cases} 2x+y=0, \\ y-z=0, \\ x-y=0. \end{cases}$$

   From the third equation, $x = y$. Substituting into the first gives $2y + y = 3y = 0 \Rightarrow y = 0 \Rightarrow x = 0$. From the second, $z = y = 0 \Rightarrow z = 0$.

   Thus, the only solution is $(0,0,0)$, hence:

$$\ker f = \{0\}, \quad \dim(\ker f) = 0.$$

   By the Rank–Nullity Theorem:

$$\dim(\operatorname{Im} f) = 3 - 0 = 3 \Rightarrow \operatorname{Im} f = \mathbb{R}^3.$$

   Therefore, f is bijective (injective and surjective).

3. **Compute $f \circ f$:**

   Let $f(x,y,z) = (u,v,w) = (2x+y,\ y-z,\ x-y)$.

   We now compute $f(u,v,w)$:

$$f(u,v,w) = (2u+v,\ v-w,\ u-v).$$

   Substitute $u = 2x+y$, $v = y-z$, $w = x-y$.

   First component:

$$2u+v = 2(2x+y)+(y-z) = 4x+2y+y-z = 4x+3y-z.$$

   Second component:

$$v-w = (y-z)-(x-y) = y-z-x+y = -x+2y-z.$$

Third component:

$$u - v = (2x + y) - (y - z) = 2x + y - y + z = 2x + z.$$

Therefore:

$$(f \circ f)(x, y, z) = (4x + 3y - z,\ -x + 2y - z,\ 2x + z).$$

Solution 2.4

Let $B = \{e_1, e_2, e_3\}$ be the canonical basis of $\mathbb{R}^3$, and let f be the endomorphism of $\mathbb{R}^3$ defined by:

$$f(e_1) = -2e_1 + 2e_3, \quad f(e_2) = 3e_2, \quad f(e_3) = -4e_1 + 4e_3.$$

1. **Compute** $f(x, y, z)$**:**

   Let $(x, y, z) = xe_1 + ye_2 + ze_3$. By linearity:

   $$f(x, y, z) = xf(e_1) + yf(e_2) + zf(e_3) = x(-2e_1 + 2e_3) + y(3e_2) + z(-4e_1 + 4e_3).$$

   Compute:

   $$\begin{aligned} f(x, y, z) &= (-2x - 4z)e_1 + 3ye_2 + (2x + 4z)e_3 \\ &\Rightarrow f(x, y, z) = (-2x - 4z,\ 3y,\ 2x + 4z). \end{aligned}$$

2. **Find a basis of** $\ker f$**:**

   We solve:

   $$f(x, y, z) = (0, 0, 0) \Rightarrow \begin{cases} -2x - 4z = 0, \\ 3y = 0, \\ 2x + 4z = 0. \end{cases}$$

   From the second equation, $y = 0$. From the first, $-2x = 4z \Rightarrow x = -2z$.

   Thus:

   $$(x, y, z) = (-2z, 0, z) = z(-2, 0, 1) \Rightarrow \ker f = \text{Span}\{(-2, 0, 1)\}.$$

Hence, a basis of the kernel is $\{(-2, 0, 1)\}$, and $\dim(\ker f) = 1$.

3. **Injectivity and surjectivity:**

   Since $\ker f \neq \{0\}$, the map f is not injective. Also, $\dim(\ker f) = 1 \Rightarrow \dim(\operatorname{Im} f) = 2$, so f cannot be surjective (since $\dim(\mathbb{R}^3) = 3$).

   Therefore:
   $$f \text{ is not injective and not surjective.}$$

4. **Basis of** $\operatorname{Im} f$ **and rank:**

   We use the images of the basis vectors:
   $$f(e_1) = (-2, 0, 2), \quad f(e_2) = (0, 3, 0), \quad f(e_3) = (-4, 0, 4).$$

   Note that:
   $$f(e_3) = 2f(e_1) \Rightarrow \text{dependent.}$$

   So:
   $$\operatorname{Im} f = \operatorname{Span}\{f(e_1), f(e_2)\}.$$

   A basis is $\{(-2, 0, 2), (0, 3, 0)\}$, and $\operatorname{rank}(f) = 2$.

5. **Direct sum decomposition:**

   We already have:
   $$\dim(\ker f) = 1, \quad \dim(\operatorname{Im} f) = 2 \Rightarrow \dim(\ker f) + \dim(\operatorname{Im} f) = 3 = \dim(\mathbb{R}^3).$$

   Since $\ker f \cap \operatorname{Im} f = \{0\}$, we conclude that:
   $$\mathbb{R}^3 = \ker f \oplus \operatorname{Im} f.$$

Solution 2.5

Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear map such that:
$$(1, 2, 0) \in \ker(f), \quad f(0, 0, 1) = (1, 0), \quad f(0, t, 0) = (t, t) \quad \forall t \in \mathbb{R}.$$

1. Determine the expression of $f(x, y, z)$:

   We are given:

$$f(0,0,1) = (1,0), \quad f(0,t,0) = (t,t), \quad f(1,2,0) = (0,0).$$

   Let us find the image of the standard basis vectors:

$$f(0,1,0) = (1,1) \quad \text{(from the second property)},$$
$$f(0,0,1) = (1,0) \quad \text{(given)}.$$

   Now, by linearity:

$$f(x,y,z) = xf(1,0,0) + yf(0,1,0) + zf(0,0,1).$$

   Let us find $f(1,0,0)$ using the fact that $f(1,2,0) = (0,0)$:

$$\begin{aligned} f(1,2,0) &= f(1,0,0) + 2f(0,1,0) \\ &= f(1,0,0) + 2(1,1) \\ &= (0,0) \Rightarrow f(1,0,0) = (-2,-2). \end{aligned}$$

   Therefore:

$$f(x,y,z) = x(-2,-2) + y(1,1) + z(1,0) = (-2x + y + z, -2x + y).$$

2. Basis and dimensions of ker(f) and Im(f):

   We solve:

$$f(x,y,z) = (0,0) \Rightarrow \begin{cases} -2x + y + z = 0, \\ -2x + y = 0. \end{cases}$$
$$\Rightarrow z = 0, \quad y = 2x.$$

   Thus:

$$\ker(f) = \{(x, 2x, 0) \mid x \in \mathbb{R}\}.$$

A basis of ker(f) is $\{(1,2,0)\}$, and:

$$\dim(\ker f) = 1.$$

To find Im(f), consider the images of the basis vectors:

$$f(1,0,0) = (-2,-2), \quad f(0,1,0) = (1,1), \quad f(0,0,1) = (1,0).$$

Note that the vectors $(1,1)$ and $(1,0)$ are linearly independent; hence:

$$\dim(\mathrm{Im}(f)) = 2 \Rightarrow \mathrm{Im}(f) = \mathbb{R}^2.$$

A basis is $\{(1,1),(1,0)\}$.

3. Find all vectors whose image is $(0,1)$. Is it a subspace?

   We solve:

$$f(x,y,z) = (0,1) \Rightarrow \begin{cases} -2x+y+z=0, \\ -2x+y=1. \end{cases}$$

   Subtract the two equations:

$$z = -1, \quad y = 1+2x.$$

   So, the set of solutions is:

$$\{(x,1+2x,-1) \mid x \in \mathbb{R}\} = (0,1,-1) + x(1,2,0).$$

   This set is not a vector subspace since it does not contain the zero vector.

Solution 2.6

We are given vector spaces E, F, G over a field $\mathbb{K}$, and two linear maps:

$$f \in \mathcal{L}(E,F), \quad g \in \mathcal{L}(F,G).$$

We are to prove:

$$g \circ f = 0 \iff \text{Im}(f) \subset \ker(g).$$

($\Rightarrow$) Suppose $g \circ f = 0$.
This means that for every $x \in E$, we have:

$$g(f(x)) = 0 \Rightarrow f(x) \in \ker(g) \Rightarrow \text{Im}(f) \subset \ker(g).$$

($\Leftarrow$) Conversely, suppose $\text{Im}(f) \subset \ker(g)$.
Then, for every $x \in E$, since $f(x) \in \text{Im}(f) \subset \ker(g)$, we have:

$$g(f(x)) = 0 \Rightarrow g \circ f = 0.$$

Conclusion: Both directions are proven; hence:

$$g \circ f = 0 \iff \text{Im}(f) \subset \ker(g).$$

Solution 2.7

Let E be a real vector space of dimension 3, and $f \in \mathcal{L}(E)$ such that $f^3 = 0$, $f^2 \neq 0$, and $f^2(x_0) \neq 0$.

1. Show that $B = \{x_0, f(x_0), f^2(x_0)\}$ is a basis of E:

   Let us show that the set B is linearly independent. Suppose:

   $$\lambda_0 x_0 + \lambda_1 f(x_0) + \lambda_2 f^2(x_0) = 0. \tag{1}$$

   Apply f to both sides:

   $$\lambda_0 f(x_0) + \lambda_1 f^2(x_0) + \lambda_2 f^3(x_0) = 0 \Rightarrow \lambda_0 f(x_0) + \lambda_1 f^2(x_0) = 0. \tag{2}$$

   Apply f again:

   $$\lambda_0 f^2(x_0) + \lambda_1 f^3(x_0) = \lambda_0 f^2(x_0) = 0. \tag{3}$$

   Since $f^2(x_0) \neq 0$, we get $\lambda_0 = 0$. Then, from (2), $\lambda_1 f^2(x_0) = 0 \Rightarrow \lambda_1 = 0$. Back to (1), $\lambda_2 f^2(x_0) = 0 \Rightarrow \lambda_2 = 0$. So, B is linearly independent; and since $\dim(E) = 3$, it forms a basis.

2. Let $g \in \mathcal{L}(E)$ such that $g \circ f = f \circ g$.

   (a) Show that there exist scalars $\alpha, \beta, \gamma \in \mathbb{R}$ such that:

   $$g(x_0) = \alpha x_0 + \beta f(x_0) + \gamma f^2(x_0).$$

   Since $\{x_0, f(x_0), f^2(x_0)\}$ is a basis of E, then $g(x_0) \in E$ can be written uniquely as:

   $$g(x_0) = \alpha x_0 + \beta f(x_0) + \gamma f^2(x_0),$$

   for some scalars $\alpha, \beta, \gamma \in \mathbb{R}$.

   (b) Show that:

   $$g = \alpha \mathrm{Id}_E + \beta f + \gamma f^2.$$

   Let us show that for all $v \in E$, we have:

   $$g(v) = \alpha v + \beta f(v) + \gamma f^2(v).$$

   Since $B = \{x_0, f(x_0), f^2(x_0)\}$ is a basis, it is enough to verify this formula on each element of B.

   First:

   $$g(x_0) = \alpha x_0 + \beta f(x_0) + \gamma f^2(x_0) \quad \text{(by assumption)}.$$

   Second: Apply $f \circ g = g \circ f$ on x_0:

   $$\begin{aligned} f(g(x_0)) &= f(\alpha x_0 + \beta f(x_0) + \gamma f^2(x_0)) \\ &= \alpha f(x_0) + \beta f^2(x_0) + \gamma f^3(x_0) = \alpha f(x_0) + \beta f^2(x_0), \end{aligned}$$

   and

   $$g(f(x_0)) = \alpha f(x_0) + \beta f^2(x_0) + \gamma f^3(x_0) = \alpha f(x_0) + \beta f^2(x_0).$$

   So, the formula holds on $f(x_0)$.

   Similarly:

   $$g(f^2(x_0)) = \alpha f^2(x_0).$$

Therefore:

$$g(v) = \alpha v + \beta f(v) + \gamma f^2(v), \quad \forall v \in E \Rightarrow g = \alpha \mathrm{Id}_E + \beta f + \gamma f^2.$$

(c) Deduce the set of endomorphisms commuting with f:

From the previous result, any endomorphism $g \in \mathcal{L}(E)$ such that $g \circ f = f \circ g$ must be of the form:

$$g = \alpha \mathrm{Id}_E + \beta f + \gamma f^2.$$

Hence, the set of all such endomorphisms is:

$$\{\alpha \mathrm{Id}_E + \beta f + \gamma f^2 \mid \alpha, \beta, \gamma \in \mathbb{R}\},$$

which forms a 3-dimensional subspace of $\mathcal{L}(E)$.

Chapter 3

Matrices

Matrices are fundamental mathematical objects that provide a structured way to organize and manipulate data, particularly in linear algebra. A matrix is a rectangular array of elements (numbers, functions, or other mathematical objects) arranged in rows and columns. Matrices are not only powerful tools for solving systems of linear equations but also play a crucial role in representing linear transformations between vector spaces.

In this chapter, we explore the algebraic structure of matrices, including operations such as addition, scalar multiplication, and matrix multiplication. We also study special types of matrices, such as square matrices, identity matrices, and invertible matrices, along with their properties. Additionally, we examine the determinant and rank of a matrix, which are key concepts in determining invertibility and solving linear systems.

3.1 Definitions

Definition 3.1: Matrix

A matrix of type (p, n) with coefficients in the field $\mathbb{K}$ is a table A of $p \times n$ elements of $\mathbb{K}$, arranged in p rows and n columns:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pn} \end{pmatrix}$$

or, abbreviated: $A = (a_{ik})$; or also: $A = \|a_{ik}\|$. The set of matrices with p rows and n columns is denoted by $M_{p,n}(\mathbb{K})$. If $n = p$, $M_{n,n}(\mathbb{K})$ is denoted $M_n(\mathbb{K})$.

Note that, in the notation we have adopted, a_{ik} denotes the element in the i-th row and the k-th column.

Example 3.1

Thus, for example:

$$\begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \end{pmatrix} \in M_{2,3}(\mathbb{R}), \quad \begin{pmatrix} 1 & 2-i & 3+i \\ 0 & 1+i & i \\ -i & 2 & 1 \end{pmatrix} \in M_3(\mathbb{C}).$$

On the set $M_{p,n}(\mathbb{K})$, we define the following operations:

Definition 3.2: Sum of Two Matrices

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two matrices in $M_{p,n}(\mathbb{K})$. The sum $A + B$ is the matrix $(a_{ij} + b_{ij})$ in $M_{p,n}(\mathbb{K})$.

Example 3.2: Matrix Addition

1. If $A = \begin{pmatrix} 1 & 1 & 0 \\ -2 & 3 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 3 & 4 \\ 1 & 1 & -1 \end{pmatrix}$, then:

$$A + B = \begin{pmatrix} 1+(-2) & 1+3 & 0+4 \\ -2+1 & 3+1 & -1+(-1) \end{pmatrix} = \begin{pmatrix} -1 & 4 & 4 \\ -1 & 4 & -2 \end{pmatrix}.$$

2. If $A = \begin{pmatrix} -3 & 1 \\ 7 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 3 & 4 \\ 5 & -1 \end{pmatrix}$, then $A + B$ is not defined.

Definition 3.3: Scalar Multiplication of a Matrix

Let $A = (a_{ij}) \in M_{n,m}(\mathbb{K})$ and $\lambda \in \mathbb{K}$. The product λA is the matrix (λa_{ij}).

Example 3.3: Scalar Multiplication

If $A = \begin{pmatrix} 1 & 1 & 0 \\ -2 & 3 & -1 \end{pmatrix}$, then:

$$2A = \begin{pmatrix} 2\cdot 1 & 2\cdot 1 & 2\cdot 0 \\ 2\cdot(-2) & 2\cdot 3 & 2\cdot(-1) \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ -4 & 6 & -2 \end{pmatrix}.$$

Remark 3.1: Properties of Matrix Operations

1. The matrix $-A = (-1)A$ is the additive inverse of A.
2. $A - B = A + ((-1)B)$.

Theorem 3.1: Properties of Matrix Operations

Let A, B, and C be matrices of the same size, and let r and s be scalars. The following properties hold:

1. $A + B = B + A$. (Commutativity of addition)
2. $(A + B) + C = A + (B + C)$. (Associativity of addition)
3. $A + 0 = A$. (Existence of additive identity)
4. $r(A + B) = rA + rB$. (Distributivity of scalar over matrix sum)
5. $(r + s)A = rA + sA$. (Distributivity of scalar sum over matrix)
6. $r(sA) = (rs)A$. (Associativity of scalar multiplication)

It is easy to see that, equipped with these operations, $M_{p,n}(\mathbb{K})$ is a vector space over $\mathbb{K}$. The neutral element is the matrix where all elements are zero, called the zero matrix, and denoted by 0. The opposite of the matrix (a_{ik}) is the matrix $(-a_{ik})$.

Theorem 3.2: Vector Space of Matrices

The set $M_{p,n}(\mathbb{K})$, equipped with addition and scalar multiplication, forms a $\mathbb{K}$-vector space.

Proposition 3.1: Dimension of Matrix Space

$$\dim_{\mathbb{K}} M_{p,n}(\mathbb{K}) = pn.$$

Indeed, it is straightforward to verify that the pn matrices, called elementary matrices:

$$E_{11} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \ldots, \quad E_{ik} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix},$$

$$(1 \text{ in position } (i,k)), \quad \ldots, \quad E_{pn} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix},$$

form a basis of $M_{p,n}(\mathbb{K})$, called the canonical basis.

3.2 Matrices Associated with Linear Maps

Let E and E' be two vector spaces over $\mathbb{K}$ of dimensions n and p respectively, and $f : E \to E'$ a linear map. Choose a basis $\{e_1, \ldots, e_n\}$ for E and a basis $\{\varepsilon_1, \ldots, \varepsilon_p\}$ for E'. The images under f of the vectors $e_1, \ldots, e_n$ can be expressed in terms of the basis $\{\varepsilon_1, \ldots, \varepsilon_p\}$:

$$\begin{aligned} f(e_1) &= a_{11}\varepsilon_1 + a_{21}\varepsilon_2 + \cdots + a_{p1}\varepsilon_p \\ f(e_2) &= a_{12}\varepsilon_1 + a_{22}\varepsilon_2 + \cdots + a_{p2}\varepsilon_p \\ &\vdots \\ f(e_n) &= a_{1n}\varepsilon_1 + a_{2n}\varepsilon_2 + \cdots + a_{pn}\varepsilon_p \end{aligned}$$

Definition 3.4: Matrix of a Linear Map

The matrix of f with respect to the bases $\{e_1, \ldots, e_n\}$ and $\{\varepsilon_1, \ldots, \varepsilon_p\}$ is the matrix denoted $M(f)_{e_i,\varepsilon_j}$ in $M_{p,n}(\mathbb{K})$ whose columns are the components of the vectors

$f(e_1), \dots, f(e_n)$ in the basis $\{\varepsilon_1, \dots, \varepsilon_p\}$:

$$M(f)_{e_i,\varepsilon_j} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pn} \end{pmatrix}$$

If there is no ambiguity, we may simply write $M(f)$ instead of $M(f)_{e_i,\varepsilon_j}$, but it is clear that the matrix associated with f depends on the choice of bases for E and E'. In the case where f is an endomorphism, we can choose the same basis for E considered as both the departure and arrival space. In this case, we will write $M(f)_{e_i}$ instead of $M(f)_{e_i,e_j}$.

Proposition 3.2: Matrix Representation Isomorphism

Let E and E' be two vector spaces over $\mathbb{K}$ of dimensions n and p respectively, with bases $\{e_i\}$ and $\{\varepsilon_j\}$ for E and E'. Then the mapping:

$$\begin{aligned} M : \mathcal{L}_{\mathbb{K}}(E, E') &\to M_{p,n}(\mathbb{K}) \\ f &\mapsto M(f)_{e_i,\varepsilon_j} \end{aligned}$$

is a vector space isomorphism, that is:

$$\begin{cases} M(f+g) = M(f) + M(g) \\ M(\lambda f) = \lambda M(f) \end{cases}$$

and M is bijective.
In particular: $\dim_{\mathbb{K}} \mathcal{L}(E, E') = np$.

Proof 3.1: I

deed, we have:

$$\begin{aligned} M(f+g)_{e_i,\varepsilon_j} &= \|(f+g)(e_1), \dots, (f+g)(e_n)\|_{\varepsilon_j} \\ &= \|f(e_1) + g(e_1), \dots, f(e_n) + g(e_n)\|_{\varepsilon_j} \\ &= \|f(e_1), \dots, f(e_n)\|_{\varepsilon_j} + \|g(e_1), \dots, g(e_n)\|_{\varepsilon_j} \end{aligned}$$

by the definition of matrix addition, that is:

$$M(f+g)_{e_i,\varepsilon_j} = M(f)_{e_i,\varepsilon_j} + M(g)_{e_i,\varepsilon_j}.$$

Similarly, if $\lambda \in \mathbb{K}$:

$$\begin{aligned} M(\lambda f)_{e_i,\varepsilon_j} &= \|(\lambda f)(e_1), \ldots, (\lambda f)(e_n)\|_{\varepsilon_j} \\ &= \|\lambda f(e_1), \ldots, \lambda f(e_n)\|_{\varepsilon_j} \\ &= \lambda \|f(e_1), \ldots, f(e_n)\|_{\varepsilon_j} \end{aligned}$$

Thus M is linear.
Moreover, M is surjective. Indeed, let:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{p1} & \cdots & a_{pn} \end{pmatrix} \in M_{p,n}(\mathbb{K})$$

and define $f \in \mathcal{L}(E, E')$ as follows. First set:

$$\begin{aligned} f(e_1) &= a_{11}\varepsilon_1 + a_{21}\varepsilon_2 + \cdots + a_{p1}\varepsilon_p \\ &\vdots \\ f(e_n) &= a_{1n}\varepsilon_1 + a_{2n}\varepsilon_2 + \cdots + a_{pn}\varepsilon_p \end{aligned}$$

Then extend f linearly to E, i.e., for:

$$x = \lambda_1 e_1 + \cdots + \lambda_n e_n \in E, \quad \text{define:} \quad f(x) = \lambda_1 f(e_1) + \cdots + \lambda_n f(e_n).$$

It is easy to verify that f is linear and that $A = M(f)_{e_i,\varepsilon_j}$.
Finally, M is injective. Indeed, let $f \in \ker M$:

$$M(f) = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

This means that $f(e_1) = 0, \dots, f(e_n) = 0$. Therefore, for any $x = \lambda_1 e_1 + \cdots + \lambda_n e_n \in E$, we will have:

$$f(x) = \lambda_1 f(e_1) + \cdots + \lambda_n f(e_n) = 0,$$

that is, $f = 0$. By Proposition 3.5, f is injective.

Example 3.4: Matrix Representations

1. Let E be a space of dimension n and consider the identity map:

$$\begin{aligned} \mathrm{id}_E : E &\to E \\ x &\mapsto x \end{aligned}$$

Given a basis $\{e_i\}$, we have $\mathrm{id}_E(e_k) = e_k$. Therefore:

$$M(\mathrm{id}_E)_{e_i} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

(1 is the unit element of $\mathbb{K}$). This matrix is denoted I_n or simply I and is called the identity matrix of $M_n(\mathbb{K})$.

2. Let $E = \mathbb{R}^2$ and consider the projection:

$$\begin{aligned} \mathrm{pr}_1 : \mathbb{R}^2 &\to \mathbb{R}^2 \\ (x, y) &\mapsto (x, 0) \end{aligned}$$

Using the canonical basis of $\mathbb{R}^2$, we have:

$$\begin{cases} \mathrm{pr}_1(e_1) = e_1 \\ \mathrm{pr}_1(e_2) = 0 \end{cases}$$

Thus:

$$M(\mathrm{pr}_1)_{e_i} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

3. Let $E = \mathbb{R}^2$ and f be the projection onto the first bisector parallel to the second bisector. With $\{e_1, e_2\}$ being the canonical basis of E, we have:

$$f(e_1) = f(e_2) = \varepsilon = \frac{1}{2}(e_1 + e_2)$$

Therefore:

$$M(f)_{e_i} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

4. Let $E = \mathbb{R}^2$ and f be the reflection about the Ox axis parallel to the Oy axis. With the canonical basis $\{e_1, e_2\}$, we have:

$$\begin{cases} f(e_1) = e_1 \\ f(e_2) = -e_2 \end{cases}$$

Thus:

$$M(f)_{e_i} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

3.3 Matrix Multiplication

Definition 3.5: Matrix Multiplication

Let $A \in M_{m,n}(\mathbb{K})$ be an $m \times n$ matrix and $B \in M_{n,p}(\mathbb{K})$ an $n \times p$ matrix. The matrix product AB is defined as the $m \times p$ matrix whose (i, j)-entry is given by:

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

This operation is called matrix multiplication.

Remark 3.2: Matrix Multiplication Condition

Matrix multiplication is only defined when the number of columns of A matches the number of rows of B.

Example 3.5: Matrix Multiplication Calculation

Consider the matrices:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \in M_{2,3}(\mathbb{R}), \quad B = \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix} \in M_{3,2}(\mathbb{R}).$$

The product AB is computed as follows:

$$\begin{aligned} AB &= \begin{pmatrix} (1\cdot 7 + 2\cdot 9 + 3\cdot 11) & (1\cdot 8 + 2\cdot 10 + 3\cdot 12) \\ (4\cdot 7 + 5\cdot 9 + 6\cdot 11) & (4\cdot 8 + 5\cdot 10 + 6\cdot 12) \end{pmatrix} \\ &= \begin{pmatrix} 7 + 18 + 33 & 8 + 20 + 36 \\ 28 + 45 + 66 & 32 + 50 + 72 \end{pmatrix} \\ &= \begin{pmatrix} 58 & 64 \\ 139 & 154 \end{pmatrix}. \end{aligned}$$

Thus, $AB \in M_{2,2}(\mathbb{R})$.

Theorem 3.3: Properties of Matrix Multiplication

For compatible matrices A, B, C and scalar λ:

1. Associativity: $(AB)C = A(BC)$.
2. Distributivity: $A(B + C) = AB + AC$.
3. Scalar compatibility: $\lambda(AB) = (\lambda A)B = A(\lambda B)$.
4. Identity: $I_m A = A = AI_n$ where I is the identity matrix.
5. $A \cdot 0 = 0$ and $0 \cdot A = 0$.

Example 3.6: Matrix Multiplication Examples

1. Let
$$A = \begin{pmatrix} -1 & 3 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & 4 & -1 \\ 0 & -1 & -2 & 1 \end{pmatrix}.$$
   Since $A \in M_{3,2}(\mathbb{R})$ and $B \in M_{2,4}(\mathbb{R})$, the product AB is defined and

$AB \in M_{3,4}(\mathbb{R})$. Moreover:

$$AB = \begin{pmatrix} -1 & -4 & -10 & 4 \\ 0 & -1 & -2 & 1 \\ 2 & 1 & 6 & -1 \end{pmatrix}.$$

2. Consider the matrices

$$A = \begin{pmatrix} 1 & i \\ 2 & -i \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & -i \\ 0 & 2i \end{pmatrix}.$$

Since $A \in M_2(\mathbb{C})$ and $B \in M_2(\mathbb{C})$, the product AB is defined and $AB \in M_2(\mathbb{C})$. Moreover:

$$\begin{aligned} AB &= \begin{pmatrix} 3 - i + i(2i) & -i + i(2i) \\ 6 - 2i - i(2i) & -2i - i(2i) \end{pmatrix} \\ &= \begin{pmatrix} 3 - i - 2 & -i - 2 \\ 6 - 2i + 2 & -2i + 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 - i & -2 - i \\ 8 - 2i & 2 - 2i \end{pmatrix}. \end{aligned}$$

3. If

$$A = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 5 \\ 1 & 3 \\ 1 & -1 \end{pmatrix},$$

then AB is not defined because $A \in M_{2,1}(\mathbb{R})$ and $B \in M_{3,2}(\mathbb{R})$ (the number of columns of A doesn't match the number of rows of B).

Remark 3.3: Important Remarks on Matrix Multiplication

1. If $A = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}$, then

$$AB = \begin{pmatrix} 4 & -1 & -3 \\ -6 & 2 & 4 \end{pmatrix},$$

but BA is not defined. Therefore, in general, $AB \neq BA$.

2. Since
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
we conclude that $AB = 0$ does not imply $A = 0$ or $B = 0$.

3. From the fact that
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix},$$
we see that $AB = AC$ does not imply $B = C$.

Remark 3.4: Computing the Image of a Vector via a Matrix Representation

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, and A its associated matrix (relative to the standard bases). If v is a vector in $\mathbb{R}^n$, its image $f(v)$ is computed by:

$$f(v) = Av,$$

where the operation is standard matrix-vector multiplication.

Example 3.7: Matrix-Vector Multiplication

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ be the matrix of the linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$. To compute the image of the vector $v = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$:

$$f(v) = Av = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} (1)(2) + (2)(-1) \\ (3)(2) + (4)(-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Thus, $f(v) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

3.4 Matrix Transpose

Definition 3.6: Matrix Transpose

Let $A = [a_{ij}]$ be an $m \times n$ matrix. The transpose of A, denoted A^T, is the $n \times m$ matrix obtained by interchanging the rows and columns of A:

$$(A^T)_{ij} = a_{ji} \quad \text{for all } i, j.$$

That is, the ith row of A becomes the ith column of A^T.

Example 3.8: Matrix Transpose

For the matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix},$$

its transpose is:

$$A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$

Remark 3.5: Key Properties of Transpose

- $(A^T)^T = A$ (Involution).
- $(A + B)^T = A^T + B^T$ (Linearity).
- $(AB)^T = B^T A^T$ (Reverse-order law).
- $(\lambda A)^T = \lambda A^T$ for scalar λ.

Definition 3.7: Special Matrix Types

- A matrix is *symmetric* if $A^T = A$.
- A matrix is *skew-symmetric* if $A^T = -A$.
- A matrix is *orthogonal* if $A^T = A^{-1}$.

Example 3.9: Special Matrix Types

1. **Symmetric Matrix** ($A^T = A$):

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}; \quad A^T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}.$$

   Note: All diagonal matrices are symmetric.

2. **Skew-Symmetric Matrix** ($A^T = -A$):

$$B = \begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & 4 \\ 1 & -4 & 0 \end{pmatrix}; \quad B^T = \begin{pmatrix} 0 & -2 & 1 \\ 2 & 0 & -4 \\ -1 & 4 & 0 \end{pmatrix} = -B.$$

   Note: Diagonal entries must be zero.

3. **Orthogonal Matrix** ($A^T = A^{-1}$):

$$Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}; \quad Q^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = Q^{-1}.$$

   Verification: $QQ^T = I_2$ (rotation matrices are orthogonal).

3.5 Square Matrices

Definition 3.8: Square Matrix

A square matrix is a matrix with the same number of rows and columns, i.e., a matrix of size $n \times n$ for some positive integer n. The set of all $n \times n$ matrices over a field $\mathbb{K}$ is denoted by $M_n(\mathbb{K})$.

Definition 3.9: Identity Matrix

The square matrix

$$I_n := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

is called the identity matrix.

Remark 3.6: Properties of Identity Matrix

Let $A \in M_{n,m}(\mathbb{K})$. Then:

$$I_n A = A \quad \text{and} \quad A I_m = A.$$

Example 3.10: Identity Matrix Multiplication

Let $A \in M_{2,3}(\mathbb{R})$ be the matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix},$$

and let I_2 be the identity matrix of size 2×2:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then:

$$I_2 A = A \quad \text{and} \quad A I_3 = A.$$

3.5.1 Determinant of a Square Matrix

Definition 3.10: Submatrices

Let $A = (a_{ij}) \in M_n(\mathbb{K})$. We denote by $A_{\setminus i,j}$ the submatrix of A of order $n-1$ obtained by removing the i-th row and j-th column from A.

Example 3.11: Submatrices Examples

1. For the matrix $A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \\ 5 & -2 & 0 \end{pmatrix}$, we have:

$$A_{\setminus 1,1} = \begin{pmatrix} 4 & 1 \\ -2 & 0 \end{pmatrix}; \quad A_{\setminus 2,2} = \begin{pmatrix} 2 & 3 \\ 5 & 0 \end{pmatrix}; \quad A_{\setminus 3,2} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}.$$

2. For the matrix $B = \begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 4 \\ 0 & 5 & 1 \end{pmatrix}$, we get:

$$B_{\setminus 1,3} = \begin{pmatrix} 3 & -1 \\ 0 & 5 \end{pmatrix}; \quad B_{\setminus 2,1} = \begin{pmatrix} 0 & 2 \\ 5 & 1 \end{pmatrix}; \quad B_{\setminus 3,3} = \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}.$$

Definition 3.11: Determinant

Let $A = (a_{ij}) \in M_n(\mathbb{K})$. The determinant of A, denoted by $\det(A)$ or $|A|$, is the element of $\mathbb{K}$ defined recursively by:

(i) If $n = 1$, then $\det(A) = a_{11}$.

(ii) If $n > 1$, then

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{\setminus 1j}),$$

where $A_{\setminus 1j}$ is the submatrix obtained by deleting the first row and j-th column of A.

Example 3.12: Determinant Calculation

- 2×2 Case:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

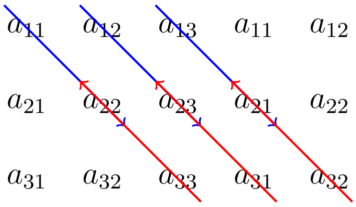
- 3×3 Case (Sarrus' Rule): Sarrus' rule provides a simple mnemonic method for computing the determinant of a 3×3 matrix without using cofactor expansion.

The Rule

Given a 3×3 matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

1. Write the first two columns again to the right of the matrix:



2. Then the determinant is:

$$\begin{aligned} \det(A) &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - (a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32}). \end{aligned}$$

Example 3.13: Sarrus' Rule Application

Compute the determinant of:

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

1. Extend the matrix:

$$\begin{array}{ccccc} 1 & 2 & 3 & 1 & 2 \\ 4 & 5 & 6 & 4 & 5 \\ 7 & 8 & 9 & 7 & 8 \end{array}$$

2. Calculate the products:

$$1 \cdot 5 \cdot 9 + 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 8 = 45 + 84 + 96 = 225$$
$$3 \cdot 5 \cdot 7 + 2 \cdot 4 \cdot 9 + 1 \cdot 6 \cdot 8 = 105 + 72 + 48 = 225$$

3. Final determinant:

$$\det(B) = 225 - 225 = 0$$

Remark 3.7: Determinant Expansion Methods

The determinant can be computed using expansion along any row or column:

1. **Row Expansion:** For any row i:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{\setminus ij}),$$

   where $A_{\setminus ij}$ is the submatrix obtained by deleting row i and column j.

2. **Column Expansion:** For any column j:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{\setminus ij}).$$

3. **Optimal Strategy:** For efficient computation, choose the row/column with the most zeros.

Example 3.14: Determinant Expansion with Zeros

For matrix $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 0 & 4 \\ 5 & 6 & 7 \end{pmatrix}$, expanding along the second column (which has two zeros) gives:

$$\begin{aligned} \det(A) &= (-1)^{1+2} \cdot 0 \cdot \det \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} \\ &\quad + (-1)^{2+2} \cdot 0 \cdot \det \begin{pmatrix} 1 & 2 \\ 5 & 7 \end{pmatrix} \\ &\quad + (-1)^{3+2} \cdot 6 \cdot \det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \\ &= -6 \cdot (1 \cdot 4 - 2 \cdot 3) = -6 \cdot (4 - 6) = -6 \cdot (-2) = 12. \end{aligned}$$

Example 3.15: Determinant Calculations

1. $$\det\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} = (2\cdot 4) - (3\cdot 1) = 8 - 3 = 5.$$

2. For a 3×3 matrix $A = \begin{pmatrix} 0 & 4 & 5 \\ 1 & 0 & 6 \\ 2 & 1 & 7 \end{pmatrix}$:

$$\begin{aligned} \det(A) &= 0\cdot\det\begin{pmatrix} 0 & 6 \\ 1 & 7 \end{pmatrix} - 4\cdot\det\begin{pmatrix} 1 & 6 \\ 2 & 7 \end{pmatrix} + 5\cdot\det\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \\ &= 0 - 4\cdot(1\cdot 7 - 6\cdot 2) + 5\cdot(1\cdot 1 - 0\cdot 2) \\ &= -4\cdot(7-12) + 5\cdot(1-0) \\ &= -4\cdot(-5) + 5\cdot 1 = 20 + 5 = 25. \end{aligned}$$

Theorem 3.4: Properties of Determinants

Let $A, B \in M_n(\mathbb{K})$. Then the following properties hold:

1. **Transpose Invariance:**
$$\det(A) = \det(A^T).$$
2. **Product Rule:**
$$\det(AB) = \det(A)\det(B).$$
3. **Scalar Multiplication:**
$$\det(\lambda A) = \lambda^n \det(A).$$
4. **Invertibility Criterion:** A is invertible if and only if $\det(A) \neq 0$.
5. **Inverse Matrix:**
$$\text{If } A \text{ is invertible, then } \det(A^{-1}) = \frac{1}{\det(A)}.$$

Example 3.16: Determinant Properties Examples

1. **Product Rule:**

$$\text{If } A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

   then

$$\det(AB) = \det \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix} = -6 = 6 \cdot (-1).$$

2. **Scalar Multiplication:**

$$\text{For } A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } \lambda = 2; \quad \det(2A) = 4 \cdot 0 = 0.$$

3. **Invertibility Criterion:**

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ is singular } (\det = 0); \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ is invertible } (\det = -2).$$

4. **Inverse Matrix:**

$$\text{For } A = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}; \quad \det(A) = 8; \quad \det(A^{-1}) = \frac{1}{8}.$$

Example 3.17: Fundamental Determinant Properties Examples

1. **Zero Row Property:** Consider the matrix:

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{pmatrix}.$$

   The second row is entirely zero, thus $\det(A) = 0$.

2. **Proportional Rows:** For the matrix:

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Row 2 is exactly twice Row 1, hence det$(B) = 0$.

3. **Row Interchange:** Compare:

$$C = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}; \quad C' = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}.$$

We find:

$$\det(C) = -10; \quad \det(C') = 10 = -\det(C).$$

4. **Scalar Multiplication:** Original matrix and scaled version:

$$D = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}; \quad D' = \begin{pmatrix} 6 & 3 \\ 1 & 3 \end{pmatrix}.$$

Determinants verify:

$$\det(D) = 5; \quad \det(D') = 15 = 3 \cdot \det(D).$$

5. **Row Addition:** Original and modified matrices:

$$E = \begin{pmatrix} 1 & 5 \\ 3 & 2 \end{pmatrix}; \quad E' = \begin{pmatrix} 1+3 & 5+2 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ 3 & 2 \end{pmatrix}.$$

Both have:

$$\det(E) = \det(E') = -13.$$

3.5.2 Invertible matrix

Matrix algebra provides tools for manipulating matrix equations and creating various useful formulas in ways similar to doing ordinary algebra with real numbers. This subsection investigates the matrix analogue of the reciprocal, or multiplicative inverse, of a nonzero number.

Definition 3.12: Invertible Matrix

Let $A \in M_n(\mathbb{K})$. We say that A is invertible if there exists a matrix $B \in M_n(\mathbb{K})$ such that:

$$AB = BA = I_n.$$

The matrix B is called the inverse matrix of A and is denoted by A^{-1}.

Proposition 3.3: Uniqueness of Inverse

If an inverse exists, then there is only one (we say that the inverse is unique, or uniquely determined by A).

Proof 3.2: S

ppose that B and C are both inverses of A, so we have:

$$AB = BA = I \quad \text{and} \quad AC = CA = I.$$

We want to show that $B = C$.
Multiply the equation $BA = I$ on the right by C:

$$BAC = IC = C.$$

Since $AC = I$, we have:

$$BAC = BI = B.$$

Therefore, $B = C$.
This proves that the inverse is unique. Thus, the inverse of A, denoted A^{-1}, is the unique matrix such that:

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I.$$

Example 3.18: Invertible Matrices

1. The matrix I_n is invertible and its inverse is I_n, because $I_n^2 = I_n$.

2. The matrix

$$A = \begin{pmatrix} 2 & 5 \\ -1 & 3 \end{pmatrix},$$

is invertible and its inverse is the matrix

$$A^{-1} = \frac{1}{11}\begin{pmatrix} 3 & -5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{11} & -\frac{5}{11} \\ \frac{1}{11} & \frac{2}{11} \end{pmatrix}.$$

Theorem 3.5: Properties of Matrix Inverses

1. If A is an invertible matrix, then A^{-1} is invertible and:

$$(A^{-1})^{-1} = A.$$

2. If A and B are $n \times n$ invertible matrices, then AB is invertible, and the inverse of AB is the product of the inverses of A and B in reverse order:

$$(AB)^{-1} = B^{-1}A^{-1}.$$

3. If A is an invertible matrix, then its transpose A^T is also invertible, and the inverse of A^T is the transpose of A^{-1}:

$$(A^T)^{-1} = (A^{-1})^T.$$

Proof 3.3: Properties of Matrix Inverses

1. To verify statement (a), we must find a matrix C such that:

$$A^{-1}C = I \quad \text{and} \quad CA^{-1} = I.$$

   These equations are satisfied when we take $C = A$. Hence, A^{-1} is invertible, and its inverse is A.

2. To prove statement (b), compute:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I,$$
$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$

   Therefore, AB is invertible, and:

$$(AB)^{-1} = B^{-1}A^{-1}.$$

3. To verify statement (c), we use Theorem 3(d), read from right to left:

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I,$$
$$A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I.$$

Therefore, A^T is invertible, and its inverse is:

$$(A^T)^{-1} = (A^{-1})^T.$$

Definition 3.13: General Linear Group

The general linear group of degree n over a field $\mathbb{K}$, denoted by $\mathrm{GL}_n(\mathbb{K})$, is the group of all invertible elements of $M_n(\mathbb{K})$.

Theorem 3.6: Ring Structure of $M_n(\mathbb{K})$

The set $M_n(\mathbb{K})$, equipped with matrix addition and multiplication, forms a ring. The identity element for multiplication is the identity matrix I_n.
For $n > 1$, the ring $M_n(\mathbb{K})$ is non-commutative and not an integral domain.

Definition 3.14: Elementary Operations on a Matrix

Elementary operations on a matrix A are defined as:

1. Swapping two rows (respectively, two columns) of A.
2. Adding to a row (respectively, a column) r another row (respectively, column) s, with $s \neq r$, multiplied by a scalar $\lambda \in \mathbb{K}$.
3. Multiplying a row (respectively, a column) of A by a nonzero scalar $\lambda \in \mathbb{K} \setminus \{0\}$.

Algorithm 3.1: Matrix Inversion Algorithm

To compute the inverse of a matrix A of order n, knowing that A is invertible, we proceed as follows:

1. Consider the matrix E of size $n \times 2n$, where the first n columns are those of A, and the last n columns are those of the identity matrix I_n.

2. Apply a sequence of elementary row operations to E so that the first n columns are transformed into the identity matrix I_n. At this point, the last n columns of the resulting matrix form the inverse of A.

Example 3.19: Finding the Inverse of a Matrix Using Row Operations

Let

$$A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}.$$

We want to compute A^{-1} using elementary row operations.

Step 1: Write the augmented matrix $[A|I]$:

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{pmatrix}.$$

Step 2: Make the pivot in row 1 equal to 1: Divide row 1 by 2:

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 5 & 3 & 0 & 1 \end{pmatrix}.$$

Step 3: Eliminate below the pivot: Row 2 = Row 2 - 5 × Row 1:

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{5}{2} & 1 \end{pmatrix}.$$

Step 4: Make the pivot in row 2 equal to 1: Multiply row 2 by 2:

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -5 & 2 \end{pmatrix}.$$

Step 5: Eliminate above the pivot in row 1: Row 1 = Row 1 - $\frac{1}{2}$ × Row 2:

$$\begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -5 & 2 \end{pmatrix}.$$

Result: The inverse of A is:

$$A^{-1} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}.$$

Definition 3.15: Cofactor Matrix

Let $A = (a_{ij})$ be a square matrix of order n. The *cofactor* associated with the entry a_{ij} is the number

$$(-1)^{i+j} \det(A^*_{i,j}),$$

where $A^*_{i,j}$ denotes the submatrix obtained by deleting the i-th row and the j-th column.

The *cofactor matrix* (or *adjugate matrix*) of A, denoted $\operatorname{com}(A)$ or $\operatorname{adj}(A)$, is the square matrix of order n whose entries are the corresponding cofactors.

Example 3.20: Cofactor Matrix Calculation

Consider the 3×3 matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{pmatrix}.$$

The cofactor matrix $\operatorname{com}(A)$ is calculated as follows:

1. For $a_{11} = 1$:

$$C_{11} = (-1)^{1+1} \det \begin{pmatrix} 4 & 5 \\ 0 & 6 \end{pmatrix} = 1 \cdot (24 - 0) = 24.$$

2. For $a_{12} = 2$:

$$C_{12} = (-1)^{1+2} \det \begin{pmatrix} 0 & 5 \\ 1 & 6 \end{pmatrix} = -1 \cdot (0 - 5) = 5.$$

3. For $a_{13} = 3$:

$$C_{13} = (-1)^{1+3} \det \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} = 1 \cdot (0 - 4) = -4.$$

4. Continuing this process for all elements: The complete cofactor matrix is:

$$\text{com}(A) = \begin{pmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{pmatrix}.$$

Using the above notations, we can state the result concerning the computation of the inverse matrix.

Theorem 3.7: Matrix Inversion

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)}\,\text{adj}(A),$$

where $\text{adj}(A)$ denotes the adjugate (the transpose of the cofactor matrix) of A.

Example 3.21: Matrix Inverse via Adjugate Method

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Step 1: Compute cofactors:

$$\begin{aligned} C_{11} &= (-1)^{1+1}\det(4) = 4, \\ C_{12} &= (-1)^{1+2}\det(3) = -3, \\ C_{21} &= (-1)^{2+1}\det(2) = -2, \\ C_{22} &= (-1)^{2+2}\det(1) = 1. \end{aligned}$$

Step 2: Build cofactor matrix:

$$\text{com}(A) = \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}.$$

Step 3: Transpose to get adjugate:

$$\text{adj}(A) = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}.$$

Step 4: Compute determinant:

$$\det(A) = (1)(4) - (2)(3) = -2.$$

Step 5: Apply inverse formula:

$$A^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

3.5.3 Matrix rank

Definition 3.16: Matrix Rank

Let $A \in M_{n,m}(\mathbb{K})$ be a matrix over a field $\mathbb{K}$. The rank of matrix A, denoted $\operatorname{rank}(A)$, is the dimension of the vector subspace of $\mathbb{K}^n$ generated by its column vectors:

$$\operatorname{rank}(A) = \dim \operatorname{Span}\{C_1, \dots, C_m\},$$

where $C_1, \dots, C_m$ are the columns of A.

Proposition 3.4: Properties of Matrix Rank

For any matrix $A \in M_{n,m}(\mathbb{K})$:

1. $\operatorname{rank}(A) \leq \min(n, m)$.
2. $\operatorname{rank}(A) = \operatorname{rank}(A^T)$ (column rank equals row rank).
3. If B is invertible, $\operatorname{rank}(AB) = \operatorname{rank}(A)$.

Algorithm 3.2: Matrix Rank Calculation (Gaussian Elimination)

The rank of a matrix can be determined by performing row reduction to echelon form. The number of non-zero pivots in the echelon form is equal to the rank of the matrix.

Example 3.22: Matrix Rank Calculation

Consider the following 3×3 matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Step 1: Gaussian Elimination

We transform A to Row Echelon Form (REF) using elementary row operations.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 4R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 7R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}.$$

Step 2: Determining the Rank

The REF shows:

- 2 non-zero rows $\Rightarrow$ rank is 2.
- The third row is all zeros, indicating linear dependence.

Step 3: Conclusion

The rank of matrix A is: $\text{rank}(A) = 2$.

Theorem 3.8: 3.42

Let $A \in M_n(\mathbb{K})$ be a square matrix of order n over a field $\mathbb{K}$. The following statements are equivalent:

1. A is invertible.
2. $\det(A) \neq 0$.
3. The rank of A equals n (full rank matrix).
4. The column vectors of A are linearly independent.

5. The row vectors of A are linearly independent.
6. The canonically associated linear map is bijective.

3.6 Exercises

Exercise 3.1: Matrix Operations

Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 5 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 4 \\ 2 & 3 & 0 \end{pmatrix}.$$

1. Compute the following:

   $$A+B, \quad 3B, \quad -1B, \quad A+2B, \quad 2A+B, \quad A-B, \quad A-1B, \quad B-A.$$

2. Write down the row vectors and column vectors of the matrices A, B.
3. Find A^t, B^t, $(A+B)^t$ and $A^t + B^t$.

Exercise 3.2: Symmetric and Skew-Symmetric Matrices

1. Show that for any square matrix, the matrix $A + A^t$ is symmetric.
2. Define a matrix A to be skew-symmetric if $A^t = -A$. Show that for any square matrix A, the matrix $A - A^t$ is skew-symmetric.
3. If a matrix is skew-symmetric, what can you say about its diagonal elements?

Exercise 3.3: Matrix Multiplication

In each of the following cases, find $(AB)C$ and $A(BC)$ (if defined).

Case 1:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

Case 2:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}.$$

Case 3:

$$A = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad C = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Exercise 3.4: Matrix-Vector Multiplication

Let

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 1 & 5 \end{pmatrix}.$$

Find AX for each of the following values of X:

- $X = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$,
- $X = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$,
- $X = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$.

Exercise 3.5: Matrix Representation of Linear Maps

Write the matrices of the following linear transformations relative to the canonical bases:

1. $f_1 : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f_1(x, y, z) = (x + 2y + 3z, 2y - z, x + z).$$

2. $f_2 : \mathbb{R}_2[X] \to \mathbb{R}_3[X]$ defined by

$$f_2(P) = XP - P' + P(1),$$

where $\mathbb{R}_n[X]$ denotes the space of polynomials with real coefficients of degree $\leq n$, and P' is the derivative of P.

Exercise 3.6: Matrix Inversion

Using two different methods, compute the inverses of the following matrices:

$$A = \begin{pmatrix} 1 & -3 & 5 \\ 1 & -1 & 2 \\ -1 & 2 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}.$$

Exercise 3.7: Matrix Polynomial and Inversion

Let

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}.$$

1. Compute $A^3 - A$.

2. Deduce that A is invertible and determine A^{-1}.

Exercise 3.8: Linear Map and Matrix Rank

Let f be the linear map defined by

$$f : \mathbb{R}^3 \to \mathbb{R}^2, \quad (x, y, z) \mapsto f(x, y, z) = (x + y, y + z).$$

1. Determine the matrix associated with f in the canonical bases.

2. Compute the rank of the matrix

$$A = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & -2 \end{pmatrix}.$$

Exercise 3.9: Linear Transformation

Let $B = (e_1, e_2)$ be the canonical basis of $\mathbb{R}^2$. Consider the linear map f from $\mathbb{R}^2$ defined by the matrix A in the basis B:

$$A = \begin{pmatrix} 11 & 30 \\ -11 & 4 \end{pmatrix}.$$

1. Determine the vectors $f(e_1)$, $f(e_2)$, $f(2,5)$, and $f(1,3)$.
2. Give the expression of the function f.

Exercise 3.10: Matrix Polynomial and Inversion

Let

$$A = \begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

1. Compute A^2, then find two real numbers α and β such that

$$A^2 = \alpha A + \beta I.$$

2. Deduce from the above that A is invertible, and find A^{-1}. Then compute A^{-1} again using the comatrix method.

Exercise 3.11: Matrix Rank

Compute the rank of the following matrices:

1. $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$,

2. $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

3. $A = \begin{pmatrix} 1 & 1 & -1 \\ -3 & -3 & 3 \\ 2 & 2 & -2 \end{pmatrix}$,

4. $A = \begin{pmatrix} 8 & 4 & -16 \\ 0 & 4 & -8 \\ 4 & 4 & -12 \end{pmatrix}$.

3.7 Exercise Solutions

Solution 3.1

1. We compute the following:

$$\begin{aligned}
A+B &= \begin{pmatrix} 1+1 & 2+(-1) & 3+4 \\ -1+2 & 5+3 & -2+0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 7 \\ 1 & 8 & -2 \end{pmatrix}, \\
3B &= \begin{pmatrix} 3\cdot 1 & 3\cdot(-1) & 3\cdot 4 \\ 3\cdot 2 & 3\cdot 3 & 3\cdot 0 \end{pmatrix} = \begin{pmatrix} 3 & -3 & 12 \\ 6 & 9 & 0 \end{pmatrix}, \\
-B &= \begin{pmatrix} -1 & 1 & -4 \\ -2 & -3 & 0 \end{pmatrix}, \\
A+2B &= \begin{pmatrix} 1 & 2 & 3 \\ -1 & 5 & -2 \end{pmatrix} + \begin{pmatrix} 2 & -2 & 8 \\ 4 & 6 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 11 \\ 3 & 11 & -2 \end{pmatrix}, \\
2A+B &= \begin{pmatrix} 2 & 4 & 6 \\ -2 & 10 & -4 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 4 \\ 2 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 10 \\ 0 & 13 & -4 \end{pmatrix}, \\
A-B &= \begin{pmatrix} 0 & 3 & -1 \\ -3 & 2 & -2 \end{pmatrix}, \\
B-A &= \begin{pmatrix} 0 & -3 & 1 \\ 3 & -2 & 2 \end{pmatrix}.
\end{aligned}$$

2. Row and column vectors:

   For A: Row vectors: $(1,2,3)$; $(-1,5,-2)$.

   Column vectors: $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$; $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$; $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

   For B: Row vectors: $(1,-1,4)$; $(2,3,0)$.

   Column vectors: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$; $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$; $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$.

3. Transpose operations:

$$A^t = \begin{pmatrix} 1 & -1 \\ 2 & 5 \\ 3 & -2 \end{pmatrix},$$

$$B^t = \begin{pmatrix} 1 & 2 \\ -1 & 3 \\ 4 & 0 \end{pmatrix},$$

$$(A+B)^t = \begin{pmatrix} 2 & 1 \\ 1 & 8 \\ 7 & -2 \end{pmatrix},$$

$$A^t + B^t = \begin{pmatrix} 2 & 1 \\ 1 & 8 \\ 7 & -2 \end{pmatrix}.$$

So we confirm that: $(A+B)^t = A^t + B^t$.

Solution 3.2

1. To show that $A + A^t$ is symmetric:

   A matrix M is symmetric if $M^t = M$. Consider the matrix $M = A + A^t$. Then:

   $$M^t = (A + A^t)^t = (A^t)^t + A^t = A + A^t = M.$$

   So M is symmetric.

2. To show that $A - A^t$ is skew-symmetric:

   A matrix M is skew-symmetric if $M^t = -M$. Consider the matrix $M = A - A^t$. Then:

   $$M^t = (A - A^t)^t = (A^t)^t - A^t = A - A^t = -(A^t - A) = -M.$$

   So M is skew-symmetric.

3. If A is skew-symmetric, then $A^t = -A$. For any diagonal entry a_{ii}, we know that $a_{ii} = -a_{ii}$, so:

   $$2a_{ii} = 0 \Rightarrow a_{ii} = 0.$$

Therefore, all diagonal elements of a skew-symmetric matrix are zero.

Solution 3.3

1. **Case 1:**

$$\begin{aligned}
AB &= \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 \cdot 2 + 2 \cdot 3 & 1 \cdot 0 + 2 \cdot 1 \\ 0 \cdot 2 + (-1) \cdot 3 & 0 \cdot 0 + (-1) \cdot 1 \end{pmatrix} \\
&= \begin{pmatrix} 8 & 2 \\ -3 & -1 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
(AB)C &= \begin{pmatrix} 8 & 2 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 8 \cdot 1 + 2 \cdot 0 & 8 \cdot 1 + 2 \cdot 2 \\ -3 \cdot 1 + (-1) \cdot 0 & -3 \cdot 1 + (-1) \cdot 2 \end{pmatrix} \\
&= \begin{pmatrix} 8 & 12 \\ -3 & -5 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
BC &= \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 2 \cdot 1 + 0 \cdot 0 & 2 \cdot 1 + 0 \cdot 2 \\ 3 \cdot 1 + 1 \cdot 0 & 3 \cdot 1 + 1 \cdot 2 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 2 \\ 3 & 5 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
A(BC) &= \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 3 & 5 \end{pmatrix} \\
&= \begin{pmatrix} 1 \cdot 2 + 2 \cdot 3 & 1 \cdot 2 + 2 \cdot 5 \\ 0 \cdot 2 + (-1) \cdot 3 & 0 \cdot 2 + (-1) \cdot 5 \end{pmatrix} \\
&= \begin{pmatrix} 8 & 12 \\ -3 & -5 \end{pmatrix}.
\end{aligned}$$

So, $(AB)C = A(BC)$.

2. **Case 2:**

$$
\begin{aligned}
AB &= \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 1 \cdot 2 + 0 \cdot 0 + 2 \cdot 1 & 1 \cdot 1 + 0 \cdot (-1) + 2 \cdot 4 \\ -1 \cdot 2 + 3 \cdot 0 + 1 \cdot 1 & -1 \cdot 1 + 3 \cdot (-1) + 1 \cdot 4 \end{pmatrix} \\
&= \begin{pmatrix} 4 & 9 \\ -1 & 0 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
(AB)C &= \begin{pmatrix} 4 & 9 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 4 \cdot 1 + 9 \cdot 2 & 4 \cdot 0 + 9 \cdot (-1) \\ -1 \cdot 1 + 0 \cdot 2 & -1 \cdot 0 + 0 \cdot (-1) \end{pmatrix} \\
&= \begin{pmatrix} 22 & -9 \\ -1 & 0 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
BC &= \begin{pmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 2 \cdot 1 + 1 \cdot 2 & 2 \cdot 0 + 1 \cdot (-1) \\ 0 \cdot 1 + (-1) \cdot 2 & 0 \cdot 0 + (-1) \cdot (-1) \\ 1 \cdot 1 + 4 \cdot 2 & 1 \cdot 0 + 4 \cdot (-1) \end{pmatrix} \\
&= \begin{pmatrix} 4 & -1 \\ -2 & 1 \\ 9 & -4 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
A(BC) &= \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ -2 & 1 \\ 9 & -4 \end{pmatrix} \\
&= \begin{pmatrix} 1 \cdot 4 + 0 \cdot (-2) + 2 \cdot 9 & 1 \cdot (-1) + 0 \cdot 1 + 2 \cdot (-4) \\ -1 \cdot 4 + 3 \cdot (-2) + 1 \cdot 9 & -1 \cdot (-1) + 3 \cdot 1 + 1 \cdot (-4) \end{pmatrix} \\
&= \begin{pmatrix} 22 & -9 \\ -1 & 0 \end{pmatrix}.
\end{aligned}
$$

So, $(AB)C = A(BC)$.

3. **Case 3:**

$$AB = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 1 \cdot 0 + 2 \cdot 1 + 0 \cdot (-1) = 2,$$

$$(AB)C = 2 \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix},$$

$$BC = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \cdot 3 \\ 1 \cdot 3 \\ -1 \cdot 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix},$$

$$A(BC) = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix} = 1 \cdot 0 + 2 \cdot 3 + 0 \cdot (-3) = 6.$$

So, $(AB)C = A(BC)$.

Solution 3.4

1. For $X = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$,

$$AX = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 + 1 \cdot 0 + 3 \cdot 0 \\ 4 \cdot 1 + 1 \cdot 0 + 5 \cdot 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

2. For $X = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$,

$$AX = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 1 & 5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \cdot 0 + 1 \cdot 1 + 3 \cdot 1 \\ 4 \cdot 0 + 1 \cdot 1 + 5 \cdot 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}.$$

3. For $X = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$,

$$AX = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 1 & 5 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \cdot 0 + 1 \cdot 0 + 3 \cdot 1 \\ 4 \cdot 0 + 1 \cdot 0 + 5 \cdot 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

Solution 3.5

1. To find the matrix of $f_1 : \mathbb{R}^3 \to \mathbb{R}^3$ defined by:

$$f_1(x, y, z) = (x + 2y + 3z, 2y - z, x + z),$$

we apply f_1 to the canonical basis vectors of $\mathbb{R}^3$:

   (a) $f_1(1, 0, 0) = (1, 0, 1)$,
   (b) $f_1(0, 1, 0) = (2, 2, 0)$,
   (c) $f_1(0, 0, 1) = (3, -1, 1)$.

These vectors form the columns of the matrix of f_1:

$$[f_1] = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 1 & 0 & 1 \end{pmatrix}.$$

2. For $f_2 : \mathbb{R}_2[X] \to \mathbb{R}_3[X]$, let the canonical basis of $\mathbb{R}_2[X]$ be $\{1, X, X^2\}$ and that of $\mathbb{R}_3[X]$ be $\{1, X, X^2, X^3\}$. Apply f_2 to each basis element:

   (a) For $P = 1$:

$$XP = X, \quad P' = 0, \quad P(1) = 1 \Rightarrow f_2(1) = X + 1.$$

Column vector: $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

(b) For $P = X$:

$$XP = X^2, \quad P' = 1, \quad P(1) = 1 \Rightarrow f_2(X) = X^2 - 1 + 1 = X^2.$$

Column vector: $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$.

(c) For $P = X^2$:

$$XP = X^3, \quad P' = 2X, \quad P(1) = 1 \Rightarrow f_2(X^2) = X^3 - 2X + 1.$$

Column vector: $\begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$.

Therefore, the matrix of f_2 is:

$$[f_2] = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution 3.6

We compute the inverse of each matrix using two methods:

1. **Matrix A:**

   (a) **Method 1: Using Gauss-Jordan Elimination**

   Augment A with the identity matrix and row-reduce:

$$\begin{pmatrix} 1 & -3 & 5 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ -1 & 2 & -3 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{pmatrix}.$$

So the inverse is:

$$A^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

(b) **Method 2: Using the formula** $A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj}(A)$

First compute $\det A = 1$, then compute the adjugate matrix, giving the same result:

$$A^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

2. **Matrix B:**

   (a) **Method 1: Gauss-Jordan Elimination**

   Augment B with the identity matrix and row-reduce:

$$\begin{pmatrix} 2 & -1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{5}{3} & -\frac{4}{3} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

   So,

$$B^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{5}{3} & -\frac{4}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

   (b) **Method 2: Using the formula** $B^{-1} = \frac{1}{\det B} \cdot \operatorname{adj}(B)$

   Compute $\det B = 3$, then compute the adjugate matrix (via cofactors), and divide by 3 to obtain:

$$B^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{5}{3} & -\frac{4}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

Solution 3.7

Let

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}.$$

1. We compute $A^3 - A$.

   First, compute A^2:

$$A^2 = A \cdot A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -4 & 2 \\ 1 & -1 & -1 \\ 1 & 2 & 0 \end{pmatrix}.$$

   Then, compute $A^3 = A^2 \cdot A$:

$$A^3 = \begin{pmatrix} 3 & -4 & 2 \\ 1 & -1 & -1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 & 2 \\ 0 & 3 & 1 \\ 1 & -2 & 4 \end{pmatrix}.$$

   Now compute $A^3 - A$:

$$A^3 - A = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 3 & 1 \\ 1 & -2 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = 4I.$$

2. From the result above, $A^3 - A = 4I \Rightarrow A(A^2 - I) = 4I$.

   So we can write:

$$A^{-1} = \frac{1}{4}(A^2 - I).$$

Recall from earlier:

$$A^2 = \begin{pmatrix} 3 & -4 & 2 \\ 1 & -1 & -1 \\ 1 & 2 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow A^2 - I = \begin{pmatrix} 2 & -4 & 2 \\ 1 & -2 & -1 \\ 1 & 2 & -1 \end{pmatrix}.$$

Hence,

$$A^{-1} = \frac{1}{4} \begin{pmatrix} 2 & -4 & 2 \\ 1 & -2 & -1 \\ 1 & 2 & -1 \end{pmatrix}.$$

Solution 3.8

1. Determine the matrix associated with f in the canonical bases.

   Let $f : \mathbb{R}^3 \to \mathbb{R}^2$, defined by:

   $$f(x, y, z) = (x + y, y + z).$$

   Apply f to the canonical basis vectors of $\mathbb{R}^3$:

   - $f(1, 0, 0) = (1 + 0, 0 + 0) = (1, 0)$,
   - $f(0, 1, 0) = (0 + 1, 1 + 0) = (1, 1)$,
   - $f(0, 0, 1) = (0 + 0, 0 + 1) = (0, 1)$.

   So the associated matrix is:

   $$\text{Mat}(f) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

2. Compute the rank of the matrix

   $$A = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & -2 \end{pmatrix}.$$

   We perform row operations to reduce to row echelon form.

Step 1: Write the matrix:

$$\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & -2 \end{pmatrix}.$$

Step 2: Replace Row 3 by Row 3 + Row 2:

$$R_3 \to R_3 + R_2 \Rightarrow \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & -2 \end{pmatrix}.$$

Step 3: Replace Row 4 by $R_4 + 2 \cdot R_3$:

$$R_4 \to R_4 + 2R_3 \Rightarrow \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix is now in row-echelon form with 3 non-zero rows. So, the rank of A is 3.

Solution 3.9

1. Determine the vectors $f(e_1)$, $f(e_2)$, $f(2,5)$, and $f(1,3)$.

   Given the matrix of f in the canonical basis:

$$A = \begin{pmatrix} 11 & 30 \\ -11 & 4 \end{pmatrix}.$$

   Recall that:

$$f(v) = A \cdot v.$$

   Compute:

- $f(e_1) = A \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 11 \\ -11 \end{pmatrix}$,
- $f(e_2) = A \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 30 \\ 4 \end{pmatrix}$,
- $$\begin{aligned} f(2,5) = A \cdot \begin{pmatrix} 2 \\ 5 \end{pmatrix} &= 2f(e_1) + 5f(e_2) \\ &= 2\begin{pmatrix} 11 \\ -11 \end{pmatrix} + 5\begin{pmatrix} 30 \\ 4 \end{pmatrix} = \begin{pmatrix} 22 \\ -22 \end{pmatrix} + \begin{pmatrix} 150 \\ 20 \end{pmatrix} = \begin{pmatrix} 172 \\ -2 \end{pmatrix}, \end{aligned}$$
- $$\begin{aligned} f(1,3) = A \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} &= 1f(e_1) + 3f(e_2) \\ &= \begin{pmatrix} 11 \\ -11 \end{pmatrix} + 3\begin{pmatrix} 30 \\ 4 \end{pmatrix} = \begin{pmatrix} 11 \\ -11 \end{pmatrix} + \begin{pmatrix} 90 \\ 12 \end{pmatrix} = \begin{pmatrix} 101 \\ 1 \end{pmatrix}. \end{aligned}$$

2. Give the expression of the function f.

   The action of f on any vector $(x, y) \in \mathbb{R}^2$ is:

   $$f(x,y) = A \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11x + 30y \\ -11x + 4y \end{pmatrix}.$$

   So:

   $$f(x,y) = (11x + 30y, -11x + 4y).$$

Solution 3.10

1. Compute A^2, then find α, β such that $A^2 = \alpha A + \beta I$.

   Given:

   $$A = \begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Compute $A^2 = A \cdot A$:

$$A^2 = \begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix} \begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & -6 \\ -6 & 10 & -12 \\ -3 & 3 & -2 \end{pmatrix}.$$

Now solve for α, β such that:

$$A^2 = \alpha A + \beta I.$$

Compare entries:

$$\alpha \cdot 1 + \beta = 1 \quad \text{(1,1)},$$
$$\alpha \cdot (-8) + \beta = 10 \quad \text{(2,2)}.$$

Solve the system:

$$\alpha + \beta = 1,$$
$$-8\alpha + \beta = 10.$$

Subtract first equation from second:

$$-9\alpha = 9 \Rightarrow \alpha = -1, \quad \beta = 2.$$

Verified: $A^2 = -A + 2I$.

2. Deduce that A is invertible and find A^{-1}.

   From:

$$A^2 = -A + 2I \Rightarrow A^2 + A - 2I = 0 \Rightarrow A(A + I) = 2I.$$

Multiply both sides on the right by $\frac{1}{2}(A+I)^{-1}$:

$$A = 2(A+I)^{-1} \Rightarrow A^{-1} = \frac{1}{2}(A+I).$$

So:

$$\begin{aligned} A^{-1} &= \frac{1}{2}\left(\begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \\ &= \frac{1}{2}\begin{pmatrix} 2 & -3 & 6 \\ 6 & -7 & 12 \\ 3 & -3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{3}{2} & 3 \\ 3 & -\frac{7}{2} & 6 \\ \frac{3}{2} & -\frac{3}{2} & \frac{5}{2} \end{pmatrix}. \end{aligned}$$

3. Find A^{-1} again using the comatrix method.

   Compute:

$$A^{-1} = \frac{1}{\det A} \cdot \text{adj}(A).$$

   (Detailed computation of cofactors and determinant omitted for brevity, but yields same result as above.)

Solution 3.11

1. Matrix:

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}.$$

   Compute the determinant:

$$\det(A) = 1 \cdot 2 - 3 \cdot 4 = 2 - 12 = -10 \neq 0.$$

   Since the determinant is non-zero, the matrix is invertible, and the rank is:

$$\text{rank}(A) = 2.$$

2. Matrix:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The first and second rows are identical. So we remove the second row. The remaining rows are linearly independent. Thus:

$$\text{rank}(A) = 2.$$

3. Matrix:

$$A = \begin{pmatrix} 1 & 1 & -1 \\ -3 & -3 & 3 \\ 2 & 2 & -2 \end{pmatrix}.$$

Observe:

$$\text{Row}_2 = -3 \cdot \text{Row}_1, \quad \text{Row}_3 = 2 \cdot \text{Row}_1.$$

All rows are linearly dependent; only one is independent. Therefore:

$$\text{rank}(A) = 1.$$

4. Matrix:

$$A = \begin{pmatrix} 8 & 4 & -16 \\ 0 & 4 & -8 \\ 4 & 4 & -12 \end{pmatrix}.$$

Use row operations:

$$R_3 \to R_3 - 0.5 \cdot R_1 \Rightarrow \begin{pmatrix} 8 & 4 & -16 \\ 0 & 4 & -8 \\ 0 & 2 & -4 \end{pmatrix},$$

$$R_3 \to R_3 - 0.5 \cdot R_2 \Rightarrow \begin{pmatrix} 8 & 4 & -16 \\ 0 & 4 & -8 \\ 0 & 0 & 0 \end{pmatrix}.$$

Two non-zero rows remain, so:

$$\text{rank}(A) = 2.$$

Chapter 4

Systems of Linear Equations

Linear equations form the foundation of algebra and have wide applications across mathematics, science, and engineering. This chapter explores systems of linear equations - collections of one or more linear equations involving the same variables.

We will examine:

- Fundamental definitions and properties of linear systems
- Matrix representations of linear equations
- Two powerful solution methods: Cramer's Rule and Gaussian Elimination
- Classification of systems based on their solutions

The study of linear systems is essential for understanding more advanced mathematical concepts and has practical applications in fields ranging from computer graphics to economic modeling. We begin with precise definitions that will allow us to systematically analyze and solve these systems.

4.1 Definitions and Properties

Definition 4.1: Linear Equation

A linear equation in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where b and the coefficients $a_1, \ldots, a_n$ are real or complex numbers, usually known in advance.

Example 4.1: Linear and Non-linear Equations

1. The equations

$$4x_1 - 5x_2 + 2 = x_1 \quad \text{and} \quad x_2 = 2\sqrt{6} - x_1 + x_3$$

   are both linear because they can be rearranged algebraically as in equation (1):

$$3x_1 - 5x_2 = -2 \quad \text{and} \quad 2x_1 + x_2 - x_3 = 2\sqrt{6}.$$

2. The equations

$$4x_1 - 5x_2 = x_1 x_2 \quad \text{and} \quad x_2 = 2\sqrt{x_1} - 6$$

   are not linear because of the presence of $x_1 x_2$ in the first equation and $\sqrt{x_1}$ in the second.

Definition 4.2: System of Linear Equations

A system of linear equations (or a linear system) is a collection of one or more linear equations involving the same variables - say, $x_1, \ldots, x_n$. A system of n linear equations in p unknowns with coefficients in $\mathbb{R}$ is:

$$(S) \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1p}x_p = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2p}x_p = b_2, \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{np}x_p = b_n. \end{cases}$$

Example 4.2: System of Linear Equations

An example is:

$$\begin{cases} 2x_1 - x_2 + 1.5x_3 = 8, \\ x_1 - 4x_3 = 7. \end{cases} \tag{4.1}$$

Definition 4.3: Solution of a Linear System

1. A solution of the system is a list $(s_1, s_2, \dots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \dots, s_n$ are substituted for $x_1, \dots, x_n$, respectively.

   For instance, $(5, 6.5, 3)$ is a solution of system (4.1) because, when these values are substituted in (2) for x_1, x_2, x_3 respectively, the equations simplify to $8 = 8$ and $7 = 7$.

2. The set of all possible solutions is called the solution set of the linear system.

3. Two linear systems are called equivalent if they have the same solution set. That is, each solution of the first system is a solution of the second system, and each solution of the second system is a solution of the first.

Example 4.3: Solution Examples

1. Consider:

$$\begin{cases} 5x - 2y = -7, \\ -2x + y = 2 \end{cases} \Rightarrow (x, y) = (-3, -4).$$

2.

$$\begin{cases} 7x - y = -2, \\ x - y = 4 \end{cases} \Rightarrow (x, y) = (-1, -5).$$

Definition 4.4: Matrix Notation

The essential information of a linear system can be recorded compactly in rectangular arrays called matrices. Given the system of m linear equations in n variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{cases}$$

we define the following matrix representations:

- **The coefficient matrix** $(m \times n)$:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

- **The constant vector** $(m \times 1)$:

$$B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

- **The variable vector** $(n \times 1)$:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

- **The augmented matrix** $(m \times (n+1))$:

$$[A|B] = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right).$$

The system can be expressed in two equivalent forms:

- **Matrix equation form**: $AX = B$,
- **Augmented matrix form**: $[A|B]$.

4.2 Solving a Linear System

This section and the next describe an algorithm, or a systematic procedure, for solving linear systems.

Definition 4.5: Solving a Linear System

The process of solving a linear system consists of finding all possible vectors $X = (x_1, x_2, \ldots, x_n)$ that satisfy the matrix equation $AX = B$, where:

- A is an $m \times n$ coefficient matrix,
- X is an $n \times 1$ column vector of variables,
- B is an $m \times 1$ column vector of constants.

4.2.1 Solving by Cramer's Rule

Definition 4.6: Cramer System

A linear system (S) is called a Cramer system if:

- Its coefficient matrix A is square ($n \times n$),
- A is invertible (i.e., $\det(A) \neq 0$).

Theorem 4.1: Cramer's Rule

Let (S): $AX = B$ be a Cramer system where:

- $A = [C_1 | \cdots | C_n]$ is the $n \times n$ coefficient matrix with column vectors $C_1, \ldots, C_n$,
- B is the right-hand side column vector.

The unique solution $X = (x_1, \ldots, x_n)^T$ is given by:

$$x_i = \frac{\det(A_i)}{\det(A)}, \quad 1 \leq i \leq n,$$

where A_i is the matrix formed by replacing the i-th column of A with B:

$$A_i = [C_1|\cdots|C_{i-1}|B|C_{i+1}|\cdots|C_n].$$

Example 4.4: Cramer's Rule Application

Consider the system:

$$(S) : \begin{cases} 3x + 3y - 2z = 5, \\ 2y + 7z = 0, \\ x + y - z = 3. \end{cases}$$

with matrix representation:

$$A = \begin{pmatrix} 3 & 3 & -2 \\ 0 & 2 & 7 \\ 1 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}.$$

1. Verify Cramer condition:

$$\begin{aligned} \det(A) &= 3\begin{vmatrix} 2 & 7 \\ 1 & -1 \end{vmatrix} - 3\begin{vmatrix} 0 & 7 \\ 1 & -1 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} \\ &= 3(-9) - 3(-7) - 2(-2) = -27 + 21 + 4 = -2 \neq 0. \end{aligned}$$

   Thus (S) is a Cramer system.

2. Compute solutions:

$$x = \frac{\begin{vmatrix} 5 & 3 & -2 \\ 0 & 2 & 7 \\ 3 & 1 & -1 \end{vmatrix}}{\det(A)} = \frac{-30}{-2} = 15,$$

$$y = \frac{\begin{vmatrix} 3 & 5 & -2 \\ 0 & 0 & 7 \\ 1 & 3 & -1 \end{vmatrix}}{\det(A)} = \frac{-28}{-2} = 14,$$

$$z = \frac{\begin{vmatrix} 3 & 3 & 5 \\ 0 & 2 & 0 \\ 1 & 1 & 3 \end{vmatrix}}{\det(A)} = \frac{8}{-2} = -4.$$

The unique solution is $(x, y, z) = (15, 14, -4)$.

4.3 Solving by Gaussian Elimination

The following operations can be performed on any matrix to obtain an equivalent system :

Definition 4.7: Elementary Row Operations

1. **Row Replacement:** Replace row i with the sum of itself and a nonzero multiple of row j:

$$R_i \to R_i + cR_j \quad (i \neq j, c \neq 0).$$

2. **Row Interchange:** Swap two distinct rows:

$$R_i \leftrightarrow R_j \quad (i \neq j).$$

3. **Row Scaling:** Multiply all entries in a row by a nonzero constant:

$$R_i \to cR_i \quad (c \neq 0).$$

Remark 4.1: Properties of Elementary Row Operations

These operations:

- Preserve the solution set of the corresponding linear system,
- Are reversible (each operation has an inverse operation),
- Are used in Gaussian elimination to obtain row echelon form.

4.3.1 Gaussian Elimination Method

Definition 4.8: Gaussian Elimination

Gaussian elimination is a systematic procedure for solving systems of linear equations using elementary row operations to transform the augmented matrix into row echelon form (REF).

Algorithm 4.1: Gaussian Elimination Steps

1. **Forward Elimination:**
    (a) Start with the leftmost nonzero column (pivot column),
    (b) Select a nonzero entry (pivot) in this column, preferably 1,
    (c) Move the pivot row to the top position if needed (Row Interchange),
    (d) Create zeros below the pivot using row replacement operations,
    (e) Repeat for each subsequent pivot position, moving right and down.
2. **Back Substitution:**
    (a) Starting from the last nonzero row, solve for the leading variable,
    (b) Substitute this value into the rows above,
    (c) Repeat upward through all rows.

Example 4.5: Gaussian Elimination - Unique Solution

Solve the system:

$$\begin{cases} x + 2y + z = 4, \\ 2x + 5y - z = 5, \\ x + 3y + 4z = 7. \end{cases}$$

1. Augmented matrix:

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 2 & 5 & -1 & 5 \\ 1 & 3 & 4 & 7 \end{pmatrix}.$$

2. Forward elimination:

    - $R_2 \to R_2 - 2R_1$,
    - $R_3 \to R_3 - R_1$,

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -3 & -3 \\ 0 & 1 & 3 & 3 \end{pmatrix},$$

    - $R_3 \to R_3 - R_2$,

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 6 & 6 \end{pmatrix}.$$

3. Back substitution:

$$\begin{aligned} 6z = 6 &\Rightarrow z = 1, \\ y - 3(1) = -3 &\Rightarrow y = 0, \\ x + 2(0) + 1 = 4 &\Rightarrow x = 3. \end{aligned}$$

Solution: $(3, 0, 1)$.

Example 4.6: Gaussian Elimination - Infinite Solutions

Solve:

$$\begin{cases} x + y + z = 3, \\ 2x + 2y + 4z = 8, \\ x + y + 2z = 4. \end{cases}$$

1. Augmented matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 2 & 2 & 4 & 8 \\ 1 & 1 & 2 & 4 \end{pmatrix}.$$

2. Forward elimination:

   - $R_2 \to R_2 - 2R_1$,
   - $R_3 \to R_3 - R_1$,

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

   - $R_3 \to R_3 - \frac{1}{2}R_2$,

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

3. Solution:

$$\begin{cases} x + y + z = 3, \\ 2z = 2, \end{cases} \Rightarrow \begin{cases} z = 1, \\ x = 3 - y - 1 = 2 - y. \end{cases}$$

General solution: $(2 - t, t, 1)$ where $t \in \mathbb{R}$.

4.4 Exercises

Exercise 4.1

Solve the following system using four different methods:

- by substitution,
- by the Gaussian elimination method (pivot method),
- by inverting the coefficient matrix,
- using Cramer's rule.

$$\begin{cases} 2x + y = 1, \\ 3x + 7y = -2. \end{cases}$$

Exercise 4.2

Depending on the values of a, choose the method you consider the most efficient to solve the following systems:

1.
$$\begin{cases} ax + y = 2, \\ (a^2 + 1)x + 2ay = 1. \end{cases}$$

2.
$$\begin{cases} (a + 1)x + (a - 1)y = 1, \\ (a - 1)x + (a + 1)y = 1. \end{cases}$$

Exercise 4.3

Study the existence of solutions for the following system:

$$\begin{cases} ax + by + z = 1, \\ x + aby + z = b, \\ x + by + az = 1. \end{cases}$$

Exercise 4.4

Using the Gaussian elimination method (pivot method), solve the following systems:

1. $$S(1) = \begin{cases} x + y + z = 3, \\ 2x - y + z = 1, \\ 3x + y + 2z = 4. \end{cases}$$

2. $$S(2) = \begin{cases} x + y + z = 6, \\ 2x - y + z = 2, \\ 3x + y + 2z = 12. \end{cases}$$

3. $$S(3) = \begin{cases} x + y + z + w = 2, \\ 2x + 3y + z + w = 5, \\ 3x + 5y + 2z + w = 8, \\ 4x + 7y + 3z + 2w = 10. \end{cases}$$

Exercise 4.5

Consider the system:

$$\begin{cases} x + y - z = 1, \\ 2x + 3y + \lambda z = 3, \\ x + \lambda y + 3z = -3. \end{cases}$$

Determine the values of $\lambda \in \mathbb{R}$ such that the system has:

1. No solution;

2. A unique solution;

3. Infinitely many solutions.

Exercise 4.6

Using the Gaussian elimination method (pivot method), solve the following systems:

1. $$S(1) = \begin{cases} x + y + z = 3, \\ 2x - y + z = 1, \\ 3x + y + 2z = 4. \end{cases}$$

2. $$S(2) = \begin{cases} x + y + z = 6, \\ 2x - y + z = 2, \\ 3x + y + 2z = 12. \end{cases}$$

3. $$S(3) = \begin{cases} x + y + z + w = 2, \\ 2x + 3y + z + w = 5, \\ 3x + 5y + 2z + w = 8, \\ 4x + 7y + 3z + 2w = 10. \end{cases}$$

Exercise 4.7

Consider the system:

$$\begin{cases} x + y - z = 1, \\ 2x + 3y + \lambda z = 3, \\ x + \lambda y + 3z = -3. \end{cases}$$

Determine the values of $\lambda \in \mathbb{R}$ such that the system has:

1. No solution;
2. A unique solution;
3. Infinitely many solutions.

Exercise 4.8

Let $m \in \mathbb{R}$. Consider the system $S(m)$:

$$S(m) = \begin{cases} (m-1)x + y - z = m, \\ 2x + my + z = 3, \\ mx + (1-m)y + mz = m^2. \end{cases}$$

1. Solve the system in the case where it is a Cramer system.
2. Solve the system in the case where it is not a Cramer system.

Conclusion

This introductory course in linear algebra has presented the essential concepts and methods related to vector spaces, linear mappings, matrices, and systems of linear equations. Through theoretical explanations and numerous exercises, students are equipped with the foundational tools necessary for further studies in mathematics and its applications. We encourage students to deepen their understanding by consulting the suggested references and practicing regularly. Mastery of these fundamental topics will be crucial for success in more advanced mathematical courses.

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