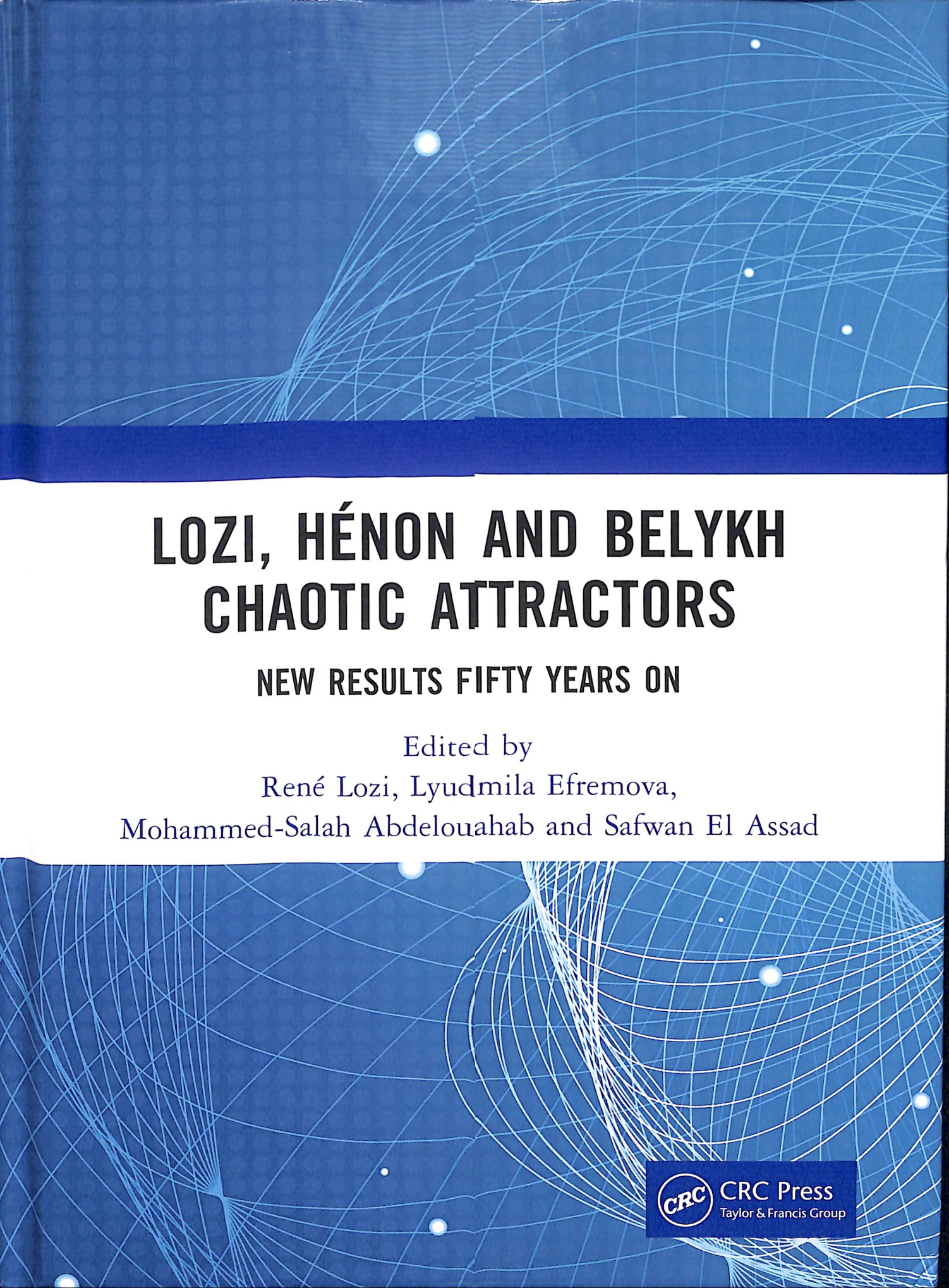


The background of the book cover is a deep blue. It features a complex pattern of thin, white, curved lines that sweep across the frame, creating a sense of motion and depth. Interspersed among these lines are several small, bright white dots, some of which appear to be part of the line patterns, while others are isolated. The overall effect is reminiscent of a mathematical plot or a celestial map.

LOZI, HÉNON AND BELYKH CHAOTIC ATTRACTORS

NEW RESULTS FIFTY YEARS ON

Edited by

René Lozi, Lyudmila Efremova,
Mohammed-Salah Abdelouahab and Safwan El Assad



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Lozi, Hénon and Belykh Chaotic Attractors

Over the past fifty years, the development of chaotic dynamical systems theory and its subsequent wide applicability in science and technology has been an extremely important achievement of modern mathematics. Chaotic attractors are not a fleeting curiosity, and their continued study is important for the progress of mathematics.

This book collects several of the new relevant results on the most important of them: the Lozi, Hénon, and Belykh attractors. Existence proofs for strange attractors in piecewise-Lozi, Hénon, and Belykh maps are given. Generalization of Lozi map in smooth nonlinear Lozi-Hénon and Belykh maps are given. Generalization of Lozi map in higher dimensions, Markov partition, or embedding into the 2D border collision normal form of this map are considered. K-symbol fractional order discrete-time and relationship between this map and maxtype difference equations are explored. Statistical self-similarity, control of chaotic transients, and target-oriented control of Hénon and Lozi attractors are presented. Controlling chimera and solitary states by additive noise in networks of chaotic maps, detecting invariant expanding cones for generating word sets to identify chaotic maps, detecting invariant expanding cones for generating word sets to identify chaos in piecewise-linear maps, and studying border collision bifurcations in a piecewise linear duopoly model complete this book.

This book is an essential companion for students and researchers in mathematics, physics, engineering, and related disciplines seeking to deepen their understanding of chaotic dynamical systems and their applications.

The chapters in this book were originally published in *Journal of Difference Equations and Applications*.

René Lozi is Emeritus Professor at Université Côte d'Azur, France, and Vice-President of the International Society of Difference Equations. His research areas include complexity and emergence theory, dynamical systems, bifurcations, control of chaos, cryptography based on chaos, and memristors.

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Lozi, Hénon and Belykh Chaotic Attractors

New Results Fifty Years On

Edited by
**René Lozi, Lyudmila Efremova,
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and Safwan El Assad**

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Chapter 11

Border collision bifurcations in a piecewise linear duopoly model

Laura Gardini and Davide Radi

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Chapter 12

Detecting invariant expanding cones for generating word sets to identify chaos in piecewise-linear maps

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