Generalized Combination Synchronization of Three Different Dimensional Fractional Chaotic and Hyperchaotic Systems Using Three Scaling Matrices

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Abstract--- In this paper, we present a new approach to investigate generalized combination synchronization (GCS) of three different dimensional fractional chaotic and hyperchaotic systems by using three scaling matrices. By exploiting the fractional Lyapunov theorem and a new property of Caputo fractional derivative, an active controller is designed to confirm the achievement of the desired synchronization. Finally, in order to prove the reliability of the theoretical results obtained, we present two numerical examples.

Keywords--- Generalized Combination Synchronization, Chaotic System, Fractional-Order System, Fractional, Lyapunov Theorem.

I. Introduction

In recent periods, fractional equations have emerged, which attract attention in various fields. Indeed, it has been found that several theoretical and experimental studies show that certain thermal systems [1], physics systems[2] and rheological systems[3] are governed by differential equations with fractional derivatives. However, its application in chaotic and hyperchaotic systems is the most attractive one, such as the fractional version of: Lorenz system[4,] Lü system [5,] modified Rossler system and Liu system[7].

Synchronization of the fractional systems is an interesting subject in nonlinear science, thanks to its many applications, especially in control processing[8] and secure communication[9].

In the majority of published works, several approaches for synchronization have been introduced, such as complete synchronization[10, hybrid] projective synchronization and generalized synchronization[12 and 13].

Recently, to achieve combination synchronization between many drive-response systems, Luo et al. [14] have proposed an active back stepping controller to illustrate its results. In[15,] function projective combination synchronization between three fractional systems are presented. The generalization of combination-combination synchronization between many fractional-order systems is studied in[16].

However, most of the previous papers have been realized to discuss the combination synchronization between same dimensional integer and fractional-order systems. Thus, a big question is asked: does the combination synchronization happen between three or more different dimensional fractional systems?

To answer the above question, in this paper, an active controller technique is adopted to fulfill the GCS of three *different dimensional* fractional chaotic and hyperchaotic systems by using three scaling matrices. In order to prove the reliability of the theoretical results obtained, we present two numerical examples.

II. Preliminaries

Several approaches and definitions of fractional operators have been employed in the literature. For example, the Caputo's fractional operator[17] is defined as:

$$d^{\alpha}y(s) = I^{n-\alpha}Y^{n}(s), \qquad (1)$$

Where $\alpha \in (n-1, n)$, I^{α} ($\alpha > 0$) is the α-order Riemann- Liouville integral operator, which is defined as:

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$$I^{\alpha} y(s) = \frac{1}{\Gamma(\alpha)} \int_{0}^{s} (s-\tau)^{\alpha-1} y(\tau) d\tau, \qquad (2)$$

And

$$\Gamma(\alpha) = \int_{0}^{+\infty} x^{\alpha-1} \exp(-x) dx, \qquad (3)$$

Is the gamma function.

Theorem 1: Consider the equation:

$$d^{\alpha} x = f(x), \qquad (4)$$

Where d^{α} is Caputo's differential operator ($0 < \alpha \leq 1$), x is the state variable and f is the continuous reel function.

When the constructed Lyapunov function V satisfies: $V(x) > 0$ and $d^{\alpha} V(x) < 0$, Equ. (4) is asymptotically stable.

Lemma 2: Consider a differentiable function ξ in the sense of Caputo, then we have

$$\frac{1}{2} d^{\alpha} \left(\xi^{T} \xi \right) \leq \xi^{T} d^{\alpha} \xi, \qquad \alpha \in (0,1). \qquad (5)$$

III. General Schemes of GCS

In this section, the general schemes of GCS between three fractional chaotic systems are considered by using three scaling matrices. For this, we consider the dynamic systems as follows:

$$d^{\alpha} x = \xi(x), \qquad (6)$$

$$d^{\alpha} y = \zeta(y), \qquad (7)$$

$$d^{\alpha} z = \psi(z) + u, \qquad (8)$$

Where $0 < \alpha \leq 1$, $x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$, $y = (y_1, y_2, ..., y_n)^T \in \mathbb{R}^n$ are the state variables of two drive systems, $z = (z_1, z_2, ..., z_m)^T \in \mathbb{R}^m$ ($n < m$) is the state variable of response system, ξ, ζ : $\mathbb{R}^n \to \mathbb{R}^n$ and ψ : $\mathbb{R}^m \to \mathbb{R}^m$ are the continuous reel functions and $u = (u_1, u_2, ..., u_m)^T \in \mathbb{R}^m$ is a controller vector which will be designed.

The definition of the proposed GCS is given as follows:

Definition 3: The drive systems (6)-(7) and the response system (8) are said to achieve GCS in dimension *n* or in *m*, if there exists three scaling matrices $Q = (Q)_{d \times n}$, $R = (R)_{d \times n}$ and $S = (S)_{d \times m}$, such that the error system:

$$e(t) = Qx(t) + Ry(t) - Sz(t), \qquad (9)$$

Satisfies:

$$\| Qx(t) + Ry(t) - Sz(t) \| \to 0, \qquad \text{when } t \to +\infty. \qquad (10)$$

Remark 4: In the previous definition, we can replace the constant matrices Q, R and S by functional matrices of the variables *x, y* and *z*.

Remark 5: If $Q=I$, $R=0$ and $S \neq 0$ (or $Q=0$, $R=I$ and $S \neq 0$), GCS problem becomes projective synchronization problem, where I is $n \times n$ identity matrix.

Remark 6: If the scaling matrix $Q=R=0$ and $S \neq 0$, then GCS problem becomes chaos control problem.

3.1 Reduced-Order GCS of Chaotic Systems

Here, we assume that the two drive systems are, respectively, given as follows:

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$$d^{\alpha}x = P_1 x + \zeta_1(x), \qquad (11)$$
$$d^{\alpha}y = P_2 y + \zeta_2(y), \qquad (12)$$

Where $P_1 = (P_1)_{n\times n}$, $P_2 = (P_2)_{n\times n}$ and ζ_1, ζ_2 :ℝⁿ→ℝⁿ are the linear parts and the nonlinear parts of (11) and (12), respectively.

The controlled response system is described as (8).

Hence:

$$\begin{aligned} d^{\alpha}e &= QD^{\alpha}x + RD^{\alpha}y - SD^{\alpha}z \\ &= (P_1+P_2)e + \left[QP_1 - (P_1+P_2)Q\right]x + \left[RP_2 - (P_1+P_2)R\right]y + \\ &+ (P_1+P_2)Sz + Q\zeta_1(x) + R\zeta_2(y) - S\left(u + \psi(z)\right). \qquad (13) \end{aligned}$$

So, the main result of this section is obtained as follows:

Theorem 9: If the control active $u = (\hat{u}, 0, \ldots, 0)^T$ is chosen as:

$$\hat{u} = -\psi(z) + S_n^{-1}\begin{bmatrix} Ce + \left[QP_1 - (P_1+P_2)Q\right]x + \left[RP_2 - (P_1+P_2)R\right]y + \\ + (P_1+P_2)Sz + Q\zeta_1(x) + R\zeta_2(y) \end{bmatrix}, \qquad (14)$$

Where $\hat{u} = (u_1, u_2, \ldots, u_n)^T \in \mathbb{R}^n$, S^{-1}_n is the inverse matrix of $S_n = (S)_{n\times n}$ and $C = (C)_{n\times n}$ is a control matrix, and if $P_1 + P_2 - C$ is a negative definite matrix, then the drive systems (11)-(12) achieve the GCS with the response systems (8).

Proof: According to (14), (13) becomes:

$$d^{\alpha}e = (P_1 + P_2 - C)e. \qquad (15)$$

We can choose the function V as:

$$V(e) = \tfrac{1}{2}e^T e. \qquad (16)$$

By using **Lemma 2**,

$$\begin{aligned} d^{\alpha}V(e) &= d^{\alpha}\left(\frac{1}{2}e^T e\right) \\ &\le e^T d^{\alpha}e \\ &\le e^T(P_1 + P_2 - C)e < 0. \qquad (17) \end{aligned}$$

According to the fractional Lyapunov **Theorem 1,** we know that the system (17) asymptotically converges to zero, which means that the systems (11), (12) and (8) achieve the GCS.

3.2 Increased-Order GCS of Chaotic Systems

Here, the two drive systems are, respectively, given as in (6)-(7), and the controlled response system is described as:

$$D^{\alpha}z = Mz + H(z) + u, \qquad (18)$$

Where $M = (M)_{m\times m}$ and H:ℝᵐ→ℝᵐ are the linear part and the nonlinear part of system (18), respectively.

Hence:

$$\begin{aligned} D^{\alpha}e &= QD^{\alpha}x + RD^{\alpha}y - SD^{\alpha}z \\ &= Me + M\left[Sz - (Qx + Ry)\right] - S\left[u + Mz + H(z)\right] + Q\xi(x) + R\zeta(y). \qquad (19) \end{aligned}$$

So, we have:

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Theorem 11: We assume that u satisfies:

$$u = -Mz - H(z) + S^{-1}\left[Ke + M\left(Sz - \left(Qx + Ry\right)\right) + Q\xi(x) + R\zeta(y)\right], \qquad (20)$$

Where S^{-1} is the inverse matrix of $S = \left(S_{m\times m}\right)$ and $K = (K)_{m\times m}$ is the control matrix.

We assume also that *M-K* is a negative definite matrix, then the systems (6), (7) and (18) achieve the GCS.

Proof. According to control (20), (19) becomes:

$$d^{\alpha}e = (M - K)e. \qquad (21)$$

Here, we can choose V as:

$$V(e) = \frac{1}{2}e^{T}e. \qquad (22)$$

By using **Lemma 2**,

$$\begin{aligned} d^{\alpha}V(e) &= \frac{1}{2}d^{\alpha}(e^{T}e) \\ &\le e^{T}d^{\alpha}e \\ &\le e^{T}(M - K)e < 0. \qquad (23) \end{aligned}$$

According to the fractional Lyapunov **Theorem 1,** we know that the system (17) asymptotically converges to zero, which means that the drive systems (6)-(7) achieve GCS with the response systems (18).

IV. Numerical Simulations

In order to prove the reliability of the theoretical results obtained, we present two numerical examples.

Consider the following Lorenz system[4] as the first drive system:

$$\begin{cases} d^{\alpha}x_1 = -\lambda(x_1 - x_2), \\ d^{\alpha}x_2 = \eta x_1 - (x_2 + x_1x_3), \\ d^{\alpha}x_3 = -\theta x_3 + x_1x_2. \end{cases} \qquad (24)$$

For $\alpha = 0.98$, the system (24), exhibits a chaotic behavior, as shown in Fig.1, when:

$$\lambda = 10,\ \eta = 28 \qquad \text{and } \theta = \frac{8}{3}, \qquad (25)$$

And the initial conditions:

$$x^{0}{}_{1} = 1,\ x^{0}{}_{2} = 2 \text{ and } x^{0}{}_{3} = 1 \qquad (26)$$

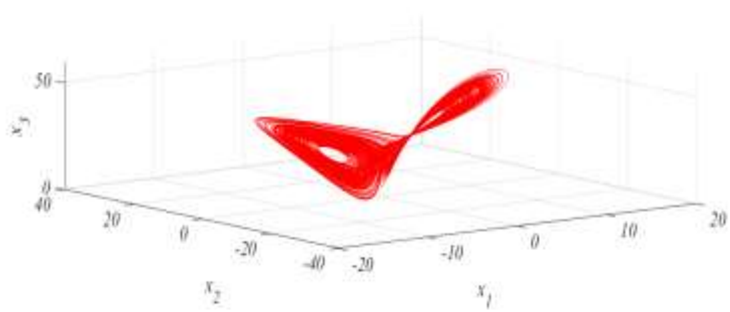

Fig. 1: Attractor of the System (25)

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Consider the following Lü system[5] as the second drive system:

$$\begin{cases} d^{\alpha} y_1 = -\delta(y_1 - y_2), \\ d^{\alpha} y_2 = -y_1 y_3 + \varepsilon y_2, \\ d^{\alpha} y_3 = y_1 y_2 - \mu y_3, \end{cases} \quad (25)$$

For $\alpha = 0.98$, the system (25) exhibits a chaotic behavior as shown in Fig. 2, when

$$\delta = 36,\ \varepsilon = 16 \quad \text{and } \mu = 3, \quad (26)$$

And the initial conditions

$$y^0{}_1 = 0.5,\ y^0{}_2 = -0.5 \quad \text{and } y^0{}_3 = 0.5 \quad (27)$$

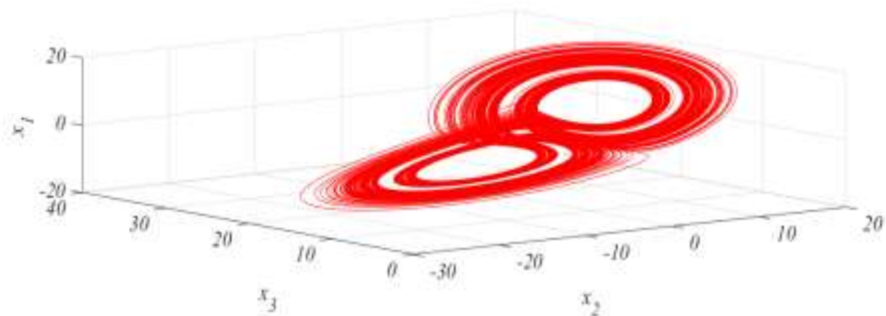

Fig. 2: Attractor of the Lü System (26)

The controlled hyperchaotic Liu system[7] is derived as:

$$\begin{cases} d^{\alpha} z_1 = \rho(z_2 - z_1) + u_1, \\ d^{\alpha} z_2 = z_1 z_3 + \sigma z_1 - z_4 + u_2, \\ d^{\alpha} z_3 = -z_1 z_2 - \omega z_3 + z_4 + u_3, \\ d^{\alpha} z_4 = \gamma z_1 + z_2 + u_4, \end{cases} \quad (28)$$

For $\alpha = 0.98$, the system (28) (without the controller u_1, u_2, u_3, u_4), exhibits a hyperchaotic behavior as shown in Fig. 3, when:

$$\rho = 10,\ \sigma = 35, \omega = 1.4 \quad \text{and } \gamma = 5, \quad (29)$$

And the initial conditions:

$$z^0{}_1 = 0.2,\ z^0{}_2 = 0.2,\ z^0{}_3 = 0.2 \text{ and } z^0{}_4 = 0.2 \quad (30)$$

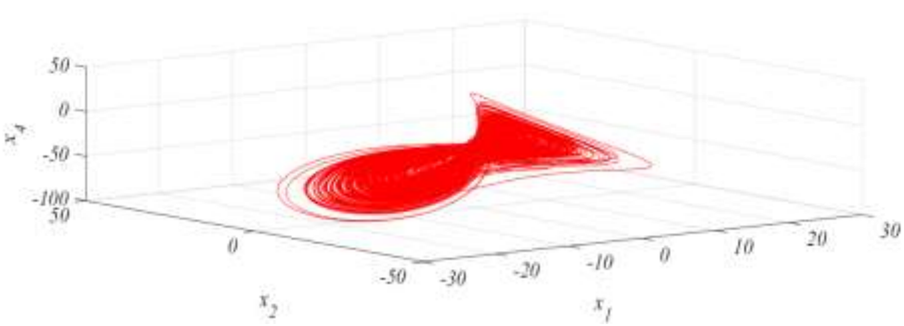

Fig. 3: Hyperhaotic Attractor of the System (28)

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4.1 Reduced-Order GCS between Systems (26)-(27) and (28)

Here, we take:

$$Q=\begin{pmatrix}1 & 2 & 2\\ 5 & -1 & 1\\ 4 & 1 & 2\end{pmatrix},\qquad R=\begin{pmatrix}0 & -2 & -2\\ 1 & 2 & 1\\ 2 & 0 & 2\end{pmatrix},$$

$$S=\begin{pmatrix}2 & 2 & 3 & 4\\ 1 & 1 & 5 & 5\\ 1 & 0 & 0 & 2\end{pmatrix}\quad and \quad C=\begin{pmatrix}2-\delta-\lambda & \lambda+\delta & 2\\ \eta+\varepsilon & 0 & -1\\ -1 & 0 & 1-\mu-\theta\end{pmatrix}.$$

According to (14), (15) becomes:

$$\begin{pmatrix}d^{\alpha}e_1\\ d^{\alpha}e_2\\ d^{\alpha}e_3\end{pmatrix}=\begin{pmatrix}-2 & 0 & -2\\ 0 & -1 & 1\\ 1 & 0 & -1\end{pmatrix}\begin{pmatrix}e_1\\ e_2\\ e_3\end{pmatrix}. \qquad (31)$$

We can show that P_1+P_2-C is a negative definite matrix. Then the condition of **Theorem 1** is satisfied. Hence the reduced-order GCS between systems (26)-(27) and (28) is achieved. Fig. 4 displays the trajectories of the synchronization error (31), with $e^0{}_1=-9, e^0{}_2=3.6 \quad and \quad e^0{}_3=9.8.$

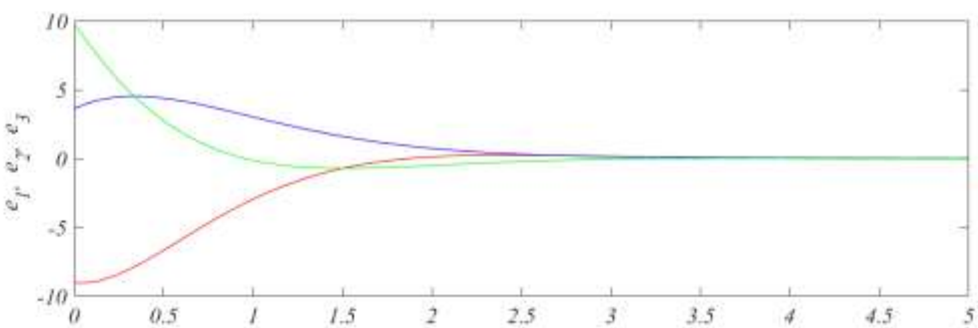

Fig. 4: The Trajectories of the Synchronization Error (31)

4.2 Increased-Order GCS between Systems (26)-(27) and (28)

Here, we take

$$Q=\begin{pmatrix}1 & -1.15 & 1\\ 2 & -0.5 & -1\\ -4 & 2 & 0\\ 1 & -1 & 4\end{pmatrix},\qquad R=\begin{pmatrix}2 & 1 & 1\\ 1 & -1 & 2\\ 1 & -2 & 3\\ -4 & 0 & 1\end{pmatrix},$$

$$S=\begin{pmatrix}-1 & 1 & 1 & 1\\ 2 & 2 & 2 & 2\\ 4 & 4 & 4 & 4\\ 2 & 2 & 2 & 2\end{pmatrix}\quad and \quad K=\begin{pmatrix}1-\rho & \rho & -1 & 0\\ \sigma & 2 & 0 & -3\\ -1 & 0 & 3-\omega & 1\\ \gamma & -1 & 0 & 4\end{pmatrix}.$$

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Therefore:

$$\begin{pmatrix} d^{\alpha}e_1 \\ d^{\alpha}e_2 \\ d^{\alpha}e_3 \\ d^{\alpha}e_4 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 2 \\ 1 & 0 & -3 & 0 \\ 0 & 2 & 0 & -4 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}. \qquad (32)$$

We can show that *M-K* is a negative definite matrix. Then the condition of **Theorem 1** is satisfied. Hence the increased-order GCS between systems (26)-(27) and (28) is achieved. Fig. 5 displays the trajectories of the synchronization error (31), with

$$e^0{}_1 = 0.3, e^0{}_2 = 0.4, e^0{}_3 = -0.2 \quad \textit{and} \quad e^0{}_4 = -0.1.$$

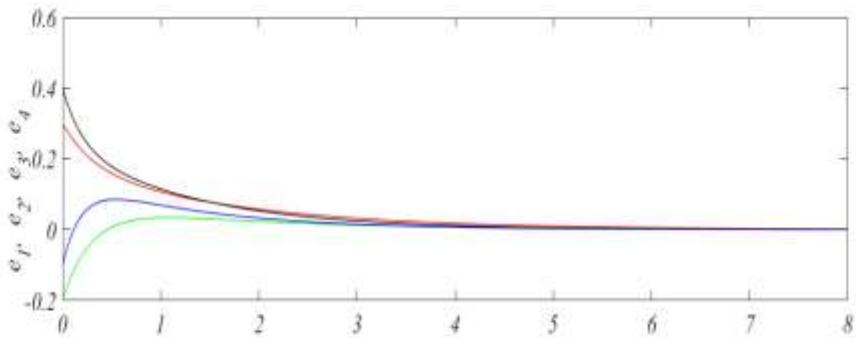

Fig. 5: The Trajectories of the synchronization error (32)

V. Conclusion

In this paper work, we have developed a new strategy to study the GCS of three different dimensional fractional chaotic systems by exploiting three scaling matrices. With the help of the fractional Lyapunov theorem and technical of active control, some sufficient hypothesizes are proposed to achieve the GCS. Two numerical examples were provided to validate the desired synchronization method.

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