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Numerical and Theoretical Study of Double Delay Integral Equation

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Soutenue devant le jury

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DEDICATION

In the name of the Most Merciful, ALLAH, I offer this modest work as a dedication to:

- Myself,
- My beloved father, Tamim, who has shown me unwavering generosity, love, and guidance throughout my journey,
- My first mother, Fouzia, and second mother, Nora, with whom I share a bond of love and respect,
- My sisters, Wafa and Nihad, who have been by my side through thick and thin,
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- Everyone who has ever extended a helping hand when I needed it,
- And lastly, to you, the reader.

Thank you sincerely for your time and consideration.

Sara

ملخص

في هذه المذكرة، تم دراسة وجود و وحدانية الحل لمعادلات فولتيرا التكاملية اللاخطية مضاعفة التاخير. تم تطبيق طريقة التجميع لتايلور لإعطاء الحل التقريبي لمعادلات فولتيرا التكاملية الخطية مع تأخيرين ثابتين، كما تم دراسة تقارب الحل التقريبي للحل الدقيق باستخدام متراجحات غرونوال و إثبات أن رتبة التقارب تساوي m-1. في الأخير تم إدراج أمثلة عددية لتأكيد النتائج النظرية وتقارب الخوارزمية المقدمة.

الكلمات المفتاحية: حل قابل للقياس، حل مستقر ، حل محدود، حل متكامل، معادلات فولتيرا الخطية ذات التأخير المزدوج، طريقة التجميع، كثيرات الحدود تايلور.

ABSTRACT

This dissertation studies the existence and uniqueness of double delay integral equations. Taylor collocation method is applied to obtain the approximate solution for linear Volterra integral equations with double constant delay. The method is based on the use of Taylor polynomials developed for the numerical solution of this type of equation. We also conducted a careful computational analysis that justifies the errors generated in the approximate solution up to the order $m-1$. Numerical examples are included to prove the convergent algorithms validity and efficacy.

Key Words: Measurable solution; Exponentially stable solution; Bounded solution; Integrable solution; Linear Volterra double delay integral equations; Collocation method; Taylor polynomials.

RÉSUMÉ

Dans ce mémoire, nous étudions l'existence et le résultat d'unicité des équations intégrales à double retard. La méthode de " Taylor collocation " est appliquée pour obtenir la solution approchée des équations intégrales linéaires de Volterra avec un double retard constant. La méthode basée sur l'utilisation des polynômes de Taylor est développée pour la résolution numérique de ce type d'équation. Nous présentons également une étude théorique qui justifie les résultats numériques pour que l'ordre de convergence soit égal à $m-1$. Des exemples illustratifs sont inclus pour prouver la validité et l'efficacité des algorithmes convergents présentés.

Mots-clés: Solution mesurable ; Solution exponentiellement stable ; Solution bornée ; Solution intégrable ; Équations intégrales linéaires à double retard de Volterra; Méthode de collocation; Polynômes de Taylor.

CONTENTS

INTRODUCTION

Integral equations are very valuable mathematical tools used in pure and applied analysis. The first mention of an integral equation in mathematical literature can be attributed to Abel in 1823, who encountered it while working on a mechanics problem. Abel's elegant solution was published in 1826.

In 1896, Vito Volterra developed a theory of integral equations, considering their solutions as the inverses of specific integral operators. Ivar Fredholm made a significant contribution in 1900, leading to an intriguing period of advancement in mathematical analysis. Poincaré, Fréchet, Hilbert, Schmidt, Hardy, and Riesz were prominent figures involved in this new field of research.

Volterra integral equations, introduced by Vito Volterra and further studied by Traian Lalescu in his 1908 thesis under Emile Picard's guidance, appear in various mathematical, scientific, and engineering problems.

Double delay integral equations (DDVIEs) are specialized integral equations involving two-time delays. They find applications in engineering, physics, and biology, such as control systems, biomedical engineering, signal processing, and finance.

The primary objective of this work is to find an approximate solution for linear

Volterra double delay integral equations (DDVIEs) using the Taylor collocation method (TCM).

The collocation method aims to approximate the exact solution of an integral equation by employing a suitable function from a chosen finite-dimensional space. The approximate solution must satisfy the integral equation at specific points on the interval, known as the set of collocation points. In this study, we consider the space of spline polynomials, denoted as $S_{m-1}^{(-1)}(\Pi_N)$, as the approximation space defined in (3.2.1)

The key advantages of this method are:

- It is a direct method providing explicit formulas for the approximate solution.

- The method has a convergence order.

- Solving an algebraic system is unnecessary, making the proposed algorithm highly effective and easy to implement.

The dissertation is structured as follows:

- **Chapter One**: This chapter covers fundamental notions, definitions, and necessary theorems that will be utilized in the subsequent chapters.
- **Chapter Two**: Here, we establish existence and uniqueness theorems for nonlinear integral equations with two constant delays.
- **Chapter Three**: We present a numerical method based on Taylor polynomials, which constructs a collocation solution in the piecewise polynomial spline space $S_{m-1}^{(-1)}(\Pi_N)$. We prove the convergence of the approximate solution to the exact solution. Theoretical results are supported by numerical examples.

CHAPTER 1

GENERALS AND NOTIONS FUNDAMENTALS

In this chapter, we define some necessary notions and theories, such as the Taylor series, integral equations, and some definitions which will prove existence theorems of DDIEs, and also comparison theorems which will demonstrate the convergence of the approximate solutions.

1.1 Taylor series

Definition 1.1.1 *(see [13]) Let the function f be n times differentiable at a. Then the polynomial*

$$T_n = \sum_{j=0}^{n} \frac{f^j(a)}{j!}(x-a)^j,$$

is called the n^{th} taylor polynomial of f at a . If f is infinitely differentiable at a, the series

$$T(x) = \sum_{j=0}^{\infty} \frac{f^j(a)}{j!}(x-a)^j,$$

is called the taylor series of f at a.

Example 1.1.1 *For a function $f(x) = 2x + 2x^2$ whose first and second partial derivatives exist at the point -3. The third-degree Taylor polynomial of f around the point -3 is :*

$$\begin{aligned} T_3(x) &= f(-3) + f^{'}(-3)(x-(-3)) + \frac{f^{''}(-3)}{2!}(x-(-3))^2 + \frac{f^{'''}(-3)}{3!}(x-(-3))^3 \\ &= -24 + 14(x+3) - 2(x+3)^2. \end{aligned}$$

1.2 Integral equations

In mathematics, an integral equation is an equation where the unknown function is inside an integral. A standard integral equation in u(x) is of the form:

$$\alpha(x)u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x, t, u(t))dt, \tag{1.2.1}$$

and if equation (1.2.1) is linear then it becomes

$$\alpha(x)u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x, t)u(t)dt, \tag{1.2.2}$$

where $g(x)$ and $h(x)$ are the limits of integration, may be variables, constants, or mixed, λ is a constant parameter, and $K(x, t)$ is a known function, of two variables x and t, called the kernel or the nucleus of the integral equation, the functions $f(x)$ and $\alpha(x)$ are a known. The unknown function $u(x)$ that will be determined appears inside the integral sign. In many other cases, the unknown function $u(x)$ appears inside and outside the integral sign.

1.2.1 Classification of linear integral equations

These equations are classified according to three dichotomies:

i) ***Location of the unknown equation***

First kind: an integral equation is called an integral equation of the first kind if $\alpha(x) = 0$, the equation is written:

$$f(x) = \lambda \int_{g(x)}^{h(x)} k(x, t)u(t)dt.$$

Second kind: an integral equation is called an integral equation of the second kind if $\alpha(x) = 1$, the equation is written:

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t)u(t)dt.$$

Third kind: an integral equation is called an integral equation of the third kind if it is a linear Integral equation of the following form :

$$\alpha(x)u(x) + \lambda \int_{g(x)}^{h(x)} k(x,t)u(t)dt = f(x).$$

where $\alpha(t)$ vanishes at a finite number of points in its domain.

ii) ***Limits of integration***

Fredholm linear integral equations (*FIEs*)**:** an equation of the form (1.2.2) whose integration bounds a and b are fixed is called a linear Fredholm integral equation.

1. Fredholm linear integral equations of the first kind:

$$f(x) = \lambda \int_{a}^{b} k(x,t)u(t)dt,$$

Example 1.2.1

$$cos(x) = \int_{0}^{1} (x-t)u(t)dt.$$

2. Fredholm linear integral equations of the seconde kind:

$$u(x) = f(x) + \lambda \int_{a}^{b} k(x,t)u(t)dt,$$

Example 1.2.2

$$u(x) = x^2 + \int_{0}^{1} xtu(t)dt.$$

Volterra linear integral equations (*VIEs*)**:** in Volterra integral equations, at least one of the bounds of the integration is a variable.

1. Volterra linear integral equations of the first kind:

$$f(x) = \lambda \int_a^x k(x,t)u(t)dt,$$

Example 1.2.3

$$e^{2x} = \frac{1}{2}\int_0^x (x+t)u(t)dt.$$

2. Volterra linear integral equations of the second kind:

$$u(x) = f(x) + \lambda \int_a^x k(x,t)u(t)dt,$$

Example 1.2.4

$$u(x) = ln(x+2) + \int_0^x (x^2t^2 + x)u(t)dt.$$

Volterra-Fredholm integral equations: the Volterra-Fredholm integral equations appear in the literature in two forms, namely:

$$\alpha(x)u(x) = f(x) + \lambda_1 \int_0^x k_1(x,t)u(t)dt + \lambda_2 \int_a^b k_2(x,t)u(t)dt,$$

and the mixed form

$$\alpha(x)u(x) = f(x) + \lambda \int_0^x \int_a^b k(r,t)u(t)dtdr,$$

where $f(x)$ and $k(x,t)$ are analytic functions. The unknown functions $u(x)$ appears inside and outside the integral signs. this is a characteristic feature of a second kind IE . If the unknown functions appear only inside the integral signs, the resulting equations are of first kind.

Example 1.2.5

$$u(x) = 6x + 5x^2 - \int_0^x xu(t)dt + \int_0^1 tu(t)dt,$$

$$u(x) = x + 18x^2 - \int_0^x \int_0^1 k(r-t)u(t)dtdr.$$

Notice that the unknown function $u(x)$ appears inside the volterra and fredholm integrals and outside both integrals.

Singular integral equations: an integral equation is called a singular integral equation if one or both limits of integration become infinite, or if the kernel $k(x,t)$ of the equation becomes infinite at one or more points in the interval of integration.

1. First style of Singular integral equations:

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t)u(t)dt,$$

   where the lower limit $g(x)$, the upper limit $h(x)$, or both limits of integration are infinite.

   Examples 1.2.1

$$u(x) = 1 + e^{-x} - \int_0^{\infty} u(t)dt,$$

$$u(x) = 3x^2 + \int_{-\infty}^{\infty} e^{-|x-t|}u(t)dt,$$

$$u(x) = x + \frac{1}{3}\int_{-\infty}^{0} cos(x+t)u(t)dt.$$

2. Second style of Singular integral equations:

$$f(x) = \lambda \int_0^x \frac{1}{(x-t)^{\alpha}}u(t)dt, 0 < \alpha < 1 \text{ (first kind)},$$

$$u(x) = f(x) + \lambda \int_0^x \frac{1}{(x-t)^{\alpha}}u(t)dt, 0 < \alpha < 1 \text{ (second kind)}.$$

   Examples 1.2.2

$$x^3 = \int_0^x \frac{1}{(x-t)^{\frac{2}{3}}}u(t)dt,$$

$$u(x) = 1 + x^2 + \int_0^x \frac{1}{(x-t)^{\frac{3}{4}}}u(t)dt.$$

• If $\alpha = \frac{1}{2}$ the singular integral equations of first type and second type are called generalized Abel integral equation.

Remark 1.2.1 *The nature of the solution of integral equations solely depends on the nature of the Kernel of the integral equation $K(x,t)$. Kernels are of the following special types:*

1. *Separable or Degenerate Kernel: If the kernel $k(x,t)$ of an integral equation is written in the form:*

$$k(x,t) = \sum_{i=1}^{n} \alpha_i(x)\beta_i(t),$$

where the functions $\alpha_i(x)$ for $i = 1, ..., n$ are free, then it is said to be separable or degenerate kernel. For example, the kernels xt, x^2t^2 are separable.

2. *Symmetric Kernel: If the kernel $k(x,t)$ is a complex-valued function such that:*

$$k(x,t) = \overline{k(t,x)},$$

then it is said to be symmetric or Hermitian. For example, $sin(x+t)$.

3. *Convolution Kernel: when $k(x,t) = k(x-t)$, then the integral equation is said integral equation with convolution kernel.*

iii) ***Nature of the known function f***

Homogeneous: an integral equation is called an integral equation homogeneous if $f(x) = 0$, the equation is written:

$$\alpha(x)u(x) = \lambda \int_{g(x)}^{h(x)} k(x,t)u(t)dt.$$

Non-homogeneous: an integral equation is called an integral equation non-homogeneous if $f(x) \neq 0$, the equation is written:

$$\alpha(x)u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t)u(t)dt.$$

1.2.2 Integral equations with delay

Definition 1.2.1 *Delay integral equations are those integral equations in which the solution of the unknown function is given in the previous time interval.*

General form of delay Volterra equations

1. The constant delay τ is visible within the integral sign.

$$x(t) = \begin{cases} x(t) = g(t) + \int_0^t k(t,s)x(s-\tau)ds & t \in [0,T], \quad \tau \in]0,t[\,, \\ \Phi(t), & t \in [-\tau, 0[\,. \end{cases}$$

   where x is an unknown function to be determined, $k(t,s)$ is a continuous kernel function, $g(t)$ represents a known function, and τ is a constant delay.

2. The constant delay τ shows up on at the edges of the integral sign.

$$x(t) = \begin{cases} g(t) + \int_0^t k_1(t,s)x(s)ds + \int_0^{t-\tau} k_2(t,s)x(s)ds, & t \in [0,T], \\ \Phi(t), & t \in [-\tau, 0[\,. \end{cases} \tag{1.2.3}$$

   Equation (1.2.3) encompasses an important particular delay equation frequently encountered in physical and biological modeling processes [14], namely

$$x(t) = \begin{cases} g(t) + \int_{t-\tau}^t k(t,s)x(s)ds, & t \in [0,T], \\ \Phi(t), & t \in [-\tau, 0[\,. \end{cases} \tag{1.2.4}$$

   it corresponds to setting $k_2 = -k_1 (= k)$ in (1.2.3) . It will be assumed that the given functions $\Phi : [-\tau, 0[\to \mathbb{R}$, $g : [0,T] \to \mathbb{R}$, $k : \Omega \to \mathbb{R}$ $(\Omega := [0,T] \times [-\tau, T] := \{(t,s) : -\tau \leq s \leq t \leq T\})$ are (at least) continuous on their domains such that (1.2.4) possesses a unique solution $u \in C(I)$.

   Examples 1.2.3

1. *Consider the following DVIEs:*

$$x(t) = \begin{cases} x(t) = \sin(t-1) + \sin(1) + \sin(t) - tcos(1) + \int_0^t (t-s)x(s-\tau)ds & t, s \in [0, T], \\ \sin(t), & t \in]-\tau, 0[\,. \end{cases}$$

2. *Consider the following DVIEs:*

$$x(t) = \begin{cases} -3t^2(1+\sin(t^3)) + \int_0^t e^{s-t}x(s)ds + \int_0^{t-1} t\sin(s)x(s)ds, & t \in [0, 1], \\ \sin(t) + 1, & t \in [-1, 0[\,. \end{cases}$$

3. *Consider the following DVIEs:*

$$x(t) = \begin{cases} \cos(t) + e^{-2t^2} + \int_{t-0.5}^t k(t,s)x(s-0.5)ds, & t \in [0, 1], \\ e^t, & t \in [-0.5, 0[\,. \end{cases}$$

Multiple delay Volterra equations

1. ***VIEs with double delays:*** The general form of double delay integral equations as the following:

$$x(t) = \begin{cases} g(t) + \int_0^{t-\tau_2} k_1(t,s)f_1(s,x(s))ds + \int_{t-\tau_2}^{t-\tau_1} k_2(t,s)f_2(s,x(s))ds \\ + \int_{t-\tau_1}^t k_3(t,s)f_3(s,x(s))ds, & t \in [\tau_2, +\infty[\,, \\ \phi(t), & t \in [0, \tau_2[\,. \end{cases}$$

where the constant delays $\tau_2 > \tau_1 > 0$, the functions $g(t), k_1(t,s), k_2(t,s), k_3(t,s)$ and $\Phi(t)$ are sufficiently smooth functions and $x(t)$ is the unknown function to be determined.

Example 1.2.6

$$x(t) = \begin{cases} \sin(t) + e^t + \int_{t-\tau_2}^{t-\tau_1} (t^2 + cos(t+s))x(s)ds, & t \in [0, 1], \\ e^t + t^2, & t \in [-1, 0[\,. \end{cases}$$

we have $\phi(t) = e^t + t^2$, $k_1(t,s) = k_3(t,s) = 0$, $g(t) = \sin(t) + e^t$, $k_2(t,s) = t^2 + cos(t+s)$ *and* $f_2(t,x) = x$.

1.3 Preliminary notions

Notations and definitions

In this section, we present the preliminary notations and definitions that are used in this thesis.

Definition 1.3.1 *(σ -algebra [8]) Let X be any non-empty set. A σ -algebra of subsets of X is a family $\mathcal{A}$ of subsets of X, with the properties:*

1. $\varnothing, X \in \mathcal{A}$.

2. *if* $E \in \mathcal{A}$, *then* $X \setminus E \in \mathcal{A}$.

3. *if* $E_1, E_2, ... \in \mathcal{A}$, *then* $\bigcup_{i\in\mathbb{N}} E_i \in \mathcal{A}$.

There are two extreme examples of sigma algebras:

- *The collection* $\{\emptyset, X\}$ *is a sigma algebra of subsets of X.*
- *The set* $\mathcal{P}(X)$ *of all subsets of X is a sigma algebra.*

Notation 1.3.1 *The pair (X,$\mathcal{A}$) is called a measurable space, and the sets in $\mathcal{A}$ are called the measurable sets.*

Definition 1.3.2 *(Positive measure [6]) Let (X,$\mathcal{A}$) be a measurable space. A positive measure on this space is a function* $\mu : \mathcal{A} \to [0, \infty]$ *such that*

1. $\mu(\emptyset) = 0$.

2. *Countable additivity: For any sequence of mutually disjoint sets $E_n \in \mathcal{A}$,*

$$\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{i=1}^{\infty} \mu(E_n).$$

Notation 1.3.2 *The set $(X, \mathcal{A}, \mu)$ will be called a measure space.*

Definition 1.3.3 *(Measurable function [9]) Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces. A fuction $f : X \to Y$ is measurable if*

$$\text{for all } B \in \mathcal{N}, \text{ the set } f^{-1}(B) \in \mathcal{M}.$$

Definition 1.3.4 *(Exponentially stable [3]) A measurable function $h : \mathbb{R}^+ \to \mathbb{R}$ is called exponentially stable, if there are $M \geq 0$ and $\gamma > 0$ such that $\forall t \in \mathbb{R}^+$, $|h(t)| \leq Me^{-\gamma t}$.*

Definition 1.3.5 *(Bounded functions) A function f defined on some set X with real or complex values is called bounded if the set of its values is bounded. In other words, there exists a real number M such that $\forall x \in X$, $| f(x) |\leq M$.*

Definition 1.3.6 *(Integrable functions [6]) Let $(X, \mathcal{A}, \mu)$ be a measure space. $f : X \to \mathbb{R}$ be measurable. Then f is called integrable if $\int_X | f | d\mu < \infty$.*

Definition 1.3.7 *(Compact set [7]) A set $S \subseteq R$ is said to be compact if and only if, for every collection C of open intervals I such that*

$$S \subseteq \bigcup_{I \in C} I,$$

there exists a finite subcollection $C_0 \subseteq C$ such that

$$S \subseteq \bigcup_{I \in C_0} I.$$

In other words, every open covering of S can be achieved by a finite subcovering.

Definition 1.3.8 *(Topological space [9]) Let X be a nonempty set. A collection $\tau \subseteq \mathcal{P}(X)$ is a topology if*

(i) $\emptyset, X \in \tau$,

(ii) *if $U_i \in \tau$ for $i = 1, ..., M$, then $U_1 \bigcap ... \bigcap U_M \in \tau$,*

(iii) *if $\{U_\alpha\}_{\alpha \in I}$ is a arbitrary collection of elements of τ then $\bigcup_{\alpha \in I} U_\alpha \in \tau$.*

Notation 1.3.3 *The pair (X, τ) is called topological space and the elements of τ open sets.*

Definition 1.3.9 *([9]) X is a topology space, the Borel σ -algebras $\mathcal{B}(x)$ is the smallest σ -algebras (the intersection of all σ -algebras) containing all open subsets of X.*

Notation 1.3.4 *The elements of $\mathcal{B}(x)$ are called Borel sets.*

Definition 1.3.10 *(Borel measure [9]) Let $(X, \mathcal{A}, \mu)$ be a measure space. If X is a topological space, then μ is a Borel measure if every Borel set is in $\mathcal{A}$.*

Definition 1.3.11 *(Nemytskii operator associated to the function f (N_f) [3])*
Let $\Im(\mathbb{R}^+, \mathbb{R})$ be the set of all measurable functions from a subset of $\mathbb{R}^+$ to $\mathbb{R}$.
Let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a measurable function. We define the operator N_f on $\Im(\mathbb{R}^+, \mathbb{R})$ by $N_f x(t) = f(t, x(t)), t \in \mathbb{R}^+$.

Definition 1.3.12 *([3]) Let $k : [\tau_2, +\infty[\times \mathbb{R}^+ \to \mathbb{R}$ be a given measurable function. We define the linear operator $\overline{K}$ on $\Im(\mathbb{R}^+, \mathbb{R})$ by the formula*

$$\overline{(Kx)}(t) = \begin{cases} \int_{t-\tau_2}^{t-\tau_1} k(t, s)x(s)ds, & t \in [\tau_2, +\infty[\,, \\ 0, & t \in [0, \tau_2[\,. \end{cases}$$

Property 1.3.1 *Let $E \subset \Im(\mathbb{R}^+, \mathbb{R})$ be a vectorial space satisfying the following property:*
If $f \in E$ and $\varnothing \neq A \subset D(f)$, $D(f)$ is the domain of f, then the function: f/A (the restriction

of f on A) belongs to E and if $f_1,\ f_2 \in E$ such that $D(f_1) \cap D(f2) = \emptyset$, then the function $f : D(f_1) \cup D(f_2) \to \mathbb{R}$ defined by:

$$f(t) = \begin{cases} f_1(t), & t \in D(f_1), \\ f_2(t), & t \in D(f_2), \end{cases}$$

belongs also to E.

Remark 1.3.1 *If $f \in E$, then the function*

$$\widehat{f}(t) = \begin{cases} f(t), & t \in D(f), \\ 0, & t \in \mathbb{R}^+ - D(f), \end{cases}$$

belongs also to E.

Lebesgue Spaces $L^P, 1 \leq p \leq \infty$

Let $(X, \mathcal{M}, \mu)$ be a measure space. Given two measurable functions $u, v : X \to [-\infty, \infty]$, we say that u is equivalent to v, and we write

$$u \sim v \text{ if } u(x) = v(x) \text{ for } \mu \text{ a.e. } \mathrm{x} \in X.$$

Note that $\sim$ is equivalence relation in the class of measurable functions.
With an abuse of notation, from now on we identify a measurable function $u : X \to [-\infty, \infty]$ with its equivalence class [u].

Definition 1.3.13 *([9]) Let $(X, \mathcal{M}, \mu)$ be a measure space.*

Let $1 \leq p < \infty$, then

$$L^P(X, \mathcal{M}, \mu) = \{u : X \to [-\infty, \infty] : u \text{ measurable and } \|u\|_{L^p(X,\mathcal{M},\mu)} < \infty\},$$

where

$$\|u\|_{L^p(X,\mathcal{M},\mu)} = \left(\int_X |u|^p d\mu \right)^{1/p}.$$

If $p = +\infty$, then

$$L^\infty(X, \mathcal{M}, \mu) = \{u : X \to [-\infty, \infty] : u \text{ measurable and } \|u\|_{L^\infty(X,\mathcal{M},\mu)} < \infty\},$$

where $\|u\|_{L^\infty(X,\mathcal{M},\mu)}$ *is the essential supremum (esssup $|u|$) of the function $|u|$.*

$$\|u\|_{L^\infty(X,\mathcal{M},\mu)} = esssup\,|u| = inf\{\alpha \in R : |u(x)| < \alpha \text{ for } \mu \text{ a.e. } x \in X\}.$$

For simplicity, and when there is no possibility of confusion, we denote the spaces $L^p(X, \mathcal{M}, \mu)$ simply by $L^p(X, \mu)$ or $L^p(X)$ and the norms $\|u\|_{L^p(X,\mathcal{M},\mu)}$ by $\|u\|_{L^p(X)}$, $\|u\|_{L^p}$ or $\|u\|_P$.

Remark 1.3.2 *$X = \mathbb{R}$ with its Euclidean topology and let $\mathcal{B}(\mathbb{R})$ be its Borel σ -algebras.*

i) *$L^1(\mathbb{R})$ the set of all Lebesgue integrable functions on $\mathbb{R}$.*

ii) *$L^\infty(\mathbb{R})$ the set of all bounded functions on $\mathbb{R}$.*

Definition 1.3.14 *(see [9]) Let $(X, \mathcal{M}, \mu)$ be a measure space, with X a topological space, $\mu : \mathcal{M} \to [0, \infty]$ a Borel measure, and $1 \leqslant p \leqslant \infty$. A measurable function $u : X \to [-\infty, \infty]$ is said to belong to $L^p_{loc}(X)$ if $u \in L^p(K)$ for every compact set $K \subset X$. A sequence $\{u_n\} \subset L^p_{loc}(X)$ is said to converge to u in $L^p_{loc}(X)$ if $u_n \to u$ in $L^p(K)$ for every compact set $K \subset X$.*

We note that the spaces $L^\infty(\mathbb{R}^+)$, $L^\infty_{loc}(\mathbb{R}^+)$, $L^1(\mathbb{R}^+)$, $L^1_{loc}(\mathbb{R}^+)$ satisfy the property (1.3.1).

1.4 Leibnitz rule for differentiation of integrals

Let $f(x, t)$ be continuous and $\frac{\partial f}{\partial t}$ be continuous in a domain of the $x-t$ plane that includes the rectangle $a \leq x \leq b$, $t_0 \leq t \leq t_1$, and let

$$F(x) = \int_{g(x)}^{h(x)} f(x, t)dt, \tag{1.4.1}$$

then differentiation of the integral in (1.4.1) exists and is given by

$$F^{'}(x) = \frac{dF}{dx} = f(x, h(x))\frac{dh(x)}{dx} - f(x, g(x))\frac{dg(x)}{dx} + \int_{g(x)}^{h(x)} \frac{\partial f(x,t)}{\partial x} dt. \qquad (1.4.2)$$

If $g(x) = a$ and $h(x) = b$ where a and b are constants, then the Leibnitz rule (1.4.2) reduces to :

$$F^{'}(x) = \frac{dF}{dx} = \int_a^b \frac{\partial f(x,t)}{\partial x} dt,$$

which means that differentiation and integration can be interchanged such as

$$\frac{d}{dx} = \int_a^b e^{xt} dt = \int_a^b t e^{xt} dt.$$

Example 1.4.1 *Consider the equation*

$$F(x) = e^x + \int_0^x F(t) dt,$$

applying Leibnitz rule, we find

$$F^{'}(x) = e^x + F(x).$$

1.5 Piecewise polynomial spaces

Let $\Pi_N = \{t_n^i : 0 = t_0^0 < t_1^0 < ... < t_{N-1}^0 < t_N^0 = t_0^1 < t_1^1 < ... < t_{N-1}^{r-1} < t_N^{r-1}\}$, denote a mesh (grid) on the given interval $I = [0, T]$, where the stepsize is given by $h = t_{n+1}^i - t_n^i$ and assume that $h = \frac{\tau_1}{N_1} = \frac{\tau_2}{N}$ with N and N_1 positive and integer.
Define the subintervals $\sigma_n^i = \left[t_n^i, t_{n+1}^i\right[, n = 0, ..., N-1, i = 0, ..., r-2$, and $\sigma_{N-1}^{r-1} = \left[t_{N-1}^{r-1}, t_N^{r-1}\right]$

Definition 1.5.1 *For a given mesh Π_N the piecewise polynomial space $S_\mu^{(d)}(\Pi_N)$ with $\mu \geq 0, -1 \leq d \leq \mu$, is given by*

$$S_\mu^{(d)}(\Pi_N) = \{v \in C^d(\Pi) : v|_{\sigma_n} \in \pi_\mu (0 \leq n \leq N-1)\}.$$

Here, Π_μ denotes the space of (real) polynomials of degree not exceeding μ.
It is readily verified that $S_\mu^{(d)}(\Pi_N)$ is a (real) linear vector space whose dimension is given by

$$dim S_\mu^{(d)}(\Pi_N) = N(\mu - d) + d + 1.$$

1.6 Collocation method

A collocation method is based on the idea of approximating the exact solution of a given integral equation with a suitable function belonging to a chosen finite dimensional space such that the approximated solution satisfies the integral equation on a certain subset of the interval on which the equation has to be solved (called the set of collocation points). Here We consider as the approximating space the polynomial spline space. In order to describe the relevant collocation method for given N , let Π_N be a uniform partition of a bounded interval $I = [0, T]$ with gride points $t_n^i = (i+1)\tau_2 + nh, n = 0, 1, ..., N$, and $i = 0, ..., r-1$ where $T = (r+1)\tau_2$ let h be the stepsize. Define the subintervals $\sigma_n^i = \left[t_n^i, t_{n+1}^i\right[, n = 0, ..., N-1, i = 0, ..., r-2$, and $\sigma_{N-1}^{r-1} = \left[t_{N-1}^{r-1}, t_N^{r-1}\right]$.

We define the real polynomial spline space of degree $m-1$ as follow :

$$S_{m-1}^{(-1)}(\Pi_N) = \{u(I, \mathbb{R}) \in C^{-1}(\Pi) : u_n^i = u|_{\sigma_n^i} \in \pi_{m-1}, n = 0, ..., N-1, i = 0, ..., r-1\}.$$

The main advantages of Taylor collocation method are :
i) This method is direct and the approximate solution is given by using explicit formulas.
ii) This method has a convergence order.
iii) There is no algebraic system needed to be solved, which makes the proposed algorithm very effective and easy to implement.

1.7 Comparison theorems

Lemma 1.7.1 *(Discrete Gronwall-type inequality [4]) Let* $\left\{k_j\right\}_{j=0}^{n}$ *be a given nonnegative sequence, and the sequence* $\{\varepsilon_n\}$ *satisfies*

$$\varepsilon_n \leq p_0 + \sum_{i=0}^{n-1} k_i \varepsilon_i, \quad n \geq 0,$$

with $p_0 \geq 0$*. Then,* ε_n *can be bounded by*

$$\varepsilon_n \leq p_0 exp\left(\sum_{j=0}^{n-1} k_j\right), \quad n \geq 0.$$

Lemma 1.7.2 *(Discrete Gronwall-type inequality [1]) If* $\{f_n\}_{n\geq 0}$*,* $\{g_n\}_{n\geq 0}$ *and* $\{\varepsilon_n\}_{n\geq 0}$ *are non-negative sequence and*

$$\varepsilon_n \leq f_n + \sum_{i=0}^{n-1} g_i \varepsilon_i, \quad n \geq 0,$$

then

$$\varepsilon_n \leq f_n + \sum_{j=0}^{n-1} f_j g_j exp\left(\sum_{k=0}^{n-1} g_k\right), \quad n \geq 0.$$

Lemma 1.7.3 *[5](Discrete Gronwall's inequality) Assume that* $(\alpha_n)_{n\geq 1}$ *and* $(q_n)_{n\geq 1}$ *are given non-negative sequences and the sequence* $(\varepsilon_n)_{n\geq 1}$ *satisfies* $\varepsilon_1 \leq \beta$ *and*

$$\varepsilon_n \leq \beta + \sum_{j=1}^{n-1} q_j + \sum_{j=1}^{n-1} \alpha_j \varepsilon_j, \quad n \geq 2,$$

then

$$\varepsilon_n \leq \left(\beta + \sum_{j=1}^{n-1} q_j\right) exp\left(\sum_{j=1}^{n-1} \alpha_j\right), \quad n \geq 2.$$

CHAPTER 2

EXISTENCE AND UNIQUENESS RESULT OF DOUBLE DELAY INTEGRAL EQUATIONS

2.1 Introduction

The aim of this chapter is to stady the existence and uniqueness theorems for the nonlinear double delay integral equation proved in [3].

$$x(t) = \begin{cases} g(t) + \int_{t-\tau_2}^{t-\tau_1} k(t,s) f(s, x(s)) ds, & t \in [\tau_2, +\infty[\,, \\ \phi(t), & t \in [0, \tau_2[\,. \end{cases} \tag{2.1.1}$$

2.2 Application of double delay integral equations

The problem (2.1.1) represents an integral formulation of the following nonlinear Gurtin-MacCamy model [10, 11].

$$\begin{aligned} B(t) &= \int_0^t K(t, t-\sigma, S) B(\sigma) d\sigma + F(t, S), \\ S(t) &= \int_0^t H(t, t-\sigma, S) B(\sigma) d\sigma + G(t, S), \end{aligned} \tag{2.2.1}$$

where

$$K(t,\sigma,S) = \beta(\sigma,S)\Pi(\sigma,t,\sigma,S), \quad H(t,\sigma,S) = \gamma(\sigma)\Pi(\sigma,t,\sigma,S),$$

$$\Pi(\sigma,t,x,S) = exp\left(-\int_0^x \mu(a-\sigma, S(t-\sigma)) d\sigma\right),$$

$$F(t,s) = \int_t^\infty \beta(a,s)\Pi(a,t,t,S) p_0(a-t) da,$$

$$G(t,s) = \int_t^\infty \gamma(a)\Pi(a,t,t,S) p_0(a-t) da.$$

Assume

$$\beta(a,S) = R_0\beta_0(a)\phi(S), \quad \mu(a,S) = \mu(a), \quad \beta_0(a) = C\gamma(a) = \chi_{[a*,a_+]}(a), \tag{2.2.2}$$

Under (2.2.2), it can be easily seen that (2.2.1) becomes

$$S(t) = \int_{t-a_+}^{t-a^m} \gamma(t-\sigma)\Pi(t\sigma, t, t-\sigma, S)B(\sigma)d\sigma, \tag{2.2.3}$$

$$\begin{aligned} B(t) &= \int_{t-a_+}^{t-a^m} R_0 C\gamma(t-\sigma)\Pi(t\sigma, t, t-\sigma, S)\phi(S(t))B(\sigma)d\sigma \\ &= R_0 C\phi(S(t))S(t) \end{aligned} \tag{2.2.4}$$

and that by putting (2.2.4) in (2.2.3) we obtain

$$\begin{aligned} S(t) &= R_0 C \int_{t-a_+}^{t-a^m} \gamma(t-\sigma)\Pi(t\sigma, t, t-\sigma, S)\phi(S(\sigma))S(\sigma)d\sigma, \\ &= R_0 C \int_{t-a_+}^{t-a^m} \gamma(t-\sigma)exp\left(-\int_0^{t-\sigma} \mu(a-t+\sigma, S(\sigma))d\sigma\right)\phi(S(\sigma))S(\sigma)d\sigma \end{aligned} \tag{2.2.5}$$

The hypotheses (2.2.2) are very common in purely logistic models describing the dynamics of a population with a finite life span in which the individuals reach maturation at the age a^m.

Table 2.1: Parameters and their explanation in the nonlinear Gurtin-MacCamy model.

Parameters	Explanation
B	birth rate
S	a weighted age distribution
β	age specific fertility
μ	age specific mortality
ϕ	nonnegative decreasing function which is responsible for the reduction of fertility by crowding effect
χ_I	characteristic function of the interval I
a_+	maximum age of the considered population
a^m	maturation age of the considered population
$\Pi(a)$	survival probability, $\Pi(a_+) = 0$
K	maternity function
R_0	Net reproduction rate: number of newborns per individual in the whole life
H	function non increasing and convex
C	positive constant
γ	weight γ reflects the different ecological impact from individuals of different ages

2.3 Existence of a measurable Solution

Let $E \subset \mathfrak{I}(\mathbb{R}^+, \mathbb{R})$ be a vectorial space satisfying the property (1.3.1).

Theorem 2.3.1 *([3]) Suppose that the following conditions are satisfied:*

i) $g : [\tau_2, +\infty[\to \mathbb{R}$ *and* $\Phi : [0, \tau_2[\to \mathbb{R}$ *are measurable functions such that* $\Phi, g \in E$.

ii) $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *is a measurable function such that the Nemytskii operator* N_f *transforms the space E into itself.*

iii) $k : [\tau_2, +\infty[\times \mathbb{R}^+ \to \mathbb{R}$ *is a measurable function and the linear integral operator* $\overline{K}$ *generated by the function k transforms the space E into itself.*

Then Problem (2.1.1) has a unique measurable solution defined on $\mathbb{R}^+$.

Proof. It is clear that there exists a unique integer $r \geq 1$ such that $r\tau_1 \leq \tau_2 < (r+1)\tau_1$. We define the function $x : \mathbb{R}^+ \to \mathbb{R}$ as follows: $x = x_n$ on the interval $[0, (r+n)\tau_1[$ for $n \geq 1$ such that

$$x(t) = \begin{cases} \Phi(t), & if \;\; t \in [0, \tau_2[\\ g(t) + \int_{t-\tau_2}^{t-\tau_1} k(t,s) f(s, \Phi(s)) ds, & if \;\; t \in [\tau_2, (r+1)\tau_1[\end{cases}$$

Using definition (1.3.11), we obtain

$$x(t) = \begin{cases} \Phi(t), & if \;\; t \in [0, \tau_2[\\ g(t) + (\overline{K} N_f \Phi)(t), & if \;\; t \in [\tau_2, (r+1)\tau_1[\end{cases} \tag{2.3.1}$$

and for $n \geq 2$

$$x(t) = \begin{cases} x_{n-1}(t), & if \;\; t \in [0, (r+n-1)\tau_1[\\ g(t) + \int_{t-\tau_2}^{t-\tau_1} k(t,s) f(s, x_{n-1}(s)) ds, & if \;\; t \in [(r+n-1)\tau_1, (r+n)\tau_1[\end{cases}$$

Using definition (1.3.11), we obtain

$$x(t) = \begin{cases} x_{n-1}(t), & if \;\; t \in [0, (r+n-1)\tau_1[\\ g(t) + (\overline{K}N_f x_{n-1})(t), & if \;\; t \in [(r+n-1)\tau_1, (r+n)\tau_1[\end{cases} \tag{2.3.2}$$

We will prove that the sequence (x_n) is well define and $x_n \in E$ for all $n \geq 1$.

1. We have $x_1 = \Phi \in E$ on $[0, \tau_2[$, and on $[\tau_2, (r+1)\tau_1[$ we have $x_1 = g + \overline{K}N_f\Phi \in E$. Then, by the property (1.3.1), we deduce that $x_1 \in E$.

2. Assume that $x_{n-1} \in E$ for $n \geq 2$, hence by the definition of x_n, we get $x_n \in E$ on $[0, (r+n-1)\tau_1[$. Moreover, by the assumptions of Theorem (2.3.1), we deduce that $x_n = g + \overline{K}N_f x_{n-1} \in E$ on $[(r+n-1)\tau_1, (r+n)\tau_1[$.
   Then, by the property (*), we get $x_n \in E$.
   Thus the sequence (x_n) is well define and $x_n \in E$ for all $n \geq 1$, therefore the function x is measurable and define on $\mathbb{R}^+$.

Now, we will prove that x is a solution of (2.1.1).

Step 1: x is a solution on $[0, (r+1)\tau_1[$. By definition, x is a solution of (2.1.1) on $[0, \tau_2[$. Moreover, for $t \in [\tau_2, (r+1)\tau_1[$ we have $0 \leq t - \tau_2 < t - \tau_1 < r\tau_1 \leq \tau_2$
which implies that

$$\begin{aligned} x(t) = x_1(t) &= g(t) + \int_{t-\tau_2}^{t-\tau_1} k(t,s) f(s, \Phi(s)) ds, \\ &= g(t) + \int_{t-\tau_2}^{t-\tau_1} k(t,s) f(s, x(s)) ds. \end{aligned}$$

Then x is a solution on $[0, (r+1)\tau_1[$.

Step 2: x is a solution on $[(r+1)\tau_1, +\infty[$. For $t \in [(r+1)\tau_1, +\infty[$, there exists a unique integer $n \geq 1$ such that $(r+n)\tau_1 \leq t < (r+n+1)\tau_1$, hence

$$\begin{aligned} x(t) = x_{n+1}(t) &= g(t) + \int_{t-\tau_2}^{t-\tau_1} k(t,s) f(s, x_n(s)) ds, \\ &= g(t) + \int_{t-\tau_2}^{t-\tau_1} k(t,s) f(s, x(s)) ds. \end{aligned}$$

Then x is a solution on $[(r+1)\tau_1, +\infty[$.

For the uniqueness, let y be a solution of (2.1.1) on $\mathbb{R}^+$, we will prove that $x = y$ by the following induction.

1. $x = y$ on $[0, (r+1)\tau_1[$.
   We have $x = y = \Phi$ on $[0, \tau_2[$ and for $t \in [\tau_2, (r+1)\tau_1[$ we have $0 \leq t - \tau_2 < t - \tau_1 < r\tau_1 \leq \tau_2$, then $y(t) = g(t) + \int_{t-\tau_2}^{t-\tau_1} k(t,s) f(s, \Phi(s)) ds = x(t)$, we deduce that $x = y$ on $[0, (r+1)\tau_1[$.

2. Assume that $x = y$ on $[0, (r+n)\tau_1[$ for $n \geq 1$, and show that $x = y$ on $[0, (r+n+1)\tau_1[$.
   Let $t \in [(r+n)\tau_1, (r+n+1)\tau_1[$, hence $0 \leq t - \tau_2 < t - \tau_1 < (r+n)\tau_1$.
   Then,
$$\begin{aligned} y(t) &= g(t) + \int_{t-\tau_2}^{t-\tau_1} k(t,s) f(s, y(s)) ds, \\ &= g(t) + \int_{t-\tau_2}^{t-\tau_1} k(t,s) f(s, x_{n-1}(s)) ds \\ &= x_n(t) = x(t), \end{aligned}$$
   which implies that $x = y$ on $[0, (r+n+1)\tau_1[$.

Then Problem (2.1.1) has a unique measurable solution defined on $\mathbb{R}^+$. ■

Remark 2.3.1 *Under the conditions of Theorem (2.3.1), the solution x need not be in the space E as in the following counterexample.*

Example 2.3.1 *[3] Consider the following double delay integral equation*

$$x(t) = \begin{cases} 1 + \int_{t-\tau_2}^{t-\tau_1} x(s)ds, & t \in [\tau_2, +\infty[\,, \\ 0, & t \in [0, \tau_2[\,. \end{cases} \tag{2.3.3}$$

such that $\tau_2 - \tau_1 \geq 1$, *we have* $\Phi(t) = 0,\ \ g(t) = 1,\ \ k(t,s) = 1$ *and* $f(t,x) = x$.
Let $E = L^\infty(\mathbb{R}^+)$, *it is clear that* E *satisfies the property* (1.3.1) *and contains the functions* Φ *and* g. *Moreover, the operators* K *and* N_f *transform the space* E *into itself. Then, by Theorem (2.3.1), Problem (3.4.1) has a unique measurable solution* x *defined on* $\mathbb{R}^+$ *by (2.3.1) and (2.3.2). Hence, for all* $t \in [\tau_2, (r+1)\tau_1[\,,\ \ x(t) = x_1(t) = 1$ *and for all* $t \in [(r+1)\tau_1, (r+2)\tau_1[\,,\ \ x(t) = x_2(t) = 1 + (\tau_2 - \tau_1)$. *So, by using the iteration, we deduce that for* $n \geq 2$ *and* $t \in [(r+n-1)\tau_1, (r+n)\tau_1)$,

$$x(t) = x_n(t) = \sum_{i=0}^{n-1} (\tau_2 - \tau_1)^i.$$

This implies that $\| x \|_{L^\infty(\mathbb{R}^+)} \geq n$ *for all* $n \geq 1$.

Consequently, we obtain $\| x \|_{L^\infty(\mathbb{R}^+)} = +\infty$ *and* $x \notin E$.

2.4 Existence of an Exponentially Stable Solution

We will need the following lemma.

Lemma 2.4.1 *[3] Suppose that the following conditions are satisfied:*

i) $g : [\tau_2, +\infty[\ \to \mathbb{R}$ *and* $\Phi : [0, \tau_2[\ \to \mathbb{R}$ *are measurable functions such that* $\Phi \in L^\infty([0, \tau_2[), g \in L^\infty_{Loc}([\tau_2, +\infty[)$.

ii) $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *is a measurable function and there exist a constant* b *and a function* $a \in L^\infty_{Loc}(\mathbb{R}^+)$ *such that* $| f(t,x) \leq a(t) + b \mid x \mid$ *for all* $t \in \mathbb{R}^+$ *and* $x \in \mathbb{R}$.

iii) $k : [\tau_2, +\infty[\times \mathbb{R}^+ \rightarrow \mathbb{R}$ *is a measurable function and the linear integral operator* $\overline{K}$ *transforms the space* $L^\infty_{Loc}(\mathbb{R}^+)$ *into itself.*

Then Problem (2.1.1) has a unique solution in $L^\infty_{Loc}(\mathbb{R}^+)$

Proof. We have the vectorial space $L^\infty_{Loc}(\mathbb{R}^+)$ verifies the property (1.3.1) and the functions $\Phi, g \in L^\infty_{Loc}(\mathbb{R}^+)$. Moreover, the assumption (ii) guarantees that the Nemytskii operator N_f transforms the space $L^\infty_{Loc}(\mathbb{R}^+)$ into itself. Additionally to the assumption (iii), we deduce, by Theorem (2.3.1), that Problem (2.1.1) has a unique measurable solution x on $\mathbb{R}^+$ defined by $x = x_n$ on $[0, (r+n)\tau_1[$ for $n \geq 1$, where the sequence (x_n) is defined by (2.3.1) and (2.3.2). Moreover, the sequence $(x_n) \in L^\infty_{Loc}(\mathbb{R}^+)$, hence for all $n \geq 2$, we have $x \in L^\infty([0, (r+n-1)\tau_1])$, which implies that $x \in L^\infty_{Loc}(\mathbb{R}^+)$.
Thus Problem (2.1.1) has a unique solution in $L^\infty_{Loc}(\mathbb{R}^+)$. ∎

The following result gives a sufficient condition on k so that the operator $\overline{K}$ transforms the space $L^\infty_{Loc}(\mathbb{R}^+)$ into itself.

Proposition 2.4.1 *[3] Assume that the function* $t \mapsto \int_{\tau_1}^{\tau_2} | k(t, t-s) | ds$ *belongs to* $L^\infty_{Loc}([\tau_2, +\infty[)$, *then the operator* $\overline{K}$ *transforms the space* $L^\infty_{Loc}(\mathbb{R}^+)$ *into itself.*

Proof. The operator $\overline{K}$ transforms the space $L^\infty_{Loc}(\mathbb{R}^+)$ into itself if and only if, for all $\alpha \geq \tau_2$ and for all $x \in L^\infty_{Loc}([\tau_2, +\infty[)$, we have $\overline{K}x \in L^\infty_{Loc}([\tau_2, \alpha[)$.
We have for all $t \in [\tau_2, \alpha[$

$$\begin{aligned}
| \overline{K}x(t) | &\leq g(t) + \int_{t-\tau_2}^{t-\tau_1} | k(t,s) || x(s) | ds, \\
&= g(t) + \int_{t-\tau_2}^{t-\tau_1} | k(t, t-s) || x(t-s) | ds, \quad (change\ of\ variable) \\
&\leq \| x \|_{L^\infty([0,\alpha-\tau_2[)} \int_{t-\tau_2}^{t-\tau_1} | k(t, t-s) | ds, \quad (| f(x) | \leq \| f \|_\infty\ a.e)
\end{aligned}$$

and since $\int_{t-\tau_2}^{t-\tau_1} | k(t, t-s) | ds \in \overline{K}x \in L^\infty([\tau_2, \alpha[)$, then $\overline{K}x \in L^\infty([\tau_2, \alpha[)$.
Thus, $\overline{K}$ transforms the space $L^\infty_{Loc}(\mathbb{R}^+)$ into itself. ∎

Example 2.4.1 *[3] Consider Problem (2.1.1) with g, Φ and f fulfilling the assumptions (i) and (ii) of Lemma (2.4.1) and $k(t,s) = \dfrac{t+s}{t-s}e^s$. Since*

$$\int_{t-\tau_2}^{t-\tau_1} \mid k(t,t-s) \mid ds = \left[2t \ln\left(\frac{\tau_2}{\tau_1}\right) - (\tau_2 - \tau_1)\right] e^t \in L^{\infty}_{Loc}([\tau_2, +\infty[),$$

then, by Proposition (2.4.1) and Lemma (2.4.1), Problem (2.1.1) has a unique solution $x \in L^{\infty}_{Loc}(\mathbb{R}^+)$.

The following result gives the existence of an exponentially stable solution of Problem (2.1.1).

Theorem 2.4.1 *[3] Suppose that the following conditions are satisfied:*

i) $g : [\tau_2, +\infty[\to \mathbb{R}$ *is exponentially stable and* $\Phi \in L^{\infty}([0, \tau_2[)$.

ii) $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *is a measurable function and there exist a constant b and a exponentially stable* $a : \mathbb{R}^+ \to \mathbb{R}$ *such that* $\mid f(t,x) \leq a(t) + b \mid x \mid$ *for all* $t \in \mathbb{R}^+$ *and* $x \in \mathbb{R}$.

iii) $k : [\tau_2, +\infty[\times \mathbb{R}^+ \to \mathbb{R}$ *is a measurable function such that the function*

$$t \mapsto \int_{\tau_1}^{\tau_2} \mid k(t,t-s) \mid ds \in L^{\infty}_{Loc}([\tau_2, +\infty[).$$

iv) *There exists* $c \geq \tau_2$ *such that* $b\alpha = b\left(\underset{t \geq c}{ess\ sup} \int_{\tau_1}^{\tau_2} \mid k(t,t-s) \mid ds\right) < 1.$

Then Problem (2.1.1) has a unique exponentially stable solution.

Proof. By Proposition (2.4.1), the assumption (iii) guarantees that the operator $\overline{K}$ transforms the space $L^{\infty}_{Loc}(\mathbb{R}^+)$ into itself, then from the above assumptions, we deduce by Lemma (2.4.1), that Problem (2.1.1) has a unique solution $x \in L^{\infty}_{Loc}(\mathbb{R}^+)$. Moreover, there exist $\gamma_1, \gamma_2 > 0$ such that $\mid g(t) \mid e^{\gamma_1 t} \in L^{\infty}(\mathbb{R}^+)$ and $a(t)e^{\gamma_2 t} \in L^{\infty}(\mathbb{R}^+)$.
Now, let $0 < \gamma \leq min(\gamma_1, \gamma_2)$ we have for all $t \geq c$

$$
\begin{aligned}
|x(t)| e^{\gamma t} &\leq |g(t)| e^{\gamma t} + \int_{t-\tau_2}^{t-\tau_1} e^{\gamma(s+\tau_2)} |k(t,s)|| f(s,x(s)| ds \\
&\leq |g(t)| e^{\gamma_1 t} + e^{\gamma_2\tau_2} \int_{t-\tau_2}^{t-\tau_1} |k(t,s)| a(s) e^{\gamma_2 s} ds \\
&+ e^{\gamma\tau_2} b \int_{t-\tau_2}^{t-\tau_1} |k(t,s)|| x(s)| e^{\gamma s} ds \quad (assumption\ (ii)) \\
&\leq |g(t)| e^{\gamma_1 t} + e^{\gamma_2\tau_2} \int_{\tau_1}^{\tau_2} |k(t,t-s)| a(t-s) e^{\gamma_2(t-s)} ds \\
&+ b e^{\gamma\tau_2} \int_{\tau_1}^{\tau_2} |k(t,t-s)|| x(t-s)| e^{\gamma(t-s)} ds \quad (change\ of\ variable) \\
&\leq \| g(z) e^{\gamma_1 z} \|_{L^\infty(\mathbb{R}^+} + \alpha e^{\gamma_2\tau_2} \| a(z) e^{\gamma_2 z} \|_{L^\infty(\mathbb{R}^+)} \\
&+ b\alpha e^{\gamma\tau_2} \| x(z) e^{\gamma z} \|_{L^\infty([c-\tau_2,t])} \quad (| f(x) | \leq \| f \|_\infty), \ (assumption\ (iv)) \\
&\leq \| g(z) e^{\gamma_1 z} \|_{L^\infty(\mathbb{R}^+)} + \alpha e^{\gamma_2\tau_2} \| a(z) e^{\gamma_2 z} \|_{L^\infty(\mathbb{R}^+)} \\
&+ b\alpha e^{\gamma_2\tau_2} \| x(z) e^{\gamma z} \|_{L^\infty([c-\tau_2,c])} + b\alpha e^{\gamma\tau_2} \| x(z) e^{\gamma z} \|_{L^\infty([c,t])},
\end{aligned}
$$

hence, for all $t \geq c$

$$
\begin{aligned}
(1 - b\alpha e^{\gamma\tau_2}) \| x(z) e^{\gamma z} \|_{L^\infty([c,t])} &\leq \| g(z) e^{\gamma_1 z} \|_{L^\infty(\mathbb{R}^+)} + \alpha e^{\gamma_2\tau_2} \| a(z) e^{\gamma_2 z} \|_{L^\infty(\mathbb{R}^+)} \\
&+ b\alpha e^{\gamma_2\tau_2} \| x(z) e^{\gamma z} \|_{L^\infty([c-\tau_2,c])} .
\end{aligned}
$$

Since $b\alpha < 1$, then there exists $0 < \gamma \leq min(\gamma_1, \gamma_2)$ such that $(1 - b\alpha e^{\gamma\tau_2} > 0$, which implies from the above estimate that $x(t)e^{\gamma t} \in L^\infty([c, +\infty])$, moreover $x(t)e^{\gamma t} \in L^\infty([0, c])$, it follows that $x(t)e^{\gamma t} \in L^\infty([0, +\infty))$. Thus Problem (2.1.1) has a unique exponentially stable solution on $\mathbb{R}^+$. ■

Example 2.4.2 *[3] Consider Problem (2.1.1) with g, Φ and f fulfilling the assumptions (i) and (ii) of Theorem (2.4.1) and $k(t,s) = \dfrac{1}{t+s}$, hence*

$$
\int_{\tau_1}^{\tau_2} |k(t,t-s)| ds = \ln\left(\frac{2t-\tau_1}{2t\tau_2}\right) \in L^\infty_{Loc}([\tau_2, +\infty[).
$$

Since, $\lim_{t\to+\infty} ln\left(\frac{2t-\tau_1}{2t\tau_2}\right) = 0$*, then there exists* $c \geq \tau_2$ *such that*

$$b\left(\operatorname{ess\,sup}_{t\geq c}\int_{\tau_1}^{\tau_2} \mid k(t,t-s)\mid ds\right) < 1.$$

Thus, by Theorem (2.4.1) , Problem (2.1.1) has a unique exponentially stable solution .

Remark 2.4.1 *[3] . If we replace the expression "exponentially stable" by 'bounded" in the assumptions (i) and (ii) of Theorem (2.4.1) and by setting* $\gamma = \gamma_1 = \gamma_2 = 0$ *in the proof, we obtain a unique bounded solution of (2.1.1).*

Theorem 2.4.2 *[3] Suppose that the following conditions are satisfied:*

i) $g : [\tau_2, +\infty[\to \mathbb{R}$ *is exponentially stable and* $\Phi \in L^\infty([0,\tau_2[)$.

ii) $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *is a measurable function and there exist a constant b and an exponentially stable* $a : \mathbb{R}^+ \to \mathbb{R}$ *such that* $\mid f(t,x) \leq a(t) + b \mid x \mid$ *for all* $t \in \mathbb{R}^+$ *and* $x \in \mathbb{R}$.

iii) $k : [\tau_2, +\infty[\times \mathbb{R}^+ \to \mathbb{R}$ *is a measurable function and* $\mid k(t,s) \mid \leq h(s)ds$ *such that* $h \in L^1_{Loc}(\mathbb{R}^+)$.

Then Problem (2.1.1) has a unique solution $x \in L^\infty_{Loc}(\mathbb{R}^+)$*. Moreover, there exist* $\gamma > 0, \lambda \geq 0$ *and* $\beta \geq 0$ *such that for all* $t \in \mathbb{R}^+$,

$$\mid x(t) \mid e^{\gamma t} \leq \left(\beta + \lambda\int_0^t h(s)ds\right) exp\left(be^{\gamma\tau_2}\int_o^t h(s)ds.\right) \tag{2.4.1}$$

Proof. We have, by the assumption (iii), for all $\alpha \geq 2$ and for all $t \in [\tau_2, \alpha]$

$$\int_{\tau_1}^{\tau_2} \mid k(t,t-s)\mid ds \leq \int_{\tau_1}^{\tau_2} h(t-s)ds \leq \int_0^{\alpha-\tau_1} h(s)ds < +\infty.$$

Then, by Proposition 2.4.1, the operator K transforms $\overline{K}$ transforms the space $L^\infty_{Loc}(\mathbb{R}^+)$ into itself, hence from the above assumptions, we deduce by Lemma 2.4.1, that Problem

(2.1.1) has a unique solution $x \in L^{\infty}_{Loc}(\mathbb{R}^{+})$. Moreover, the solution is given by the following iteration: $x = x_n$ on the interval $[0, (r+n)\tau_1), n \geq 1$ such that

$$x(t) = \begin{cases} \Phi(t), & if \ \ t \in [0, \tau_2[\\ g(t) + \int_{t-\tau_2}^{t-\tau_1} k(t,s) f(s, \Phi(s)) ds, & if \ \ t \in [\tau_2, (r+1)\tau_1[\end{cases}$$

and for $n \geq 2$

$$x(t) = \begin{cases} x_{n-1}(t), & if \ \ t \in [0, (r+n-1)\tau_1[\\ g(t) + \int_{t-\tau_2}^{t-\tau_1} k(t,s) f(s, x_{n-1}(s)) ds, & if \ \ t \in [(r+n-1)\tau_1, (r+n)\tau_1[\end{cases}$$

On the other hand, there exist $\gamma_1, \gamma_2 > 0$ such that $\mid g(t) \mid e^{\gamma_1 t} \in L^{\infty}(\mathbb{R}^{+})$ and $a(t)e^{\gamma_2 t} \in L^{\infty}(\mathbb{R}^{+})$.

Let $\gamma = min(\gamma_1, \gamma_2)$ and define the sequence $(\epsilon_n)_{n \geq 1}$ as follows: for $n \geq 2$

$$\epsilon_n = ess\ sup \left\{ \mid x(t) \mid e^{\gamma t}, t \in [(r+n-1)\tau_1, (r+n)\tau_1[\right\}$$

and $\epsilon_1 = ess\ sup \{ \mid x(t) \mid e^{\gamma t}, t \in [0, (r+1)\tau_1[\}$

Now, for $n \geq 2$ and $t \in [(r+n-1)\tau_1, (r+n)\tau_1[$, we have

$$\begin{aligned}
\mid x(t) \mid e^{\gamma t} &\leq \mid g(t) \mid e^{\gamma t} + \int_{t-\tau_2}^{t-\tau_1} e^{\gamma(s+\tau_2)} \mid k(t,s) \mid (a(s) + b \mid x(s) \mid) ds \\
&\leq \parallel g(z)e^{\gamma_1 z} \parallel_{L^\infty([\tau_2,+\infty[)} + e^{\gamma_2\tau_2} \parallel a(z)e^{\gamma_2 z} \parallel_{L^\infty(\mathbb{R}^+)} \int_0^{(r+n-1)\tau_1} h(s)ds \\
&+ be^{\gamma\tau_2} \int_0^{(r+n-1)\tau_1} h(s)ds \mid x(s) \mid e^{\gamma s} ds \\
&\leq \parallel g(z)e^{\gamma_1 z} \parallel_{L^\infty([\tau_2,+\infty[)} + e^{\gamma_2\tau_2} \parallel a(z)e^{\gamma_2 z} \parallel_{L^\infty(\mathbb{R}^+)} \int_0^{(r+1)\tau_1} h(s)ds \\
&+ e^{\gamma_2\tau_2} \parallel a(z)e^{\gamma_2 z} \parallel_{L^\infty(\mathbb{R}^+)} \sum_{j=2}^{n-1} \int_{(r+j-1)\tau_1}^{(r+j)\tau_1} h(s)ds \\
&+ be^{\gamma\tau_2} \int_0^{(r+1)\tau_1} h(s)ds \mid x(s) \mid e^{\gamma s} ds + be^{\gamma\tau_2} \sum_{j=2}^{n-1} \int_{(r+j-1)\tau_1}^{(r+j)\tau_1} h(s) \mid x(s) \mid e^{\gamma s} ds \\
&\leq \parallel g(z)e^{\gamma_1 z} \parallel_{L^\infty([\tau_2,+\infty[)} + \sum_{j=1}^{n-1} q_j + \sum_{j=1}^{n-1} \alpha_j \epsilon_j,
\end{aligned}$$

where

$q_1 = e^{\gamma_2\tau_2} \parallel a(z)e^{\gamma_2 z} \parallel_{L^\infty(\mathbb{R}^+)} e^{\gamma_2\tau_2} \int_0^{(r+1)\tau_1} h(s)ds$ and for $j \geq 2$

$$q_j = e^{\gamma_2\tau_2} \parallel a(z)e^{\gamma_2 z} \parallel_{L^\infty(\mathbb{R}^+)} \int_{(r+j-1)\tau_1}^{(r+j)\tau_1} h(s)ds$$

$\alpha_1 = be^{\gamma\tau_2} \int_0^{(r+1)\tau_1} h(s)ds$ and for $j \geq 2$

$$\alpha_j = be^{\gamma\tau_2} \int_{(r+j-1)}^{(r+j)\tau_1} h(s)ds$$

On the other hand, for $t \in [0, \tau_2[$,we have $\mid x(t) \mid e^{\gamma t} \leq e^{\gamma\tau_2} \parallel \Phi \parallel_{L^\infty([0,\tau_2[)}$, and for $t \in [\tau_2, (r+1)\tau_1[$, we have

$$\begin{aligned}
\mid x(t) \mid e^{\gamma t} \leq \parallel g(z)e^{\gamma_1 z} \parallel_{L^\infty([\tau_2,(r+1)\tau_1[)} + e^{\gamma\tau_2} \parallel h \parallel_{L^\infty([0,r\tau_1[)} \\
\times (\parallel a(z)e^{\gamma_2 z} \parallel_{L^\infty([0,r\tau_1[)} + be^{\gamma r\tau_1} \parallel \Phi \parallel_{L^\infty([0,r\tau_1[)}),
\end{aligned}$$

hence,

$$\begin{aligned}\epsilon_1 \leq max\{e^{\gamma\tau_2} \parallel \Phi \parallel_{L^\infty([0,\tau_2[)}, \parallel g(z)e^{\gamma_1 z} \parallel_{L^\infty([\tau_2,(r+1)\tau_1[)} \\ + e^{\gamma\tau_2} \parallel h \parallel_{L^1([0,r\tau_1[)} (\parallel a(z)e^{\gamma_2 z} \parallel_{L^\infty([0,r\tau_1[)} + be^{\gamma r\tau_1} \parallel \Phi \parallel_{L^\infty([0,r\tau_1[)})\} \equiv \rho,\end{aligned}$$

Let $\beta = max\{\rho, \parallel g(z)e^{\gamma_1 z} \parallel_{L^\infty([\tau_2,+\infty[)}\}$, then for all $n \geq 2$

$$\epsilon_n \leq \beta + \sum_{i=1}^{n-1} q_i + \sum_{i=1}^{n-1} \alpha_i \epsilon_i$$

with $\epsilon_1 \leq \beta$, we deduce, by Lemma (1.7.3), that for all $n \geq 2$

$$\begin{aligned}\epsilon_n &\leq \left(\beta + \sum_{i=1}^{n-1} q_i\right) exp\left(\sum_{i=1}^{n-1} \alpha_i\right) \\ &= \left(\beta + e^{\gamma_2\tau_2} \parallel a(z)e^{\gamma_2 z} \parallel_{L^\infty(\mathbb{R}^+} \int_0^{(r+n-1)\tau_1} h(s)ds\right) \\ &\times exp\left(be^{\gamma}\tau_2 \int_0^{(r+n-1)\tau_1} h(s)ds\right).\end{aligned}$$

Then, for $\lambda = e^{\gamma_2\tau_2} \parallel a(z)e^{\gamma_2 z} \parallel_{L^\infty(\mathbb{R}^+}$ and $t \in [(r = n-1)\tau_1, (r+n)\tau_1[$, we obtain

$$\begin{aligned}\mid x(t) \mid e^{\gamma t} \leq \epsilon_n &\leq \left(\beta + \lambda \int_0^{(r+n-1)\tau_1} h(s)ds\right) exp\left(be^{\gamma\tau_2} \int_0^{(r+n-1)\tau_1} h(s)ds\right) \\ &\leq \left(\beta + \lambda \int_0^t h(s)ds\right) exp\left(be\gamma\tau_2 \int_0^t h(s)ds\right).\end{aligned}$$

Moreover, for $t \in [0, (r+1)\tau_1[$, we obtain

$$\mid x(t) \mid e^{\gamma t} \leq \beta \leq \left(\beta + \lambda \int_0^{(r+n-1)\tau_1} h(s)ds\right) exp\left(be^{\gamma\tau_2} \int_0^{(r+n-1)\tau_1} h(s)ds\right).$$

This completes the proof of the theorem. ■

Remark 2.4.2 *[3]*

1. *If $h \in L^1(\mathbb{R}^+)$ we deduce, by the inequality (3.2.2), that the solution is exponentially stable.*

2. *If we replace the expression "exponentially stable" by "bounded" in the assumptions (i) and (ii) of Theorem (2.4.1), then, by setting $\gamma = \gamma_1 = \gamma_2 = 0$ in the proof, we obtain the inequality (2.4.1), with $\gamma = 0$. Moreover, if $h \in L^1(\mathbb{R}^+)$, then the solution is bounded.*

Example 2.4.3 *Consider Problem (2.1.1) with g, Φ and f fulfilling the assumptions (i) and (ii) of Theorem 2.5.1 and $k(t,s) = tse^{-(t+s)}$. Since*

$$| k(t,s) |\leq se^{-s} = h(s) \in L^1(\mathbb{R}^+),$$

then, by Theorem 2.5.1, Problem (2.1.1) has a unique exponentially stable solution.

2.5 Existence of an integrable Solution

Arguing as in Lemma 2.4.1, we deduce the following result.

Lemma 2.5.1 *[3] Suppose that the following conditions are satisfied:*

i) $g : [\tau_2, +\infty[\to \mathbb{R}$ *and* $\Phi : [0, \tau_2[\to \mathbb{R}$ *are measurable functions such that* $\Phi \in L^\infty([0,\tau_2[), g \in L^\infty_{Loc}([\tau_2, +\infty[)$.

ii) $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *is a measurable function and there exist a constant b and a function $a \in L^\infty_{Loc}(\mathbb{R}^+)$ such that $| f(t,x) \leq a(t) + b \mid x \mid$ for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$.*

iii) $k : [\tau_2, +\infty[\times \mathbb{R}^+ \to \mathbb{R}$ *is a measurable function and the operator $\overline{K}$ transforms the space $L^\infty_{Loc}(\mathbb{R}^+)$ into itself.*

Then Problem (2.1.1) has a unique solution in $L^\infty_{Loc}(\mathbb{R}^+)$

The following result gives a sufficient condition on k so that the operator $\overline{K}$ transforms the space $L^\infty_{Loc}(\mathbb{R}^+)$ into itself.

Proposition 2.5.1 *[3] Let $\widetilde{k}$ be the function defined on by $\mathbb{R}^+$ by $\widetilde{k} = \int_{\Theta(s)}^{\tau_2} | k(t+s,s) | dt$ such that*

$$\Theta(s) = \begin{cases} \tau_1, & s \geq \tau_2 - \tau_1, \\ \tau_2 - s, & 0 \leq s \leq \tau_2 - \tau_1. \end{cases}$$

If the function $\widetilde{k} \in L^\infty_{Loc}(\mathbb{R}^+)$, then the operator $\overline{K}$ transforms the space $L^\infty_{Loc}(\mathbb{R}^+)$ into itself.

Proof. The operator $\overline{K}$ transforms the space $L^1_{Loc}(\mathbb{R}^+)$ into itself if and only if, for all $\alpha \geq 2\tau_2 - \tau_1$ and $x \in L^1_{Loc}(\mathbb{R}^+)$, we have $\overline{K}x \in L^1([\tau_2, \alpha])$. Assume that $\widetilde{k} \in L^\infty_{Loc}(\mathbb{R}^+)$, then for $\alpha \geq 2\tau_2 - \tau_1$ and $x \in L^1_{Loc}(\mathbb{R}^+)$, we have

$$\begin{aligned}
\int_{\tau_2}^{\alpha} | \overline{K}x(t) | dt &\leq \int_{\tau_2}^{\alpha} \int_{t-\tau_2}^{t-\tau_1} | k(t,s) || x(s) | dsdt \\
&\leq \int_{\tau_2}^{2\tau_2-\tau_1} \int_{t-\tau_2}^{t-\tau_1} | k(t,s) || x(s) | dsdt + \int_{2\tau_2-\tau_1}^{\alpha} \int_{t-\tau_2}^{t-\tau_1} | k(t,s) || x(s) | dsdt \\
&\leq \int_{\tau_2}^{2\tau_2-\tau_1} \int_{t-\tau_2}^{\tau_2-\tau_1} | k(t,s) || x(s) | dsdt + \int_{\tau_2}^{2\tau_2-\tau_1} \int_{\tau_2-\tau_1}^{t-\tau_1} | k(t,s) || x(s) | dsdt \\
&+ \int_{\tau_2-\tau_1}^{\alpha} \int_{t-\tau_2}^{t-\tau_1} | k(t,s) || x(s) | dsdt \\
&\leq \int_{0}^{2\tau_2-\tau_1} \int_{\tau_2}^{s+\tau_2} | k(t,s) || x(s) | dtds + \int_{\tau_2-\tau_1}^{2\tau_2-2\tau_1} \int_{s+\tau_1}^{2\tau_2-\tau_1} | k(t,s) || x(s) | dtds \\
&+ \int_{\tau_2-\tau_1}^{\alpha-\tau_1} \int_{s+\tau_1}^{s+\tau_2} | k(t,s) || x(s) | dtds \\
&\leq \int_{0}^{\tau_2-\tau_1} | x(s) | \int_{\tau_2-s}^{\tau_2} | k(t,s) | dtds + \int_{\tau_2-\tau_1}^{2\tau_2-2\tau_1} | x(s) | \int_{\tau_1}^{\tau_2} | k(t,s) | dtds \\
&+ \int_{\tau_2-\tau_1}^{\alpha-\tau_1} | x(s) | \int_{\tau_1}^{\tau_2} | k(t,s) | dtds \\
&\leq \| \widetilde{k} \|_{L^\infty([0,\tau_2-\tau_1])} \| x \|_{L^1([0,\tau_2-\tau_1])} + \| \widetilde{k} \|_{L^\infty([\tau_2-\tau_1,2(\tau_2-\tau_1)])} \| x \|_{L^1([\tau_2-\tau_1,2(\tau_2-\tau_1)])} \\
&\quad + \| \widetilde{k} \|_{L^\infty([\tau_2-\tau_1,\alpha-\tau_1])} \| x \|_{L^1([\tau_2-\tau_1,\alpha-\tau_1])} .
\end{aligned}$$

This shows that $\overline{K}x \in L^1([\tau_2, \alpha])$.
Thus, $\overline{K}$ transforms the space $L^1_{Loc}(\mathbb{R}^+)$ into itself. ■

Example 2.5.1 *[3] Consider Problem (2.1.1) with g, Φ and f fulfilling the assumptions (i) and* *(ii) of Lemma 2.5.1 and* $k(t,s) = (t-s)e^s$. *Since*

$$\widetilde{k}(s) = \begin{cases} (\tau_2 - \tau_1)e^s, & s \geq \tau_2 - \tau_1, \\ se^s, & 0 \leq s \leq \tau_2 - \tau_1. \end{cases}$$

then, $\widetilde{k} \in L^{\infty}_{Loc}(\mathbb{R}^+)$, *this implies, by proportion 2.5.1 and Lemma 2.5.1, that Problem (2.1.1) has a unique solution* $x \in L^1_{Loc}(\mathbb{R}^+)$.

The following result gives the existence of an integrable solution of (2.1.1).

Theorem 2.5.1 *[3] Suppose that the following conditions are satisfied:*

i) $g : [\tau_2, +\infty[\to \mathbb{R}$ *and* $\Phi : [0, \tau_2[\to \mathbb{R}$ *are measurable function such that* $g \in L^1([\tau_2, +\infty[)$ *and* $\Phi \in L^1([0, \tau_2[)$.

ii) $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *is a measurable function and there exist a constant* b *and a function* $a \in L^1(\mathbb{R}^+)$ *such that* $| f(t,x) \leq a(t) + b \mid x \mid$ *for all* $t \in \mathbb{R}^+$ *and* $x \in \mathbb{R}$.

iii) $k : [\tau_2, +\infty[\times \mathbb{R}^+ \to \mathbb{R}$ *is a measurable function such that* $\widetilde{k} \in L^{\infty}(\mathbb{R}^+)$.

iv) *There exists* $c \geq \tau_2 - \tau_1$ *such that* $b \parallel \widetilde{k} \parallel_{L^{\infty}([c,+\infty[)} < 1$.

Then Problem (2.1.1) has a unique solution $x \in L^1(\mathbb{R}^+)$.

Proof. By Proposition 2.5.1, the assumption (iii) guarantees that the operator $\overline{K}$ transforms $L^1_{Loc}(\mathbb{R}^+)$ into itself, then from the above assumptions, we deduce by Lemma 2.5.1, that Problem (2.1.1) has a unique solution $x \in L^1_{Loc}(\mathbb{R}^+)$.

We will show that $x \in L^1(\mathbb{R}^+)$. We have for all $t \geq c + \tau_2$

$$\begin{aligned}
\int_{c+\tau_2}^{t} | x(s) | ds &\leq \int_{c+\tau_2}^{t} | g(s) | ds + \int_{c+\tau_2}^{t} \int_{s-\tau_2}^{s-\tau_1} | k(s,r) \| a(r) | drds \\
&\quad + \int_{c+\tau_2}^{t} \int_{s-\tau_2}^{s-\tau_1} | k(s,r) \| x(r) | drds \\
&\leq \int_{c+\tau_2}^{t} | g(s) | ds + \int_{c}^{t-\tau_1} \int_{\tau_1}^{\tau_2} | k(r+s,s) \| a(s) | drds \\
&\quad + b \int_{c}^{t-\tau_1} \int_{\tau_1}^{\tau_2} | k(r+s,s) \| x(s) | drds \quad (change\ of\ variable) \\
&\leq \| g \|_{L^1(\mathbb{R}^+)} + \| \widetilde{k} \|_{L^\infty(\mathbb{R}^+)} + b \| \widetilde{k} \|_{L^\infty([c,t])} \| x \|_{L^1([c,t])} \\
&\leq \| g \|_{L^1(\mathbb{R}^+)} + \| \widetilde{k} \|_{L^\infty(\mathbb{R}^+)} + b \| \widetilde{k} \|_{L^\infty([c,+\infty])} \| x \|_{L^1([c,c+\tau_2])} \\
&\quad + b \| \widetilde{k} \|_{L^\infty([c,+\infty])} \| x \|_{L^1([c+\tau_2,c])},
\end{aligned}$$

hence, for all $t \geq c + \tau_2$

$$\begin{aligned}
\left(1 - b \| \widetilde{k} \|_{L^\infty([c,+\infty])}\right) \int_{c+\tau_2}^{t} | x(s) | ds \leq & \| g \|_{L^1(\mathbb{R}^+)} + \| \widetilde{k} \|_{L^\infty(\mathbb{R}^+)} \| a \|_{L^1(\mathbb{R}^+)} \\
& + b \| \widetilde{k} \|_{L^\infty([c,+\infty])} \| x \|_{L^1([c,c+\tau_2])} .
\end{aligned}$$

This shows that $L^1([\tau_2, +\infty])$, moreover $\Phi \in L^1([0, \tau_2])$ and $x \in L^1([\tau_2, c + \tau_2])$.

Then Problem (2.1.1) has a unique integrable solution on $\mathbb{R}^+$. ■

Example 2.5.2 *[3] Consider Problem (2.1.1) with g, Φ and f fulfilling the assumptions (i) and (ii) of Theorem 2.5.1 and $k(t,s) = (t+s)e^{-t}$, hence*

$$\widetilde{k}(s) = \begin{cases} [(\tau_1+1)e^{-\tau_1} - (\tau_2+1)e^{-\tau_2}]e^{-s} + 2(e^{-\tau_1} - e^{-\tau_2})se^{-s}, & s \geq \tau_2 - \tau_1, \\ e^{-\tau_2}[\tau_2 + s + 1 - \tau_2 e^{-s} - e^{-s} - 2se^{-s}], & 0 \leq s \leq \tau_2 - \tau_1. \end{cases}$$

We have, $\widetilde{k}$ is continuous and $\lim_{s \to +\infty} \widetilde{k}(s)$ then, $\widetilde{k}$ is bounded and there exists $c \leq \tau_2$ such that $b \| \widetilde{k} \|_{L^\infty([c,+\infty])} < 1$. Thus, by Theorem 2.5.1, Problem (2.1.1) has a unique solution $x \in L^1(\mathbb{R}^+)$.

2.6 Double delay convolution integral equations

Consider the following nonlinear double delay integral equation:

$$x(t) = \begin{cases} g(t) + \int_{t-\tau_2}^{t-\tau_1} h(t-s) f(s, x(s)) ds, & t \in [\tau_2, +\infty[\,, \\ \Phi(t), & t \in [0, \tau_2[\,, \end{cases} \tag{2.6.1}$$

where $h : \mathbb{R}^+ \to \mathbb{R}$ is a measurable function. We have (2.6.1) is of the form (2.1.1) such that $k(t,s) = h(t-s)$. Then $\int_{\tau_1}^{\tau_2} \mid k(t, t-s) \mid ds = \int_{\tau_1}^{\tau_2} \mid h(s) \mid ds$ and

$$\widetilde{k}(s) = \begin{cases} \int_{\tau_1}^{\tau_2} \mid h(t) \mid dt, & s \geq \tau_2 - \tau_1, \\ \int_{\tau_2 - s}^{\tau_2} \mid h(t) \mid dt, & 0 \leq s \leq \tau_2 - \tau_1. \end{cases}$$

The following result is directly yielded by applying Theorem 2.5.1 and by using Remark

Theorem 2.6.1 *[3] Suppose that the following conditions are satisfied:*

i) $g : [\tau_2, +\infty[\to \mathbb{R}$ *is exponentially stable(resp. bounded) and* $\Phi \in L^1([0, \tau_2[)$.

ii) $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *is a measurable function and there exist a constant b and an exponentially stable function (resp. bounded) a such that* $\mid f(t, x) \leq a(t) + b \mid x \mid$ *for all* $t \in \mathbb{R}^+$ *and* $x \in \mathbb{R}$.

iii) $h : \mathbb{R}^+ \to \mathbb{R}$ *is a measurable function such that* $\int_{\tau_1}^{\tau_2} \mid h(t) \mid dt < +\infty$ *and* $b \int_{\tau_1}^{\tau_2} \mid h(t) \mid dt < 1$.

Then Problem (2.1.1) has a unique exponentially stable(resp. bounded)solution . Also, by applying Theorem (2.5.1), the following result takes place.

Theorem 2.6.2 *[3] Suppose that the following conditions are satisfied:*

i) $g : [\tau_2, +\infty[\to \mathbb{R}$ *and* $\Phi : ([0, \tau_2[) \to \mathbb{R}$ *are measurable functions such that* $g \in L^1([\tau_2, +\infty[)$ *and* $\Phi \in L^1([0, \tau_2[)$.

ii) *$f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a measurable function and there exist a constant b and a function $a \in L^1(\mathbb{R}^+)$ such that $\mid f(t,x) \leq a(t) + b \mid x \mid$ for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$.*

iii) *$h : \mathbb{R}^+ \to \mathbb{R}$ is a measurable function such that $\int_{\tau_1}^{\tau_2} \mid h(t) \mid dt < +\infty$ and $b \int_{\tau_1}^{\tau_2} \mid h(t) \mid dt < 1$.*

Then Problem (2.1.1) has a unique solution $x \in L^1(\mathbb{R}^+)$.

Finally, we consider the following double delay integral equations of the form

$$x(t) = \begin{cases} R_0 C \int_{t-\tau_2}^{t-\tau_1} \gamma(t-\sigma) exp\left(-\int_0^{t-\sigma} \mu(a-t+\sigma, x(\sigma)) d\sigma\right) \phi(x(\sigma)) x(\sigma) d\sigma, \\ \qquad\qquad t \in [\tau_2, +\infty[\,, \\ \phi(t), \qquad\qquad t \in [0, \tau_2[\,. \end{cases} \tag{2.6.2}$$

problem (2.6.2) will be studied under the following assumptions:

1. R_0 , $C \in \mathbb{R}^+$.

2. γ is a non negative function on $\mathbb{R}^+$.

3. $\mu(a,b) = \alpha(a)$ such that α is a non negative function.

4. ϕ is a non negative decreasing function on $\mathbb{R}^+$.

5. Φ is a non negative function on $[0, \tau_2[$.

Then Problem (2.6.2) is a double delay convolution integral equation of the form (2.6.1) such that

$$h(s) = R_0 C \gamma(s) exp\left(-\int_0^s \alpha(a-s) da\right), f(s,x) = \phi(x)x.$$

Moreover, it is clear, by above assumptions, that if (2.6.2) has a measurable solution $x \in \mathbb{R}^+$, then x is non negative .

The following corollaries are directly yielded by applying Theorem 2.6.1(resp. Theorem 2.6.2).

Corollary 2.6.1 *[3] Suppose that the following conditions are satisfied:*

i) $\Phi \in L^{\infty}([0, \tau_2[)$.

ii) $\gamma, \alpha : \mathbb{R}^{+} \to \mathbb{R}^{+}$ *are measurable functions such that*

$$R_0 C\phi(0) \int_{\tau_1}^{\tau_2} \gamma(t) exp\left(-\int_0^t \alpha(a-t)da\right) dt < 1.$$

Then Problem (2.6.1) has a unique non negative bounded solution.

Corollary 2.6.2 *[3] Suppose that the following conditions are satisfied:*

i) $\Phi \in L^{1}([0, \tau_2[)$.

ii) $\gamma, \alpha : \mathbb{R}^{+} \to \mathbb{R}^{+}$ *are measurable functions such that*

$$R_0 C\phi(0) \int_{\tau_1}^{\tau_2} \gamma(t) exp\left(-\int_0^t \alpha(a-t)da\right) dt < 1.$$

Then Problem (2.6.1) has a unique non negative solution $x \in L^1(\mathbb{R}^+)$.

CHAPTER 3

NUMERICAL SOLUTION OF DOUBLE DELAY INTEGRAL EQUATIONS USING TAYLOR COLLOCATION METHOD

3.1 Introduction

In this chapter, we apply a direct collocation method based on the use of Taylor polynomials to approximate the solution of linear Volterra integral equations with two constant delays in the polynomial spline $S_{m-1}^{(-1)}(\Pi_N)$. The approximate solution is given by using iterative formulas, and we prove the convergence of the approximate solution to the exact solution.

We consider the linear Volterra integral equation with two constant delays τ_1, τ_2 of the form:

$$x(t) = g(t) + \int_0^{t-\tau_2} k_3(t,s)x(s)ds + \int_{t-\tau_2}^{t-\tau_1} k_2(t,s)x(s)ds + \int_{t-\tau_1}^{t} k_1(t,s)x(s)ds, \tag{3.1.1}$$

for $t \in [\tau_2, T]$ and $x(t) = \Phi(t)$ for $t \in [0, \tau_2]$. In the following we assume that the given functions g, k_1, k_2, k_3 and Φ are sufficiently smooth. Furthermore, we suppose that

$$\Phi(\tau_2) = g(\tau_2) + \int_{\tau_2-\tau_1}^{\tau_2} k_1(\tau_2, s)\Phi(s)ds + \int_0^{\tau_2-\tau_1} k_2(\tau_2, s)\Phi(s)ds.$$

3.2 Description of the method

We suppose that $T = (r+1)\tau_2$, where $r \in \{1, 2, 3, ...\}$. Let Π_N be a uniform partition of the interval $I = [\tau_2, T]$ defined by $t_n^i = (i+1)\tau_2 + nh, n = 0, 1, .., N,\ i = 0, 1, .., r-1$, where the step-size is given by $h = t_{n+1}^i - t_n^i$ and assume that $h = \frac{\tau_1}{N_1} = \frac{\tau_2}{N}$ with N and N_1 positive and integer. Define the subintervals $\sigma_n^i = [t_n^i; t_{n+1}^i[, n = 0, 1, .., N-1,\ i = 0, 1, .., r-2$ and $\sigma_{N-1}^{r-1} = [t_{N-1}^{r-1}, t_N^{r-1}]$. Moreover, denote by π_m the set of all real polynomials of degree not exceeding m. We define the real polynomial spline space of degree $m-1$ as follows:

$$S_{m-1}^{(-1)}(\Pi_N) = \{u(I, \mathbb{R}) \in C^{-1}(\Pi) : u_n^i = u|_{\sigma_n^i} \in \pi_{m-1}, n = 0, ..., N-1,\ i = 0, 1, .., r-1\}. \tag{3.2.1}$$

This is the space of piecewise polynomials of degree (at most) $m-1$. Its dimension is rNm, i.e., the same as the total number of the coefficients of the polynomials $u_n^p, n =$

$0, ..., N-1, p = 0, 1, .., r-1$. To find these coefficients, we use Taylor polynomial on each subinterval.

3.2.1 Approximate solution in the interval σ_0^0

First, we approximate x in the interval σ_0^0 by the polynomial

$$u_0^0(t) = \sum_{j=0}^{m-1} \frac{x^{(j)}(\tau_2)}{j!}(t-\tau_2)^j \, ; \quad t \in \sigma_0^0, \tag{3.2.2}$$

where $x^{(j)}(\tau_2), j = 0, ..., m-1$ is the exact value of $x^{(j)}$ at τ_2 and the function x must be differentiable around the τ_2 point. By differentiate equation (3.1.1) j-times, we get, for $j = 0, 1, ..., m-1$,

$$\begin{aligned} x^{(j)}(t) =& g^{(j)}(t) + \left(\int_{t-\tau_1}^{\tau_2} k_1(t,s)\Phi(s)ds + \int_{t-\tau_2}^{t-\tau_1} k_2(t,s)\Phi(s)ds + \int_0^{t-\tau_2} k_3(t,s)\Phi(s)ds \right)^{(j)}(t) \\ &+ \sum_{i=0}^{j-1} \left[\partial_1^{(j-1-i)} k_1(t,t)x(t) \right]^{(i)} + \int_{\tau_2}^{t} \partial_1^{(j)} k_1(t,s)x(s)ds, \end{aligned}$$

which implies,

$$\begin{aligned} x^{(j)}(t) =& g^{(j)}(t) + \left(\int_{t-\tau_1}^{\tau_2} k_1(t,s)\Phi(s)ds + \int_{t-\tau_2}^{t-\tau_1} k_2(t,s)\Phi(s)ds + \int_0^{t-\tau_2} k_3(t,s)\Phi(s)ds \right)^{(j)}(t) \\ &+ \sum_{i=0}^{j-1} \sum_{l=0}^{i} \binom{i}{l} \left[\partial_1^{(j-1-i)} k_1(t,t) \right]^{(i-l)}(t)x^{(l)}(t) + \int_{\tau_2}^{t} \partial_1^{(j)} k_1(t,s)x(s)ds, \end{aligned}$$

hence,

$$\begin{aligned} x^{(j)}(\tau_2) = g^{(j)}(\tau_2) &+ \left(\int_{t-\tau_1}^{\tau_2} k_1(t,s)\Phi(s)ds + \int_{t-\tau_2}^{t-\tau_1} k_2(t,s)\Phi(s)ds + \int_0^{t-\tau_2} k_3(t,s)\Phi(s)ds \right)^{(j)}(\tau_2) \\ &+ \sum_{i=0}^{j-1} \sum_{l=0}^{i} \binom{i}{l} \left[\partial_1^{(j-1-i)} k_1(t,t) \right]^{(i-l)}(\tau_2)x^{(l)}(\tau_2). \end{aligned}$$

3.2.2 Approximate solution in the interval σ_n^0

Second, for x to be approximated by u_n^0 ($n \in \{1, 2, ..., N-1\}$) on the interval σ_n^0, x must be approximated by u_k^0 ($0 \le k < n$) on each interval σ_k^0, such that

$$u_n^0(t) = \sum_{j=0}^{m-1} \frac{\hat{u}_{n,0}^{(j)}(t_n^0)}{j!}(t - t_n^0)^j; \quad t \in \sigma_n^0, \tag{3.2.3}$$

where $\hat{u}_{n,0}$ is the exact solution of the integral equations
for $t \in \sigma_n^0$, $n \in \{1, 2, ..., N_1 - 1\}$,

$$\begin{aligned}\hat{u}_{n,0}(t) = g(t) &+ \int_{t-\tau_1}^{\tau_2} k_1(t,s)\Phi(s)ds + \int_{t-\tau_2}^{t-\tau_1} k_2(t,s)\Phi(s)ds + \int_0^{t-\tau_2} k_3(t,s)\Phi(s)ds \\ &+ \sum_{i=0}^{n-1} \int_{t_i^0}^{t_{i+1}^0} k_1(t,s)u_i^0(s)ds + \int_{t_n^0}^{t} k_1(t,s)\hat{u}_{n,0}(s)ds,\end{aligned} \tag{3.2.4}$$

and for $t \in \sigma_n^0$, $n \in \{N_1, N_1 + 1, ..., N-1\}$,

$$\begin{aligned}\hat{u}_{n,0}(t) = g(t) &+ \int_0^{t-\tau_2} k_3(t,s)\Phi(s)ds + \int_{t-\tau_2}^{\tau_2} k_2(t,s)\Phi(s)ds \\ &+ \sum_{i=0}^{n-N_1-1} \int_{t_i^0}^{t_{i+1}^0} k_2(t,s)u_i^0(s)ds + \int_{t_n^0-\tau_1}^{t-\tau_1} k_2(t,s)u_{n-N_1}^0(s)ds \\ &+ \int_{t-\tau_1}^{t_{n+1}^0-\tau_1} k_1(t,s)u_{n-N_1}^0(s)ds + \sum_{i=n+1-N_1}^{n-1} \int_{t_i^0}^{t_{i+1}^0} k_1(t,s)u_i^0(s)ds + \\ &\int_{t_n^0}^{t} k_1(t,s)\hat{u}_{n,0}(s)ds.\end{aligned} \tag{3.2.5}$$

Now, for all $j = 0, 1, ..., m-1$, the formula for computing the values of the coefficients $\hat{u}_{n,0}^{(j)}(t_n^0)$ can be obtained by employing similar arguments to those used for obtaining the values of $x^{(j)}(\tau_2)$ above, we get the following formulas for $n \in \{1, 2, ..., N_1 - 1\}$,

$$\begin{aligned}\hat{u}_{n,0}^{(j)}(t) &= g^{(j)}(t) + \left(\int_{t-\tau_1}^{\tau_2} k_1(t,s)\Phi(s)ds + \int_{t-\tau_2}^{t-\tau_1} k_2(t,s)\Phi(s)ds + \int_0^{t-\tau_2} k_3(t,s)\Phi(s)ds\right)^{(j)}(t)\\ &+ \sum_{i=0}^{n-1}\int_{t_i^0}^{t_{i+1}^0} \partial_1^{(j)} k_1(t,s)u_i^0(s)ds + \sum_{i=0}^{j-1}[\partial_1^{(j-1-i)}k_1(t,t)\hat{u}_{n,0}(t)]^{(i)}\\ &+ \int_{t_n^0}^{t} \partial_1^{(j)} k_1(t,s)\hat{u}_{n,0}(s)ds,\end{aligned}$$

which implies,

$$\begin{aligned}\hat{u}_{n,0}^{(j)}(t) &= g^{(j)}(t_n^0) + \left(\int_{t-\tau_1}^{\tau_2} k_1(t,s)\Phi(s)ds + \int_{t-\tau_2}^{t-\tau_1} k_2(t,s)\Phi(s)ds + \int_0^{t-\tau_2} k_3(t,s)\Phi(s)ds\right)^{(j)}(t)\\ &+ \sum_{i=0}^{n-1}\sum_{l=0}^{m-1}\frac{\hat{u}_{i,0}^{(l)}(t_i^0)}{l!}\int_{t_i^0}^{t_{i+1}^0} \partial_1^{(j)} k_1(t,s)(s-t_i^0)^l ds + \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}[\partial_1^{(j-1-i)}k_1(t,t)]^{(i-l)}(t_n^0)\hat{u}_{n,0}^{(l)}(t)\\ &+ \int_{t_n^0}^{t} \partial_1^{(j)} k_1(t,s)\hat{u}_{n,0}(s)ds,\end{aligned} \tag{3.2.6}$$

hence,

$$\begin{aligned}\hat{u}_{n,0}^{(j)}(t_n^0) &= g^{(j)}(t_n^0) + \left(\int_{t-\tau_1}^{\tau_2} k_1(t,s)\Phi(s)ds + \int_{t-\tau_2}^{t-\tau_1} k_2(t,s)\Phi(s)ds + \int_0^{t-\tau_2} k_3(t,s)\Phi(s)ds\right)^{(j)}(t_n^0)\\ &+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}[\partial_1^{(j-1-i)}k_1(t,t)]^{(i-l)}(t_n^0)\hat{u}_{n,0}^{(l)}(t_n^0)\\ &+ \sum_{i=0}^{n-1}\sum_{l=0}^{m-1}\frac{\hat{u}_{i,0}^{(l)}(t_i^0)}{l!}\int_{t_i^0}^{t_{i+1}^0} \partial_1^{(j)} k_1(t_n^0,s)(s-t_i^0)^l ds,\end{aligned}$$

and for $n \in \{N_1, N_1+1, ..., N-1\}$,

$$
\begin{aligned}
\hat{u}_{n,0}^{(j)}(t) &= g^{(j)}(t) + \left(\int_0^{t-\tau_2} k_3(t,s)\Phi(s)ds + \int_{t-\tau_2}^{\tau_2} k_2(t,s)\Phi(s)ds +\right)^{(j)}(t) \\
&+ \sum_{i=0}^{n-N_1-1} \int_{t_i^0}^{t_{i+1}^0} \partial_1^{(j)} k_2(t,s) u_i^0(s)ds + \sum_{i=0}^{j-1} [\partial_1^{(j-1-i)} k_2(t, t-\tau_1) u_{n-N_1}^0 (t-\tau_1)]^{(i)} \\
&+ \int_{t_n^0-\tau_1}^{t-\tau_1} \partial_1^{(j)} k_2(t,s) u_{n-N_1}^0(s)ds - \sum_{i=0}^{j-1} [\partial_1^{(j-1-i)} k_1(t, t-\tau_1) u_{n-N_1}^0 (t-\tau_1)]^{(i)} \\
&+ \int_{t-\tau_1}^{t_{n+1}^0-\tau_1} \partial_1^{(j)} k_1(t,s) u_{n-N_1}^0(s)ds + \sum_{i=n+1-N_1}^{n-1} \int_{t_i^0}^{t_{i+1}^0} \partial_1^{(j)} k_1(t,s) u_i^0(s)ds \\
&+ \sum_{i=0}^{j-1} [\partial_1^{(j-1-i)} k_1(t,t) \hat{u}_{n,0}(t)]^{(i)} + \int_{t_n^0}^{t} \partial_1^{(j)} k_1(t,s) \hat{u}_{n,0}(s)ds,
\end{aligned}
$$

which implies,

$$
\begin{aligned}
\hat{u}_{n,0}^{(j)}(t) &= g^{(j)}(t) + \left(\int_0^{t-\tau_2} k_3(t,s)\Phi(s)ds + \int_{t-\tau_2}^{\tau_2} k_2(t,s)\Phi(s)ds +\right)^{(j)}(t) \\
&+ \sum_{i=0}^{n-N_1-1} \sum_{l=0}^{m-1} \frac{\hat{u}_{i,0}^{(l)}(t_i^0)}{l!} \int_{t_i^0}^{t_{i+1}^0} \partial_1^{(j)} k_2(t,s)(s-t_i^0)^l ds \\
&+ \sum_{i=0}^{j-1} \sum_{l=0}^{i} \binom{i}{l} [\partial_1^{(j-1-i)} k_2(t, t-\tau_1)]^{(i-l)}(t) [u_{n-N_1}^0(t-\tau_1)]^{(l)} \\
&+ \int_{t_n^0-\tau_1}^{t-\tau_1} \partial_1^{(j)} k_2(t,s) u_{n-N_1}^0(s)ds \\
&- \sum_{i=0}^{j-1} \sum_{l=0}^{i} \binom{i}{l} [\partial_1^{(j-1-i)} k_1(t, t-\tau_1)]^{(i-l)}(t) [u_{n-N_1}^0(t-\tau_1)]^{(l)} \\
&+ \int_{t-\tau_1}^{t_{n+1}^0-\tau_1} \partial_1^{(j)} k_1(t,s) u_{n-N_1}^0 ds + \sum_{i=n+1-N_1}^{n-1} \int_{t_i^0}^{t_{i+1}^0} \partial_1^{(j)} k_1(t,s) u_i^0 ds \\
&+ \sum_{i=0}^{j-1} \sum_{l=0}^{i} \binom{i}{l} [\partial_1^{(j-1-i)} k_1(t,t)]^{(i-l)}(t) \hat{u}_{n,0}^{(l)}(t) + \int_{t_n^0}^{t} \partial_1^{(j)} k_1(t,s) \hat{u}_{n,0}(s)ds,
\end{aligned}
$$

which implies,

$$
\begin{aligned}
\hat{u}_{n,0}^{(j)}(t) &= g^{(j)}(t) + \left(\int_0^{t-\tau_2} k_3(t,s)\Phi(s)ds + \int_{t-\tau_2}^{\tau_2} k_2(t,s)\Phi(s)ds+\right)^{(j)}(t) \\
&+ \sum_{i=0}^{n-N_1-1}\sum_{l=0}^{m-1}\frac{\hat{u}_{i,0}^{(l)}(t_i^0)}{l!}\int_{t_i^0}^{t_{i+1}^0}\partial_1^{(j)}k_2(t,s)(s-t_i^0)^l ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m-1}\binom{i}{l}\frac{\hat{u}_{n-N_1,0}^{(r)}(t_{n-N_1}^0)}{(r-l)!}[\partial_1^{(j-1-i)}k_2(t,t-\tau_1)]^{(i-l)}(t)(t-\tau_1-t_{n-N_1}^0)^{r-l} \\
&+ \sum_{l=0}^{m-1}\frac{\hat{u}_{n-N_1,0}^{(l)}(t_{n-N_1}^0)}{l!}\int_{t_n^0-\tau_1}^{t-\tau_1}\partial_1^{(j)}k_2(t,s)(s-t_{n-N_1}^0)^l ds \\
&- \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m-1}\binom{i}{l}\frac{\hat{u}_{n-N_1,0}^{(r)}(t_{n-N_1}^0)}{(r-l)!}[\partial_1^{(j-1-i)}k_1(t,t-\tau_1)]^{(i-l)}(t)(t-\tau_1-t_{n-N_1}^0)^{r-l} \qquad (3.2.7)\\
&+ \sum_{l=0}^{m-1}\frac{\hat{u}_{n-N_1,0}^{(l)}(t_{n-N_1}^0)}{l!}\int_{t-\tau_1}^{t_{n+1}^0-\tau_1}\partial_1^{(j)}k_1(t,s)(s-t_{n-N_1}^0)^l ds \\
&+ \sum_{i=n+1-N_1}^{n-1}\sum_{l=0}^{m-1}\frac{\hat{u}_{i,0}^{(l)}(t_i^0)}{l!}\int_{t_i^0}^{t_{i+1}^0}\partial_1^{(j)}k_1(t,s)(s-t_i^0)^l ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}[\partial_1^{(j-1-i)}k_1(t,t)]^{(i-l)}(t)\hat{u}_{n,0}^{(l)}(t) \\
&+ \int_{t_n^0}^{t}\partial_1^{(j)}k_1(t,s)\hat{u}_{n,0}(s)ds,
\end{aligned}
$$

hence,

$$
\begin{aligned}
\hat{u}_{n,0}^{(j)}(t_n^0) &= g^{(j)}(t_n^0) + \left(\int_0^{t-\tau_2} k_3(t,s)\Phi(s)ds + \int_{t-\tau_2}^{\tau_2} k_2(t,s)\Phi(s)ds\right)^{(j)}(t_n^0) \\
&+ \sum_{i=0}^{n-N_1-1}\sum_{l=0}^{m-1}\frac{\hat{u}_{i,0}^{(l)}(t_i^0)}{l!}\int_{t_i^0}^{t_{i+1}^0}\partial_1^{(j)}k_2(t_n^0,s)(s-t_i^0)^l ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\hat{u}_{n-N_1,0}^{(l)}(t_{n-N_1}^0)[\partial_1^{(j-1-i)}k_2(t,t-\tau_1)]^{(i-l)}(t_n^0) \\
&- \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\hat{u}_{n-N_1,0}^{(l)}(t_{n-N_1}^0)[\partial_1^{(j-1-i)}k_1(t,t-\tau_1)]^{(i-l)}(t_n^0) \\
&+ \sum_{l=0}^{m-1}\frac{\hat{u}_{n-N_1,0}^{(l)}(t_{n-N_1}^0)}{l!}\int_{t_n^0-\tau_1}^{t_{n+1}^0-\tau_1}\partial_1^{(j)}k_1(t_n^0,s)(s-t_{n-N_1}^0)^l ds \\
&+ \sum_{i=n+1-N_1}^{n-1}\sum_{l=0}^{m-1}\frac{\hat{u}_{i,0}^{(l)}(t_i^0)}{l!}\int_{t_i^0}^{t_{i+1}^0}\partial_1^{(j)}k_1(t_n^0,s)(s-t_i^0)^l ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}[\partial_1^{(j-1-i)}k_1(t,t)]^{(i-l)}(t_n^0)\hat{u}_{n,0}^{(l)}(t_n^0).
\end{aligned}
$$

3.2.3 Approximate solution in the interval σ_0^p

Third, for x to be approximated by u_0^p ($p \in \{1, 2, ..., r-1\}$) on the interval σ_0^p, x must be approximated by u_k^j ($0 \le k \le N-1$ and $0 \le j < p$) on each interval σ_k^j such that,

$$u_0^p(t) = \sum_{j=0}^{m-1}\frac{\hat{u}_{0,p}^{(j)}(t_0^p)}{j!}(t-t_0^p)^j; \quad t \in \sigma_0^p, \tag{3.2.8}$$

where $\hat{u}_{0,p}$ is the exact solution of the integral equation

$$\begin{aligned}
\hat{u}_{0,p}(t) = g(t) &+ \int_0^{\tau_2} k_3(t,s)\phi(s)ds + \sum_{i=0}^{p-2}\sum_{d=0}^{N-1}\int_{t_d^i}^{t_{d+1}^i} k_3(t,s)u_d^i(s)ds \\
&+ \int_{t_0^{p-1}}^{t-\tau_2} k_3(t,s)u_0^{p-1}(s)ds + \int_{t-\tau_2}^{t_1^{p-1}} k_2(t,s)u_0^{p-1}(s)ds \\
&+ \sum_{d=1}^{N-N_1-1}\int_{t_d^{p-1}}^{t_{d+1}^{p-1}} k_2(t,s)u_d^{p-1}(s)ds + \int_{t_0^p-\tau_1}^{t-\tau_1} k_2(t,s)u_{N-N_1}^{p-1}(s)ds \\
&+ \int_{t-\tau_1}^{t_1^p-\tau_1} k_1(t,s)u_{N-N_1}^{p-1}(s)ds + \sum_{d=N-N_1+1}^{N-1}\int_{t_d^{p-1}}^{t_{d+1}^{p-1}} k_1(t,s)u_d^{p-1}(s)ds \\
&+ \int_{t_0^p}^{t} k_1(t,s)\hat{u}_{0,p}(s)ds.
\end{aligned} \tag{3.2.9}$$

The coefficients $\hat{u}_{0,p}^{(j)}(t_0^p)$ is given by the following formula

$$\begin{aligned}
\hat{u}_{0,p}^{(j)}(t) = g^{(j)}(t) &+ \left(\int_0^{\tau_2} k_3(t,s)\Phi(s)ds\right)^{(j)}(t) + \sum_{i=0}^{p-2}\sum_{d=0}^{N-1}\int_{t_d^i}^{t_{d+1}^i} \partial_1^{(j)}k_3(t,s)u_d^i(s)ds \\
&+ \sum_{i=0}^{j-1}[\partial_1^{(j-1-i)}k_3(t,t-\tau_2)u_0^{p-1}(t-\tau_2)]^{(i)} + \int_{t_0^{p-1}}^{t-\tau_2} \partial_1^{(j)}k_3(t,s)u_0^{p-1}(s)ds \\
&- \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}[\partial_1^{(j-1-i)}k_2(t,t-\tau_2)]^{(i-l)}(t)[u_0^{p-1}(t-\tau_2)]^{(l)} \\
&+ \int_{t-\tau_2}^{t_1^{p-1}} \partial_1^{(j)}k_2(t,s)u_0^{p-1}(s)ds + \sum_{d=1}^{N-N_1-1}\int_{t_d^{p-1}}^{t_{d+1}^{p-1}} \partial_1^{(j)}k_2(t,s)u_d^{p-1}(s)ds \\
&+ \sum_{i=0}^{j-1}[\partial_1^{(j-1-i)}k_2(t,t-\tau_1)u_{N-N_1}^{p-1}(t-\tau_1)]^{(i)} + \int_{t_0^p-\tau_1}^{t-\tau_1} \partial_1^{(j)}k_2(t,s)u_{N-N_1}^{p-1}(s)ds \\
&- \sum_{i=0}^{j-1}[\partial_1^{(j-1-i)}k_1(t,t-\tau_1)u_{N-N_1}^{p-1}(t-\tau_1)]^{(i)} \\
&+ \int_{t-\tau_1}^{t_1^p-\tau_1} \partial_1^{(j)}k_1(t,s)u_{N-N_1}^{p-1}(s)ds + \sum_{d=N-N_1+1}^{N-1}\int_{t_d^{p-1}}^{t_{d+1}^{p-1}} \partial_1^{(j)}k_1(t,s)u_d^{p-1}(s)ds \\
&+ \sum_{i=0}^{j-1}[\partial_1^{(j-1-i)}k_1(t,t)\hat{u}_{0,p}^{(l)}(t)]^{(i)} + \int_{t_0^p}^{t} \partial_1^{(j)}k_1(t,s)\hat{u}_{0,p}(s)ds,
\end{aligned}$$

which implies,

$$
\begin{aligned}
\hat{u}_{0,p}^{(j)}(t) = g^{(j)}(t) &+ \left(\int_0^{\tau_2} k_3(t,s)\Phi(s)ds\right)^{(j)}(t) + \sum_{i=0}^{p-2}\sum_{d=0}^{N-1}\int_{t_d^i}^{t_{d+1}^i} \partial_1^{(j)} k_3(t,s)u_d^i(s)ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}[\partial_1^{(j-1-i)}k_3(t,t-\tau_2)]^{(i-l)}(t)[u_0^{p-1}(t-\tau_2)]^{(l)} \\
&+ \int_{t_0^{p-1}}^{t-\tau_2} \partial_1^{(j)} k_3(t,s)u_0^{p-1}(s)ds \\
&- \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}[\partial_1^{(j-1-i)}k_2(t,t-\tau_2)]^{(i-l)}(t)[u_0^{p-1}(t-\tau_2)]^{(l)} \\
&+ \int_{t-\tau_2}^{t_1^{p-1}} \partial_1^{(j)} k_2(t,s)u_0^{p-1}(s)ds + \sum_{d=1}^{N-N_1-1}\int_{t_d^{p-1}}^{t_{d+1}^{p-1}} \partial_1^{(j)} k_2(t,s)u_d^{p-1}(s)ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}[\partial_1^{(j-1-i)}k_2(t,t-\tau_1)]^{(i-l)}(t)[u_{N-N_1}^{p-1}(t-\tau_1)]^{(l)} \\
&+ \int_{t_0^{p}-\tau_1}^{t-\tau_1} \partial_1^{(j)} k_2(t,s)u_{N-N_1}^{p-1}(s)ds \\
&- \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}[\partial_1^{(j-1-i)}k_1(t,t-\tau_1)]^{(i-l)}(t)[u_{N-N_1}^{p-1}(t-\tau_1)]^{(l)} \\
&+ \int_{t-\tau_1}^{t_1^{p}-\tau_1} \partial_1^{(j)} k_1(t,s)u_{N-N_1}^{p-1}(s)ds + \sum_{d=N-N_1+1}^{N-1}\int_{t_d^{p-1}}^{t_{d+1}^{p-1}} \partial_1^{(j)} k_1(t,s)u_d^{p-1}(s)ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}[\partial_1^{(j-1-i)}k_1(t,t)]^{(i-l)}(t)\hat{u}_{0,p}^{(l)}(t) + \int_{t_0^{p}}^{t} \partial_1^{(j)} k_1(t,s)\hat{u}_{0,p}(s)ds,
\end{aligned}
$$

which implies,

$$
\begin{aligned}
\hat{u}_{0,p}^{(j)}(t) = g^{(j)}(t) &+ \left(\int_0^{\tau_2} k_3(t,s)\Phi(s)ds\right)^{(j)}(t) \\
&+ \sum_{i=0}^{p-2}\sum_{d=0}^{N-1}\sum_{l=0}^{m-1} \frac{\hat{u}_{d,i}^{(l)}(t_d^i)}{l!}\int_{t_d^i}^{t_{d+1}^i} \partial_1^{(j)} k_3(t,s)(s-t_d^i)^l ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m-1}\binom{i}{l}\frac{\hat{u}_{0,p-1}^{(r)}(t_0^{p-1})}{(r-l)!}[\partial_1^{(j-1-i)}k_3(t,t-\tau_2)]^{(i-l)}(t)(t-\tau_2-t_0^{p-1})^{r-l} \\
&+ \sum_{l=0}^{m-1}\frac{\hat{u}_{0,p-1}^{(l)}(t_0^{p-1})}{l!}\int_{t_0^{p-1}}^{t-\tau_2} \partial_1^{(j)} k_3(t,s)(s-t_0^{p-1})^l ds \\
&- \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m-1}\binom{i}{l}\frac{\hat{u}_{0,p-1}^{(r)}(t_0^{p-1})}{(r-l)!}[\partial_1^{(j-1-i)}k_2(t,t-\tau_2)]^{(i-l)}(t)(t-\tau_2-t_0^{p-1})^{r-l} \\
&+ \sum_{l=0}^{m-1}\frac{\hat{u}_{0,p-1}^{(l)}(t_0^{p-1})}{l!}\int_{t-\tau_2}^{t_1^{p-1}} \partial_1^{(j)} k_2(t,s)(s-t_0^{p-1})^l ds \\
&+ \sum_{d=1}^{N-N_1-1}\sum_{l=0}^{m-1}\frac{\hat{u}_{d,p-1}^{(l)}(t_d^{p-1})}{l!}\int_{t_d^{p-1}}^{t_{d+1}^{p-1}} \partial_1^{(j)} k_2(t,s)(s-t_d^{p-1})^l ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m-1}\binom{i}{l}\frac{\hat{u}_{N-N_1,p-1}^{(r)}(t_{N-N_1}^{p-1})}{(r-l)!}[\partial_1^{(j-1-i)}k_2(t,t-\tau_1)]^{(i-l)}(t)(t-\tau_1-t_{N-N_1}^{p-1})^{r-l} \\
&+ \sum_{l=0}^{m-1}\frac{\hat{u}_{N-N_1,p-1}^{(l)}(t_{N-N_1}^{p-1})}{l!}\int_{t_0^p-\tau_1}^{t-\tau_1} \partial_1^{(j)} k_2(t,s)(s-t_{N-N_1}^{p-1})^l ds \\
&- \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m-1}\binom{i}{l}\frac{\hat{u}_{N-N_1,p-1}^{(r)}(t_{N-N_1}^{p-1})}{(r-l)!}[\partial_1^{(j-1-i)}k_1(t,t-\tau_1)]^{(i-l)}(t)(t-\tau_1-t_{N-N_1}^{p-1})^{r-l} \\
&+ \sum_{l=0}^{m-1}\frac{\hat{u}_{N-N_1,p-1}^{(l)}(t_{N-N_1}^{p-1})}{l!}\int_{t-\tau_1}^{t_{N-N_1+1}^{p-1}} \partial_1^{(j)} k_1(t,s)(s-t_{N-N_1}^{p})^l ds \\
&+ \sum_{d=N-N_1+1}^{N-1}\sum_{l=0}^{m-1}\frac{\hat{u}_{d,p-1}^{(l)}(t_d^{p-1})}{l!}\int_{t_d^{p-1}}^{t_{d+1}^{p-1}} \partial_1^{(j)} k_1(t,s)(s-t_d^{p-1})^l ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}[\partial_1^{(j-1-i)}k_1(t,t)]^{(i-l)}(t)\hat{u}_{0,p}^{(l)}(t) \\
&+ \int_{t_0^p}^{t} \partial_1^{(j)} k_1(t,s)\hat{u}_{0,p}(s)ds,
\end{aligned}
\tag{3.2.10}
$$

hence,

$$
\begin{aligned}
\hat{u}^{(j)}_{0,p}(t^p_0) = g^{(j)}(t^p_0) &+ \left(\int_0^{\tau_2} k_3(t,s)\Phi(s)ds\right)^{(j)}(t^p_0) \\
&+ \sum_{i=0}^{p-2}\sum_{d=0}^{N-1}\sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{d,i}(t^i_d)}{l!}\int_{t^i_d}^{t^i_{d+1}} \partial_1^{(j)} k_3(t^p_0,s)(s-t^i_d)^l ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\hat{u}^{(l)}_{0,p-1}(t^{p-1}_0)[\partial_1^{(j-1-i)}k_3(t,t-\tau_2)]^{(i-l)}(t^p_0) \\
&- \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\hat{u}^{(l)}_{0,p-1}(t^{p-1}_0)[\partial_1^{(j-1-i)}k_2(t,t-\tau_2)]^{(i-l)}(t^p_0) \\
&+ \sum_{l=0}^{m-1}\frac{\hat{u}^{(l)}_{0,p-1}(t^{p-1}_0)}{l!}\int_{t^p_0-\tau_2}^{t^{p-1}_1}\partial_1^{(j)}k_2(t^p_0,s)(s-t^{p-1}_0)^l ds \\
&+ \sum_{d=1}^{N-N_1-1}\sum_{l=0}^{m-1}\frac{\hat{u}^{(l)}_{d,p-1}(t^{p-1}_d)}{l!}\int_{t^{p-1}_d}^{t^{p-1}_{d+1}}\partial_1^{(j)}k_2(t^p_0,s)(s-t^{p-1}_d)^l ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\hat{u}^{(l)}_{N-N_1,p-1}(t^{p-1}_{N-N_1})[\partial_1^{(j-1-i)}k_2(t,t-\tau_1)]^{(i-l)}(t^p_0) \\
&- \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\hat{u}^{(l)}_{N-N_1,p-1}(t^{p-1}_{N-N_1})[\partial_1^{(j-1-i)}k_1(t,t-\tau_1)]^{(i-l)}(t^p_0) \\
&+ \sum_{l=0}^{m-1}\frac{\hat{u}^{(l)}_{N-N_1,p-1}(t^{p-1}_{N-N_1})}{l!}\int_{t^{p-1}_{N-N_1}}^{t^{p-1}_{N-N_1+1}}\partial_1^{(j)}k_1(t^p_0,s)(s-t^p_{N-N_1})^l ds \\
&+ \sum_{d=N-N_1+1}^{N-1}\sum_{l=0}^{m-1}\frac{\hat{u}^{(l)}_{d,p-1}(t^{p-1}_d)}{l!}\int_{t^{p-1}_d}^{t^{p-1}_{d+1}}\partial_1^{(j)}k_1(t^p_0,s)(s-t^{p-1}_d)^l ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}[\partial_1^{(j-1-i)}k_1(t,t)]^{(i-l)}(t^p_0)\hat{u}^{(l)}_{0,p}(t^p_0).
\end{aligned}
$$

3.2.4 Approximate solution in the interval σ^p_n

Finally, for x to be approximated by u^p_n ($n \in \{1,...,N-1\}$ and $p \in \{1,2,...,r-1\}$) on the interval σ^p_n, x must be approximated by u^j_k ($0 \le k < n$ and $0 \le j \le p$) on each interval σ^j_k

such that,

$$u_n^p(t) = \sum_{j=0}^{m-1} \frac{\hat{u}_{n,p}^{(j)}(t_n^p)}{j!}(t - t_n^p)^j; \quad t \in \sigma_n^p, \tag{3.2.11}$$

where $\hat{u}_{n,p}$ is the exact solution of the integral equations
for $t \in \sigma_n^p$, $n \in \{1, 2, ..., N_1 - 1\}$,

$$\begin{aligned}
\hat{u}_{n,p}(t) = g(t) &+ \int_0^{\tau_2} k_3(t,s)\phi(s)ds + \sum_{i=0}^{p-2}\sum_{d=0}^{N-1}\int_{t_d^i}^{t_{d+1}^i} k_3(t,s)u_d^i(s)ds \\
&+ \sum_{i=0}^{n-1}\int_{t_i^{p-1}}^{t_{i+1}^{p-1}} k_3(t,s)u_i^{p-1}(s)ds + \int_{t_n^{p-1}}^{t-\tau_2} k_3(t,s)u_n^{p-1}(s)ds \\
&+ \int_{t-\tau_2}^{t_{n+1}^{p-1}} k_2(t,s)u_n^{p-1}(s)ds + \sum_{d=n+1}^{N-N_1+n-1}\int_{t_d^{p-1}}^{t_{d+1}^{p-1}} k_2(t,s)u_d^{p-1}(s)ds \\
&+ \int_{t_n^p-\tau_1}^{t-\tau_1} k_2(t,s)u_{N-N_1+n}^{p-1}(s)ds + \int_{t-\tau_1}^{t_{n+1}^p-\tau_1} k_1(t,s)u_{N-N_1+n}^{p-1}(s)ds \\
&+ \sum_{i=N-N_1+n+1}^{N-1}\int_{t_i^{p-1}}^{t_{i+1}^{p-1}} k_1(t,s)u_i^{p-1}(s)ds + \sum_{i=0}^{n-1}\int_{t_i^p}^{t_{i+1}^p} k_1(t,s)u_i^p(s)ds \\
&+ \int_{t_n^p}^{t} k_1(t,s)\hat{u}_{n,p}(s)ds,
\end{aligned} \tag{3.2.12}$$

and for $t \in \sigma_n^p$, $n \in \{N_1, N_1 + 1, ..., N - 1\}$,

$$\begin{aligned}
\hat{u}_{n,p}(t) = g(t) &+ \int_0^{\tau_2} k_3(t,s)\phi(s)ds + \sum_{i=0}^{p-2}\sum_{d=0}^{N-1}\int_{t_d^i}^{t_{d+1}^i} k_3(t,s)u_d^i(s)ds \\
&+ \sum_{i=0}^{n-1}\int_{t_i^{p-1}}^{t_{i+1}^{p-1}} k_3(t,s)u_i^{p-1}(s)ds + \int_{t_n^{p-1}}^{t-\tau_2} k_3(t,s)u_n^{p-1}(s)ds \\
&+ \int_{t-\tau_2}^{t_{n+1}^{p-1}} k_2(t,s)u_n^{p-1}(s)ds + \sum_{d=n+1}^{N-1}\int_{t_d^{p-1}}^{t_{d+1}^{p-1}} k_2(t,s)u_d^{p-1}(s)ds \\
&+ \sum_{i=0}^{n-N_1-1}\int_{t_i^p}^{t_{i+1}^p} k_2(t,s)u_i^p(s)ds + \int_{t_n^p-\tau_1}^{t-\tau_1} k_2(t,s)u_{n-N_1}^p(s)ds \\
&+ \int_{t_n^p-\tau_1}^{t_{n+1}^p-\tau_1} k_1(t,s)u_{n-N_1}^p(s)ds + \sum_{i=n+1-N_1}^{n-1}\int_{t_i^p}^{t_{i+1}^p} k_1(t,s)u_i^p(s)ds \\
&+ \int_{t_n^p}^{t} k_1(t,s)\hat{u}_{n,p}(s)ds,
\end{aligned} \tag{3.2.13}$$

The coefficients $\hat{u}^{(j)}_{n,p}(t^p_n)$ is given by the following formula
for $t \in \sigma^p_n$, $n \in \{1, 2, ..., N_1 - 1\}$,

$$\begin{aligned}
\hat{u}^{(j)}_{n,p}(t) = g^{(j)}(t) &+ \left(\int_0^{\tau_2} k_3(t,s)\Phi(s)ds\right)^{(j)}(t) + \sum_{i=0}^{p-2}\sum_{d=0}^{N-1}\int_{t^i_d}^{t^i_{d+1}} \partial_1^{(j)} k_3(t,s)u^i_d(s)ds \\
&+ \sum_{i=0}^{n-1}\int_{t^{p-1}_i}^{t^{p-1}_{i+1}} \partial_1^{(j)} k_3(t,s)u^{p-1}_i(s)ds + \sum_{i=0}^{j-1}[\partial_1^{(j-1-i)} k_3(t, t-\tau_2)u^{p-1}_n(t-\tau_2)]^{(i)} \\
&+ \int_{t^{p-1}_n}^{t-\tau_2} \partial_1^{(j)} k_3(t,s)u^{p-1}_n(s)ds - \sum_{i=0}^{j-1}[\partial_1^{(j-1-i)} k_2(t, t-\tau_2)u^{p-1}_n(t-\tau_2)]^{(i)} \\
&+ \int_{t-\tau_2}^{t^{p-1}_{n+1}} \partial_1^{(j)} k_2(t,s)u^{p-1}_n(s)ds + \sum_{d=n+1}^{N-N_1+n-1}\int_{t^{p-1}_d}^{t^{p-1}_{d+1}} \partial_1^{(j)} k_2(t,s)u^{p-1}_d(s)ds \\
&+ \sum_{i=0}^{j-1}[\partial_1^{(j-1-i)} k_2(t, t-\tau_1)u^{p-1}_{N-N_1+n}(t-\tau_1)]^{(i)} + \int_{t^p_n-\tau_1}^{t-\tau_1} \partial_1^{(j)} k_2(t,s)u^{p-1}_{N-N_1+n}(s)ds \\
&- \sum_{i=0}^{j-1}[\partial_1^{(j-1-i)} k_1(t, t-\tau_1)u^{p-1}_{N-N_1+n}(t-\tau_1)]^{(i)} + \int_{t-\tau_1}^{t^{p-1}_{n+1}-\tau_1} \partial_1^{(j)} k_1(t,s)u^{p-1}_{N-N_1+n}(s)ds \\
&+ \sum_{i=N-N_1+n+1}^{N-1}\int_{t^{p-1}_i}^{t^{p-1}_{i+1}} \partial_1^{(j)} k_1(t,s)u^{p-1}_i(s)ds + \sum_{i=0}^{n-1}\int_{t^p_i}^{t^p_{i+1}} \partial_1^{(j)} k_1(t,s)u^p_i(s)ds \\
&+ \sum_{i=0}^{j-1}[\partial_1^{(j-1-i)} k_1(t,t)\hat{u}_{n,p}(t)]^{(i)} + \int_{t^p_n}^{t} \partial_1^{(j)} k_1(t,s)\hat{u}_{n,p}(s)ds,
\end{aligned}$$

which implies,

$$
\begin{aligned}
\hat{u}_{n,p}^{(j)}(t) = g^{(j)}(t) &+ \left(\int_0^{\tau_2} k_3(t,s)\Phi(s)ds\right)^{(j)}(t) + \sum_{i=0}^{p-2}\sum_{d=0}^{N-1}\int_{t_d^i}^{t_{d+1}^i} \partial_1^{(j)} k_3(t,s)u_d^i(s)ds \\
&+ \sum_{i=0}^{n-1}\int_{t_i^{p-1}}^{t_{i+1}^{p-1}} \partial_1^{(j)} k_3(t,s)u_i^{p-1}(s)ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}[\partial_1^{(j-1-i)}k_3(t,t-\tau_2)]^{(i-l)}(t)[u_n^{p-1}(t-\tau_2)]^{(l)} \\
&+ \int_{t_n^{p-1}}^{t-\tau_2} \partial_1^{(j)} k_3(t,s)u_n^{p-1}(s)ds \\
&- \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}[\partial_1^{(j-1-i)}k_2(t,t-\tau_2)]^{(i-l)}(t)[u_n^{p-1}(t-\tau_2)]^{(l)} \\
&+ \int_{t-\tau_2}^{t_{n+1}^{p-1}} \partial_1^{(j)} k_2(t,s)u_n^{p-1}(s)ds \\
&+ \sum_{d=n+1}^{N-N_1+n-1}\int_{t_d^{p-1}}^{t_{d+1}^{p-1}} \partial_1^{(j)} k_2(t,s)u_d^{p-1}(s)ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}[\partial_1^{(j-1-i)}k_2(t,t-\tau_1)]^{(i-l)}(t)[u_{N-N_1+n}^{p-1}(t-\tau_1)]^{(l)} \\
&+ \int_{t_n^p-\tau_1}^{t-\tau_1} \partial_1^{(j)} k_2(t,s)u_{N-N_1+n}^{p-1}(s)ds \\
&- \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}[\partial_1^{(j-1-i)}k_1(t,t-\tau_1)]^{(i-l)}(t)[u_{N-N_1+n}^{p-1}(t-\tau_1)]^{(l)} \\
&+ \int_{t-\tau_1}^{t_{n+1}^p-\tau_1} \partial_1^{(j)} k_1(t,s)u_{N-N_1+n}^{p-1}(s)ds + \sum_{i=N-N_1+n+1}^{N-1}\int_{t_i^{p-1}}^{t_{i+1}^{p-1}} \partial_1^{(j)} k_1(t,s)u_i^{p-1}(s)ds \\
&+ \sum_{i=0}^{n-1}\int_{t_i^p}^{t_{i+1}^p} \partial_1^{(j)} k_1(t,s)u_i^p(s)ds + \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}[\partial_1^{(j-1-i)}k_1(t,t)]^{(i-l)}(t)\hat{u}_{n,p}^{(l)}(t) \\
&+ \int_{t_n^p}^{t} \partial_1^{(j)} k_1(t,s)\hat{u}_{n,p}(s)ds,
\end{aligned}
$$

which implies,

$$
\begin{aligned}
\hat{u}^{(j)}_{n,p}(t) = {} & g^{(j)}(t) + \left(\int_0^{\tau_2} k_3(t,s)\Phi(s)ds\right)^{(j)}(t) + \sum_{i=0}^{p-2}\sum_{d=0}^{N-1}\sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{d,i}(t^i_d)}{l!}\int_{t^i_d}^{t^i_{d+1}} \partial_1^{(j)} k_3(t,s)(s-t^i_d)^l ds \\
& + \sum_{i=0}^{n-1}\sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{i,p-1}(t^{p-1}_i)}{l!}\int_{t^{p-1}_i}^{t^{p-1}_{i+1}} \partial_1^{(j)} k_3(t,s)(s-t^{p-1}_i)^l ds \\
& + \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m-1} \binom{i}{l}\frac{\hat{u}^{(r)}_{n,p-1}(t^{p-1}_n)}{(r-l)!}[\partial_1^{(j-1-i)} k_3(t,t-\tau_2)]^{(i-l)}(t)(t-\tau_2-t^{p-1}_n)^{r-l} \\
& + \sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{n,p-1}(t^{p-1}_n)}{l!}\int_{t^{p-1}_n}^{t-\tau_2} \partial_1^{(j)} k_3(t,s)(s-t^{p-1}_n)^l ds \\
& - \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m-1} \binom{i}{l}\frac{\hat{u}^{(r)}_{n,p-1}(t^{p-1}_n)}{(r-l)!}[\partial_1^{(j-1-i)} k_2(t,t-\tau_2)]^{(i-l)}(t)(t-\tau_2-t^{p-1}_n)^{r-l} \\
& + \sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{n,p-1}(t^{p-1}_n)}{l!}\int_{t-\tau_2}^{t^{p-1}_{n+1}} \partial_1^{(j)} k_2(t,s)(s-t^{p-1}_n)^l ds \\
& + \sum_{d=n+1}^{N-N_1+n-1}\sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{d,p-1}(t^{p-1}_d)}{l!}\int_{t^{p-1}_d}^{t^{p-1}_{d+1}} \partial_1^{(j)} k_2(t,s)(s-t^{p-1}_d)^l ds \\
& + \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m-1} \binom{i}{l}\frac{\hat{u}^{(r)}_{N-N_1+n,p-1}(t^{p-1}_{N-N_1+n})}{(r-l)!}[\partial_1^{(j-1-i)} k_2(t,t-\tau_1)]^{(i-l)}(t)(t-\tau_1-t^{p-1}_{N-N_1+n})^{r-l} \\
& + \sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{N-N_1+n,p-1}(t^{p-1}_{N-N_1+n})}{l!}\int_{t^p_n-\tau_1}^{t-\tau_1} \partial_1^{(j)} k_2(t,s)(s)[(t-\tau_1-t^{p-1}_{N-N_1+n})^l ds \\
& - \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m-1} \binom{i}{l}\frac{\hat{u}^{(r)}_{N-N_1+n,p-1}(t^{p-1}_{N-N_1+n})}{(r-l)!}[\partial_1^{(j-1-i)} k_1(t,t-\tau_1)]^{(i-l)}(t)(t-\tau_1-t^{p-1}_{N-N_1+n})^{r} \\
& + \sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{N-N_1+n,p-1}(t^{p-1}_{N-N_1+n})}{l!}\int_{t-\tau_1}^{t^p_{n+1}-\tau_1} \partial_1^{(j)} k_1(t,s)(s-t^{p-1}_{N-N_1+n})^l ds \\
& + \sum_{i=N-N_1+n+1}^{N-1}\sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{i,p-1}(t^{p-1}_i)}{l!}\int_{t^{p-1}_i}^{t^{p-1}_{i+1}} \partial_1^{(j)} k_1(t,s)(s)(s-t^{p-1}_i)^l ds \\
& + \sum_{i=0}^{n-1}\sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{i,p}(t^{p}_i)}{l!}\int_{t^{p}_i}^{t^{p}_{i+1}} \partial_1^{(j)} k_1(t,s)(s-t^{p}_i)^l ds \\
& + \sum_{i=0}^{j-1}\sum_{l=0}^{i} \binom{i}{l}[\partial_1^{(j-1-i)} k_1(t,t)]^{(i-l)}(t)\hat{u}^{(l)}_{n,p}(t) \\
& + \int_{t^p_n}^{t} \partial_1^{(j)} k_1(t,s)\hat{u}_{n,p}(s)ds,
\end{aligned}
\tag{3.2.14}
$$

hence,

$$
\begin{aligned}
\hat{u}^{(j)}_{n,p}(t^p_n) = g^{(j)}(t^p_n) &+ \left(\int_0^{\tau_2} k_3(t^p_n, s)\Phi(s)ds\right)^{(j)}(t^p_n) + \sum_{i=0}^{p-2}\sum_{d=0}^{N-1}\sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{d,i}(t^i_d)}{l!}\int_{t^i_d}^{t^i_{d+1}} \partial_1^{(j)} k_3(t^p_n, s)(s - t^i_d)^l ds \\
&+ \sum_{i=0}^{n-1}\sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{i,p-1}(t^{p-1}_i)}{l!}\int_{t^{p-1}_i}^{t^{p-1}_{i+1}} \partial_1^{(j)} k_3(t^p_n, s)(s - t^{p-1}_i)^l ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\hat{u}^{(l)}_{n,p-1}(t^{p-1}_n)[\partial_1^{(j-1-i)}k_3(t, t-\tau_2)]^{(i-l)}(t^p_n) \\
&- \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\hat{u}^{(l)}_{n,p-1}(t^{p-1}_n)[\partial_1^{(j-1-i)}k_2(t, t-\tau_2)]^{(i-l)}(t^p_n) \\
&+ \sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{n,p-1}(t^{p-1}_n)}{l!}\int_{t^p_n-\tau_2}^{t^{p-1}_{n+1}} \partial_1^{(j)} k_2(t^p_n, s)(s - t^{p-1}_n)^l ds \\
&+ \sum_{d=n+1}^{N-N_1+n-1}\sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{d,p-1}(t^{p-1}_d)}{l!}\int_{t^{p-1}_d}^{t^{p-1}_{d+1}} \partial_1^{(j)} k_2(t^p_n, s)(s - t^{p-1}_d)^l ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\hat{u}^{(l)}_{N-N_1+n,p-1}(t^{p-1}_{N-N_1+n})[\partial_1^{(j-1-i)}k_2(t, t-\tau_1)]^{(i-l)}(t^p_n) \\
&- \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\hat{u}^{(l)}_{N-N_1+n,p-1}(t^{p-1}_{N-N_1+n})[\partial_1^{(j-1-i)}k_1(t, t-\tau_1)]^{(i-l)}(t^p_n) \\
&+ \sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{N-N_1+n,p-1}(t^{p-1}_{N-N_1+n})}{l!}\int_{t^p_n-\tau_1}^{t^p_{n+1}-\tau_1} \partial_1^{(j)} k_1(t^p_n, s)(s - t^{p-1}_{N-N_1+n})^l ds \\
&+ \sum_{i=N-N_1+n+1}^{N-1}\sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{i,p-1}(t^{p-1}_i)}{l!}\int_{t^{p-1}_i}^{t^{p-1}_{i+1}} \partial_1^{(j)} k_1(t^p_n, s)(s - t^{p-1}_i)^l ds \\
&+ \sum_{i=0}^{n-1}\sum_{l=0}^{m-1} \frac{\hat{u}^{(l)}_{i,p}(t^p_i)}{l!}\int_{t^p_i}^{t^p_{i+1}} \partial_1^{(j)} k_1(t^p_n, s)(s - t^p_i)^l ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}[\partial_1^{(j-1-i)}k_1(t, t)]^{(i-l)}(t^p_n)\hat{u}^{(l)}_{n,p}(t^p_n),
\end{aligned}
$$

and for $t \in \sigma_n^p$, $n \in \{N_1, N_1+1, ..., N-1\}$

$$
\begin{aligned}
\hat{u}_{n,p}^{(j)}(t) = g^{(j)}(t) &+ \left(\int_0^{\tau_2} k_3(t,s)\Phi(s)ds \right)^{(j)}(t) + \sum_{i=0}^{p-2} \sum_{d=0}^{N-1} \int_{t_d^i}^{t_{d+1}^i} \partial_1^{(j)} k_3(t,s) u_d^i(s)ds \\
&+ \sum_{i=0}^{n-1} \int_{t_i^{p-1}}^{t_{i+1}^{p-1}} \partial_1^{(j)} k_3(t,s) u_i^{p-1}(s)ds \\
&+ \sum_{i=0}^{j-1} [\partial_1^{(j-1-i)} k_3(t, t-\tau_2) u_n^{p-1}(t-\tau_2)]^{(i)} \\
&+ \int_{t_n^{p-1}}^{t-\tau_2} \partial_1^{(j)} k_3(t,s) u_n^{(p-1)}(s)ds \\
&- \sum_{i=0}^{j-1} [\partial_1^{(j-1-i)} k_2(t, t-\tau_2) u_n^{p-1}(t-\tau_2)]^{(i)} \\
&+ \int_{t-\tau_2}^{t_{n+1}^{p-1}} \partial_1^{(j)} k_2(t,s) u_n^{p-1}(s)ds + \sum_{d=n+1}^{N-1} \int_{t_d^{p-1}}^{t_{d+1}^{p-1}} \partial_1^{(j)} k_2(t,s) u_d^{p-1}(s)ds \\
&+ \sum_{i=0}^{n-N_1-1} \int_{t_i^p}^{t_{i+1}^p} \partial_1^{(j)} k_2(t,s) u_i^p(s)ds \\
&+ \sum_{i=0}^{j-1} [\partial_1^{(j-1-i)} k_2(t, t-\tau_1) u_{n-N_1}^p(t-\tau_1)]^{(i)} \\
&+ \int_{t_n^p-\tau_1}^{t-\tau_1} \partial_1^{(j)} k_2(t,s) u_{n-N_1}^p(s)ds \\
&- \sum_{i=0}^{j-1} [\partial_1^{(j-1-i)} k_1(t, t-\tau_1) u_{n-N_1}^p(t-\tau_1)]^{(i)} \\
&+ \int_{t_n^p-\tau_1}^{t_{n+1}^p-\tau_1} \partial_1^{(j)} k_1(t,s) u_{n-N_1}^p(s)ds + \sum_{i=n+1-N_1}^{n-1} \int_{t_i^p}^{t_{i+1}^p} \partial_1^{(j)} k_1(t,s) u_i^p(s)ds \\
&+ \sum_{i=0}^{j-1} [\partial_1^{(j-1-i)} k_1(t,t) \hat{u}_{n,p}(t)]^{(i)} + \int_{t_n^p}^{t} \partial_1^{(j)} k_1(t,s) \hat{u}_{n,p}(s)ds,
\end{aligned}
$$

which implies,

$$
\begin{aligned}
\hat{u}_{n,p}^{(j)}(t) = g^{(j)}(t) &+ \left(\int_0^{\tau_2} k_3(t,s)\Phi(s)ds\right)^{(j)}(t) + \sum_{i=0}^{p-2}\sum_{d=0}^{N-1}\int_{t_d^i}^{t_{d+1}^i} \partial_1^{(j)} k_3(t,s)u_d^i(s)ds \\
&+ \sum_{i=0}^{n-1}\int_{t_i^{p-1}}^{t_{i+1}^{p-1}} \partial_1^{(j)} k_3(t,s)u_i^{p-1}(s)ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}[\partial_1^{(j-1-i)}k_3(t,t-\tau_2)]^{(i-l)}(t)[u_n^{p-1}(t-\tau_2)]^{(l)} \\
&+ \int_{t_n^{p-1}}^{t-\tau_2} \partial_1^{(j)} k_3(t,s)u_n^{(p-1)}(s)ds \\
&- \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}[\partial_1^{(j-1-i)}k_2(t,t-\tau_2)]^{(i-l)}(t)[u_n^{p-1}(t-\tau_2)]^{(l)} \\
&+ \int_{t-\tau_2}^{t_{n+1}^{p-1}} \partial_1^{(j)} k_2(t,s)u_n^{p-1}(s)ds + \sum_{d=n+1}^{N-1}\int_{t_d^{p-1}}^{t_{d+1}^{p-1}} \partial_1^{(j)} k_2(t,s)u_d^{p-1}(s)ds \\
&+ \sum_{i=0}^{n-N_1-1}\int_{t_i^p}^{t_{i+1}^p} \partial_1^{(j)} k_2(t,s)u_i^p(s)ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}[\partial_1^{(j-1-i)}k_2(t,t-\tau_1)]^{(i-l)}(t)[u_{n-N_1}^{p}(t-\tau_1)]^{(l)} \\
&+ \int_{t_n^p-\tau_1}^{t-\tau_1} \partial_1^{(j)} k_2(t,s)u_{n-N_1}^p(s)ds \\
&- \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}[\partial_1^{(j-1-i)}k_1(t,t-\tau_1)]^{(i-l)}(t)[u_{n-N_1}^{p}(t-\tau_1)]^{(l)} \\
&+ \int_{t_n^p-\tau_1}^{t_{n+1}^p-\tau_1} \partial_1^{(j)} k_1(t,s)u_{n-N_1}^p(s)ds + \sum_{i=n+1-N_1}^{n-1}\int_{t_i^p}^{t_{i+1}^p} \partial_1^{(j)} k_1(t,s)u_i^p(s)ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}[\partial_1^{(j-1-i)}k_1(t,t)]^{(i-l)}(t)\hat{u}_{n,p}^{(l)}(t) + \int_{t_n^p}^{t} \partial_1^{(j)} k_1(t,s)\hat{u}_{n,p}(s)ds,
\end{aligned}
$$

which implies,

$$
\begin{aligned}
\hat{u}_{n,p}^{(j)}(t) &= g^{(j)}(t) + \left(\int_0^{\tau_2} k_3(t,s)\Phi(s)ds\right)^{(j)}(t) + \sum_{i=0}^{p-2}\sum_{d=0}^{N-1}\sum_{l=0}^{m-1} \frac{\hat{u}_{d,i}^{(l)}(t_d^i)}{l!}\int_{t_d^i}^{t_{d+1}^i} \partial_1^{(j)} k_3(t,s)(s-t_d^i)^l ds \\
&+ \sum_{i=0}^{n-1}\sum_{l=0}^{m-1} \frac{\hat{u}_{i,p-1}^{(l)}(t_i^{p-1})}{l!}\int_{t_i^{p-1}}^{t_{i+1}^{p-1}} \partial_1^{(j)} k_3(t,s)(s-t_i^{p-1})^l ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m-1} \binom{i}{l} \frac{\hat{u}_{n,p-1}^{(r)}(t_n^{p-1})}{(r-l)!}[\partial_1^{(j-1-i)} k_3(t,t-\tau_2)]^{(i-l)}(t)(t-\tau_2-t_n^{p-1})^{r-l} \\
&+ \sum_{l=0}^{m-1} \frac{\hat{u}_{n,p-1}^{(l)}(t_n^{p-1})}{l!}\int_{t_n^{p-1}}^{t-\tau_2} \partial_1^{(j)} k_3(t,s)(s-t_n^{p-1})^l ds \\
&- \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m-1} \binom{i}{l} \frac{\hat{u}_{n,p-1}^{(r)}(t_n^{p-1})}{(r-l)!}[\partial_1^{(j-1-i)} k_2(t,t-\tau_2)]^{(i-l)}(t)(t-\tau_2-t_n^{p-1})^{r-l} \\
&+ \sum_{l=0}^{m-1} \frac{\hat{u}_{n,p-1}^{(l)}(t_n^{p-1})}{l!}\int_{t-\tau_2}^{t_{n+1}^{p-1}} \partial_1^{(j)} k_2(t,s)(s-t_n^{p-1})^l ds \\
&+ \sum_{d=n+1}^{N-1}\sum_{l=0}^{m-1} \frac{\hat{u}_{d,p-1}^{(l)}(t_d^{p-1})}{l!}\int_{t_d^{p-1}}^{t_{d+1}^{p-1}} \partial_1^{(j)} k_2(t,s)(s-t_d^{p-1})^l ds \\
&+ \sum_{i=0}^{n-N_1-1}\sum_{l=0}^{m-1} \frac{\hat{u}_{i,p}^{(l)}(t_i^{p})}{l!}\int_{t_i^{p}}^{t_{i+1}^{p}} \partial_1^{(j)} k_2(t,s)(s-t_i^{p})^l ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m-1} \binom{i}{l} \frac{\hat{u}_{n-N_1,p}^{(r)}(t_{n-N_1}^{p})}{(r-l)!}[\partial_1^{(j-1-i)} k_2(t,t-\tau_1)]^{(i-l)}(t)(t-\tau_1-t_{N-N_1}^{p})^{r-l} \\
&+ \sum_{l=0}^{m-1} \frac{\hat{u}_{n-N_1,p}^{(l)}(t_{n-N_1}^{p})}{l!}\int_{t_n^p-\tau_1}^{t-\tau_1} \partial_1^{(j)} k_2(t,s)(s-t_{n-N_1}^{p})^l ds \\
&- \sum_{i=0}^{j-1}\sum_{l=0}^{i}\sum_{r=0}^{m-1} \binom{i}{l} \frac{\hat{u}_{n-N_1,p}^{(r)}(t_{n-N_1}^{p})}{(r-l)!}[\partial_1^{(j-1-i)} k_1(t,t-\tau_1)]^{(i-l)}(t)(t-\tau_1-t_{n-N_1}^{p})^{r-l} \\
&+ \sum_{l=0}^{m-1} \frac{\hat{u}_{n-N_1,p}^{(l)}(t_{n-N_1}^{p})}{l!}\int_{t_n^p-\tau_1}^{t_{n+1}^p-\tau_1} \partial_1^{(j)} k_1(t,s)(s-t_{n-N_1}^{p})^l ds \\
&+ \sum_{i=n+1-N_1}^{n-1}\sum_{l=0}^{m-1} \frac{\hat{u}_{i,p}^{(l)}(t_i^{p})}{l!}\int_{t_i^{p}}^{t_{i+1}^{p}} \partial_1^{(j)} k_1(t,s)(s-t_i^{p})^l ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i} \binom{i}{l}[\partial_1^{(j-1-i)} k_1(t,t)]^{(i-l)}(t)\hat{u}_{n,p}^{(l)}(t) \\
&+ \int_{t_n^p}^{t} \partial_1^{(j)} k_1(t,s)\hat{u}_{n,p}(s)ds,
\end{aligned}
\tag{3.2.15}
$$

hence,

$$
\begin{aligned}
\hat{u}_{n,p}^{(j)}(t_n^p) = g^{(j)}(t_n^p) &+ \left(\int_0^{\tau_2} k_3(t,s)\Phi(s)ds\right)^{(j)}(t_n^p) + \sum_{i=0}^{p-2}\sum_{d=0}^{N-1}\sum_{l=0}^{m-1} \frac{\hat{u}_{d,i}^{(l)}(t_d^i)}{l!}\int_{t_d^i}^{t_{d+1}^i} \partial_1^{(j)} k_3(t_n^p,s)(s-t_d^i)^l ds \\
&+ \sum_{i=0}^{n-1}\sum_{l=0}^{m-1} \frac{\hat{u}_{i,p-1}^{(l)}(t_i^{p-1})}{l!}\int_{t_i^{p-1}}^{t_{i+1}^{p-1}} \partial_1^{(j)} k_3(t_n^p,s)(s-t_i^{p-1})^l ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\hat{u}_{n,p-1}^{(l)}(t_n^{p-1})[\partial_1^{(j-1-i)} k_3(t,t-\tau_2)]^{(i-l)}(t_n^p) \\
&- \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\hat{u}_{n,p-1}^{(l)}(t_n^{p-1})[\partial_1^{(j-1-i)} k_2(t,t-\tau_2)]^{(i-l)}(t_n^p) \\
&+ \sum_{l=0}^{m-1} \frac{\hat{u}_{n,p-1}^{(l)}(t_n^{p-1})}{l!}\int_{t_n^p-\tau_2}^{t_{n+1}^{p-1}} \partial_1^{(j)} k_2(t_n^p,s)(s-t_n^{p-1})^l ds \\
&+ \sum_{d=n+1}^{N-1}\sum_{l=0}^{m-1} \frac{\hat{u}_{d,p-1}^{(l)}(t_d^{p-1})}{l!}\int_{t_d^{p-1}}^{t_{d+1}^{p-1}} \partial_1^{(j)} k_2(t_n^p,s)(s-t_d^{p-1})^l ds \\
&+ \sum_{i=0}^{n-N_1-1}\sum_{l=0}^{m-1} \frac{\hat{u}_{i,p}^{(l)}(t_i^p)}{l!}\int_{t_i^p}^{t_{i+1}^p} \partial_1^{(j)} k_2(t_n^p,s)(s-t_i^p)^l ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\hat{u}_{n-N_1,p}^{(l)}(t_{n-N_1}^p)[\partial_1^{(j-1-i)} k_2(t,t-\tau_1)]^{(i-l)}(t_n^p) \\
&- \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\hat{u}_{n-N_1,p}^{(l)}(t_{n-N_1}^p)[\partial_1^{(j-1-i)} k_1(t,t-\tau_1)]^{(i-l)}(t_n^p) \\
&+ \sum_{l=0}^{m-1} \frac{\hat{u}_{n-N_1,p}^{(l)}(t_{n-N_1}^p)}{l!}\int_{t_n^p-\tau_1}^{t_{n+1}^p-\tau_1} \partial_1^{(j)} k_1(t_n^p,s)(s-t_{n-N_1}^p)^l ds \\
&+ \sum_{i=n+1-N_1}^{n-1}\sum_{l=0}^{m-1} \frac{\hat{u}_{i,p}^{(l)}(t_i^p)}{l!}\int_{t_i^p}^{t_{i+1}^p} \partial_1^{(j)} k_1(t_n^p,s)(s-t_i^p)^l ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}[\partial_1^{(j-1-i)} k_1(t,t)]^{(i-l)}(t_n^p)\hat{u}_{n,p}^{(l)}(t_n^p).
\end{aligned}
\tag{3.2.16}
$$

3.3 Analysis of convergence

3.3.1 Boundedness of the approximate polynomial's coefficients

We need the following lemma to prove the convergence of the method.

Lemma 3.3.1 *Let g, k_1, k_2 and k_3 be m times continuously differentiable on their respective domains. Then, there exists a positive number $\alpha(m)$ such that for all $n = 0, 1, ..., N-1$, $p = 0, 1, ..., r-1$, and $j = 0, ..., m$, we have*

$$\| \hat{u}_{n,p}^{(j)} \|_{L^{\infty}(\sigma_n^p)} \leq \alpha(m),$$

where $\hat{u}_{0,0}(t) = x(t)$ for $t \in \sigma_0^0$.

Proof. The proof is split into two steps.

Claim 1. Let $a_n^j = \| \hat{u}_{n,0}^{(j)} \|_{L^{\infty}(\sigma_n^0)}$, we have $j = 0, ..., m$,

$$a_0^j \leq max\left\{\| x^{(j)} \|_{L^{\infty}(\sigma_0^0)}, j = 0, ..., m\right\} = \alpha_1^1(m). \tag{3.3.1}$$

from (3.2.6), we have for all $n = 0, 1, ..., N_1 - 1$ and $j = 1, ..., m-1$,

$$a_n^j \leq c_1 + mb_1^1 \sum_{l=0}^{j-1} a_n^l + b_1^2 a_n^0 + hd_1^1 \sum_{i=0}^{n-1} \sum_{l=0}^{m-1} a_i^l,$$

and from(3.2.7), we have for all $n = N_1, N_1 + 1, ..., N-1$ and $j = 1, ..., m-1$,

$$a_n^j \leq c + mb_1^1 \sum_{l=0}^{j-1} a_n^l + b_1^2 a_n^0 + 2hm^2 d_1^2 \sum_{l=0}^{m-1} a_{n-N_1}^l + hd_1^1 \sum_{i=0}^{n-1} \sum_{l=0}^{m-1} a_i^l,$$

where, for all $j = 0, ..., m-1$
$c = max\left\{\|g^{(j)} + \left(\int_0^{t-\tau_2} k_3(t,s)\Phi(s)ds + \int_{t-\tau_2}^{\tau_2} k_2(t,s)\Phi(s)ds\right)^{(j)} \|_{L^{\infty}(\sigma_n^0)}, j = 0, ..., m-1\right\},$

$c_1 = max\left\{\|g^{(j)} + \left(\int_{t-\tau_1}^{\tau_2} k_1(t,s)\Phi(s)ds + \int_{t-\tau_2}^{t-\tau_1} k_2(t,s)\Phi(s)ds + \int_0^{t-\tau_2} k_3(t,s)\Phi(s)ds\right)^{(j)}\|_{L^\infty(\sigma_n^0)}, j = 0,...,m-1\right\}$,
$b_1^1 = max\left\{\binom{i}{l}\|[\partial_1^{(j-1-i)}k_1(t,t)]^{(i-l)})\|_{L^\infty(\sigma_n^0)}, j = 1,...,m-1; i = 0,...,j-1; l = 0,...,i\right\}$,
$b_1^2 = max\left\{\| \int_0^t \mid \partial_1^{(j)}k_1(t,s) \mid ds\|_{L^\infty(\sigma_n^0)}, j = 0,...,m-1\right\}$,
$d_1^1 = max\left\{\frac{1}{l!}\|\partial_1^{(j)}k_i(t,s)(s-t_n^0)^l ds\|_{L^\infty(\sigma_n^0)}, j = 0,...,m-1; n = 0,...,N-1; i = 1,2,3\right\}$,
and
$d_1^2 = max\left\{\binom{i}{l}\frac{1}{(r-l)!}\|[\partial_1^{(j-1-i)}k_s(t,t-\tau_1)]^{(i-l)})\|_{L^\infty(\sigma_n^0)}, l = 0,...,i; i = 0,...,j-1; s = 1,2\right\}$.

The constants c,c_1, b_1^1, b_1^2, d_1^1 and d_1^2 are positive and independent of N.
Hence,
for all $n = 0,1,...,N_1-1$,

$$a_n^j \le c_1 + mb_1^1\sum_{l=0}^{j-1} a_n^l + b_1^2\sum_{l=0}^{j-1} a_n^l + hd_1^1\sum_{i=0}^{n-1}\sum_{l=0}^{m-1} a_i^l,$$

and for all $n = N_1, N_1+1,...,N-1$,

$$a_n^j \le c + mb_1^1\sum_{l=0}^{j-1} a_n^l + b_1^2\sum_{l=0}^{j-1} a_n^l + 2hm^2d_1^2\sum_{l=0}^{m-1} a_{n-N_1}^l + hd_1^1\sum_{i=0}^{n-1}\sum_{l=0}^{m-1} a_i^l,$$

which implies that, for all $j \ge 1$
for all $n = 0,1,...,N_1-1$,

$$\begin{aligned} a_n^j \le & c_1 + \underbrace{\left(mb_1^1 + b_1^2\right)}_{b_1}\sum_{k=0}^{j-1} a_n^k + hd_1^1\sum_{i=0}^{n-1}\sum_{k=0}^{m-1} a_i^k \\ & \le c_1 + b_1a_n^0 + b_1\sum_{k=1}^{j-1} a_n^k + hd_1^1\sum_{i=0}^{n-1}\sum_{k=0}^{m-1} a_i^k, \end{aligned} \tag{3.3.2}$$

and for all $n = N_1, N_1+1, ..., N-1$,

$$
\begin{aligned}
a_n^j \leq& c + \underbrace{\left(mb_1^1 + b_1^2\right)}_{b_1} \sum_{k=0}^{j-1} a_n^k + 2hm^2 d_1^2 \sum_{l=0}^{m-1} a_{n-N_1}^l + hd_1^1 \sum_{i=0}^{n-1} \sum_{k=0}^{m-1} a_i^k \\
&\leq c + b_1 a_n^0 + b_1 \sum_{k=1}^{j-1} a_n^k + 2hm^2 d_1^2 \sum_{l=0}^{m-1} a_{n-N_1}^l + hd_1^1 \sum_{i=0}^{n-1} \sum_{k=0}^{m-1} a_i^k \\
\leq& c + b_1 a_n^0 + b_1 \sum_{k=1}^{j-1} a_n^k + \left(2m^2 d_1^2 + d_1^1\right) h \sum_{i=0}^{n-1} \sum_{k=0}^{m-1} a_i^k.
\end{aligned}
\tag{3.3.3}
$$

Moreover, for $j = 0$, for all $n = 0, 1, ..., N_1 - 1$, we have from (3.2.4)

$$
a_n^0 \leq c_1 + b_1 a_n^0 + hd_1^1 \sum_{i=0}^{n-1} \sum_{k=0}^{m-1} a_i^k, \tag{3.3.4}
$$

and for all $n = N_1, N_1+1, ..., N-1$, we have from (3.2.5)

$$
a_n^0 \leq c + b_1 a_n^0 + (2m^2 d_1^2 + d_1^1) h \sum_{i=0}^{n-1} \sum_{k=0}^{m-1} a_i^k. \tag{3.3.5}
$$

Now, for each fixed $n \geq 1$, we consider the sequence a_n^j for $j = 0, ..., m$. That satisfies for all $n = 0, 1, ..., N-1$,

$$
a_n^j \leq c_1^1 + b_1 a_n^0 + b_1 \sum_{k=1}^{j-1} a_n^k + hd_1 \sum_{i=0}^{n-1} \sum_{k=0}^{m-1} a_i^k, \tag{3.3.6}
$$

where $d_1 = 2m^2 d_1^2 + d_1^1$ and $c_1^1 = max(c, c_1)$.

Using Lemma(1.7.1), we find for all $j = 0, ..., m$ and $n = 0, 1, ..., N-1$,

$$\begin{aligned} a_n^j &\leq \left(c_1^1 + b_1 a_n^0 + h d_1 \sum_{i=0}^{n-1} \sum_{k=0}^{m-1} a_i^k \right) exp \left(\sum_{k=0}^{j-1} b_1 \right) \\ &\leq \underbrace{c_1^1 exp\,(mb_1)}_{c_2} + \underbrace{b_1 exp\,(mb_1)}_{b_2} a_n^0 + h\, \underbrace{d_1 exp\,(mb_1)}_{d_2} \sum_{i=0}^{n-1} \sum_{k=0}^{m-1} a_i^k \\ &\leq c_2 + b_2 a_n^0 + h d_2 \sum_{i=0}^{n-1} \sum_{k=0}^{m} a_i^k. \end{aligned} \tag{3.3.7}$$

Consider the sequence $z_n = \sum_{j=1}^{m} a_n^j$ for $n \geq 0$.
Then, by (3.3.7), we have

$$\begin{aligned} z_n &\leq \underbrace{mc_2}_{c_3^1} + \underbrace{mb_2}_{b_3} a_n^0 + h \underbrace{md_2}_{d_3} \sum_{i=0}^{n-1} \sum_{k=0}^{m} a_i^k \\ &\leq c_3^1 + b_3 a_n^0 + h d_3 \sum_{i=0}^{n-1} a_i^0 + h d_3 \sum_{i=0}^{n-1} \sum_{k=1}^{m} a_i^k \\ &\leq c_3^1 + b_3 a_n^0 + h d_3 \sum_{i=0}^{n} a_i^0 + h d_3 \sum_{i=0}^{n-1} z_i. \end{aligned} \tag{3.3.8}$$

Moreover, from (3.3.1), we obtain

$$z_0 \leq m\alpha_1^1(m) = c_3^2. \tag{3.3.9}$$

We set $c_3 = max(c_3^1, c_3^2)$. Then, from (3.3.8) and (3.3.9), we have for all $n = 0, 1, ..., N-1$,

$$z_n \leq c_3 + b_3 a_n^0 + h d_3 \sum_{i=0}^{n} a_i^0 + h d_3 \sum_{i=0}^{n-1} z_i.$$

Using the notations of Lemma (1.7.2), for all $n = 0, 1, ..., N-1$, we put

$$\varepsilon_n = z_n, \quad f_n = c_3 + b_3 a_n^0 + hd_3 \sum_{i=0}^{n} a_i^0, \quad g_i = hd_3.$$

Then, for all $n = 0, 1, ..., N-1$, we obtain,

$$\begin{aligned}
z_n \leq & c_3 + b_3 a_n^0 + hd_3 \sum_{i=0}^{n} a_i^0 + \sum_{j=0}^{n-1} hd_3 \left(c_3 + b_3 a_j^0 + hd_3 \sum_{i=0}^{j} a_i^0 \right) exp \left(\sum_{k=0}^{n-1} hd_3 \right) \\
\leq & c_3 + b_3 a_n^0 + hd_3 \sum_{i=0}^{n} a_i^0 + \sum_{j=0}^{n-1} hd_3 \left(c_3 + b_3 a_j^0 + hd_3 \sum_{i=0}^{j} a_i^0 \right) exp\left(\tau_2 d_3\right) \\
\leq & \underbrace{c_3 \left(1 + \tau_2 d_3\right)}_{c_4} + \left(b_3 + hd_3\right) a_n^0 + h \underbrace{\left(b_3 + b_3 d_3 exp\left(\tau_2 d_3\right) + \tau_2 d_3^2 exp\left(\tau_2 d_3\right) \right)}_{d_4} \sum_{i=0}^{n-1} a_i^0 \\
\leq & c_4 + \underbrace{\left(b_3 + hd_3\right)}_{b_4} a_n^0 + hd_4 \sum_{i=0}^{n-1} a_i^0.
\end{aligned} \tag{3.3.10}$$

On the other hand,for $j = 0$, for all $n = 0, ..., N-1$, we have from (3.3.6)

$$a_n^0 \leq c_1^1 + b_1 a_n^0 + hd_1 \sum_{i=0}^{n-1} \sum_{k=0}^{m-1} a_i^k.$$

Using (3.3.10), we deduce that

$$\begin{aligned}
a_n^0 \leq & c_1^1 + b_1 a_n^0 + hd_1 \sum_{i=0}^{n-1} a_i^0 + hd_1 \sum_{i=0}^{n-1} \left(c_4 + b_4 a_i^0 + hd_4 \sum_{k=0}^{n-1} a_k^0 \right) \\
\leq & \underbrace{\left(c_1^1 + \tau_2 d_1 c_4\right)}_{c_5} + b_1 a_n^0 + h \underbrace{\left(d_1 + d_1 b_4 + \tau_2 d_4 d_1\right)}_{d_5} \sum_{i=0}^{n-1} a_i^0.
\end{aligned}$$

This implies that for all $n = 0, 1, ..., N-1$,

$$\left(1 - b_1\right) a_n^0 \leq c_5 + hd_5 \sum_{i=0}^{n-1} a_i^0.$$

Hence, for all $1 - b_1 > 0$ and $n = 0, 1, ..., N - 1$ we have

$$a_n^0 \leq \frac{c_5}{1 - b_1} + \frac{hd_5}{1 - b_1} \sum_{i=0}^{n-1} a_i^0.$$

Using the notation of Lemma (1.7.1), for all $n = 0, 1, ..., N - 1$, we put

$$\varepsilon_n = a_n^0, \quad p_0 = \frac{c_5}{1 - b_1}, \quad k_i = \frac{hd_5}{1 - b_1}.$$

Then, for all $n = 0, 1, ..., N - 1$, we obtain

$$\begin{aligned}
a_n^0 \leq & \left(\frac{c_5}{1 - b_1}\right) exp\left(\sum_{i=0}^{n-1} \frac{hd_5}{1 - b_1}\right) \\
\leq & \underbrace{\left(\frac{c_5}{1 - b_1}\right)}_{c_6} exp\left(\tau_2 \underbrace{d_5 \frac{1}{1 - b_1}}_{c_7}\right) \\
\leq & \underbrace{c_6 + exp\left(\tau_2 c_7\right)}_{c_8}.
\end{aligned}$$

Which implies, from (3.3.10), that for all $j = 1, ...m$ and $n = 0, ..., N - 1$,

$$\begin{aligned}
a_n^j \leq z_n \leq & c_4 + b_4 c_8 + hd_4 \sum_{i=0}^{n-1} c_8 \\
\leq & c_4 + b_4 c_8 + d_4 \tau_1 c_8 \\
\leq & c_9.
\end{aligned}$$

Hence, the first step is completed by setting

$$\alpha_1(m) = max\left(\alpha_1^1(m), c_9\right).$$

Claim2. There exists a positive constant $\alpha(m)$ such that $\| \hat{u}_{n,p}^{(j)} \|_{L^\infty(\sigma_n^p)} \leq \alpha(m)$ for all $n = 0, 1, ..., N-1$, $j = 0, ..., m$ and $p = 1, 2, ..., r-1$.
Let $a_{n,p}^{(j)} = \| \hat{u}_{n,p}^{(j)} \|_{L^\infty(\sigma_n^p)} \leq \alpha(m)$ and $\varepsilon_p = max \left\{ a_{i,p}^j, j = 0, .., m, i = 0, ..., N-1 \right\}$ for $p = 1, ..., r-1$. Similarly to Claim 1, from (3.2.10), we obtain for all $j = 1, .., m$,

$$a_{0,p}^j \leq c_{10} + b_5 \sum_{i=0}^{p-1} \varepsilon_i + d_5 \sum_{l=0}^{j-1} a_{0,p}^l,$$

where c_{10}, b_5, and d_5 are positive numbers.

Using the notation of Lemma (1.7.1), for all $n = 0, 1, ..., N-1$, we put

$$\varepsilon_n = a_{0,p}^j, \quad p_0 = c_{10} + b_5 \sum_{i=0}^{p-1} \varepsilon_i, \quad k_i = d_5.$$

Then, for all $n = 0, 1, ..., N-1$ and $j = 0, 1, ..., m$, we obtain

$$\begin{aligned} a_{0,p}^j &\leq \left(c_{10} + b_5 \sum_{i=0}^{p-1} \varepsilon_i \right) exp \left(\sum_{k=0}^{j-1} d_5 \right) \\ &\leq \underbrace{c_{10} exp\,(md_5)}_{c_{11}} + \underbrace{b_5 exp\,(md_5)}_{b_6} \sum_{i=0}^{p-1} \varepsilon_i \\ &\leq c_{11} + b_6 \sum_{i=0}^{p-1} \varepsilon_i. \end{aligned}$$

Then, by Lemma (1.7.1), we obtain for all $p = 0, ..., r-1$, $n = 0, ..., N-1$, and $j = 0, ..., m$,

$$
\begin{aligned}
a_{0,p}^{j} \leq & c_{11} exp\left(\sum_{k=0}^{p-1} b_6\right) \\
\leq & c_{11} exp\left((r-1)b_6\right) \\
\leq & C.
\end{aligned}
\tag{3.3.11}
$$

Next, from (3.2.14) and (3.2.15), we obtain for all $n = 1, ..., N-1$,

$$
a_{n,p}^{j} \leq c_{12} + b_7 \sum_{i=0}^{p-1} \varepsilon_i + e_7 \sum_{l=0}^{j-1} a_{n,p}^{l} + d_7 h \sum_{i=0}^{n-1} \sum_{l=0}^{m-1} a_{i,p}^{l}, \tag{3.3.12}
$$

where c_{12}, b_7, e_7 and d_7 are positive numbers.

Then, by Lemma (1.7.1), for all $j = 0, ..., m$,

$$
\begin{aligned}
a_{n,p}^{j} \leq & \left(c_{12} + b_7 \sum_{i=0}^{p-1} \varepsilon_i + d_7 h \sum_{i=0}^{n-1} \sum_{l=0}^{m-1} a_{i,p}^{l}\right) exp\left(\sum_{k=0}^{j-1} e_7\right) \\
\leq & \underbrace{c_{12} exp\left(me_7\right)}_{c_{13}} + \underbrace{b_7 exp\left(me_7\right)}_{b_8} \sum_{i=o}^{p-1} \varepsilon_i + \underbrace{e_7 exp\left(me_7\right)}_{e_8} a_{n,p}^{0} + \underbrace{d_7 exp\left(me_7\right)}_{d_8} h \sum_{i=o}^{n-1} \sum_{l=o}^{m-1} a_{i,p}^{l} \\
\leq & c_{13} + b_8 \sum_{i=0}^{p-1} \varepsilon_i + e_8 a_{n,p}^{0} + d_8 h \sum_{i=0}^{n-1} \sum_{l=0}^{m-1} a_{i,p}^{l}.
\end{aligned}
\tag{3.3.13}
$$

Consider the sequence $y_n = \sum_{j=1}^{m} a_{n,p}^{j}$, $n - 0, 1, ..., N-1$,
hence, by the aforementioned inequality, the sequence y_n satisfies for all $n = 1, .., N-1$,

$$
y_n \leq \underbrace{mc_{13}}_{c_{14}1} + \underbrace{mb_8}_{b_9} \sum_{i=0}^{p-1} \varepsilon_i + \underbrace{me_8}_{e_9} a_{n,p}^{0} + \underbrace{md_8}_{d_9} h \sum_{i=0}^{n} a_{i,p}^{0} + \underbrace{md_8}_{d_9} h \sum_{i=0}^{n-1} y_i. \tag{3.3.14}
$$

Moreover, from (3.3.11), we obtain

$$y_0 \leq m\alpha_1^1(m) = c_{14}^2. \tag{3.3.15}$$

Let $c_{14} = max(c_{14}^1, c_{14}^2)$.

Then, from (3.3.14) and (3.3.16), we obtain for all $n = 0, 1, ..., N-1$

$$y_n \leq c_{14} + b_9 \sum_{i=0}^{p-1} \varepsilon_i + e_9 a_{n,p}^0 + d_9 h \sum_{i=0}^{n} a_{i,p}^0 + d_9 h \sum_{i=0}^{n-1} y_i.$$

Using the notations of Lemma (1.7.2), for all $n = 0, ..., N-1$, we put

$$\varepsilon_n = y_n, \quad f_n = c_{14} + b_9 \sum_{i=0}^{p-1} \varepsilon_i + e_9 a_{n,p}^0 + d_9 h \sum_{i=0}^{n} a_{i,p}^0, \quad g_i = d_9 h.$$

Then, for all $n = 0, 1, ..., N-1$, we obtain

$$\begin{aligned} y_n \leq & c_{14} + b_9 \sum_{i=0}^{p-1} \varepsilon_i + e_9 a_{n,p}^0 + d_9 h \sum_{i=0}^{n} a_{i,p}^0 + \sum_{j=0}^{n-1} h d_9 \left(c_{14} + b_9 \sum_{i=0}^{p-1} \varepsilon_i + e_9 a_{j,p}^0 + d_9 h \sum_{i=0}^{j} a_{i,p}^0 \right) exp\left(\sum_{k=0}^{n-1} h d_9 \right) \\ \leq & \underbrace{c_{14} \left(1 + \tau_2 d_9 exp(\tau_1 d_9)\right)}_{c_{15}} + \underbrace{b_9 \left(1 + \tau_2 d_9 exp(\tau_2 d_9)\right)}_{b_{10}} \sum_{i=0}^{p-1} \varepsilon_i + \underbrace{(e_9 + \tau_2 d_9)}_{e_{10}} a_{n,p}^0 \\ & + h \underbrace{\left(d_9 + d_9 e_9 exp(\tau_2 d_9) + \tau_2 d_9^2 exp(\tau_2 d_9) \right)}_{d_{10}} \sum_{i=0}^{n-1} a_{i,p}^0 \\ \leq & c_{15} + b_{10} \sum_{i=0}^{p-1} \varepsilon_i + e_{10} a_{n,p}^0 + h d_{10} \sum_{i=0}^{n-1} a_{i,p}^0. \end{aligned} \tag{3.3.16}$$

On the other hand, for $j = 0$, we have from (3.3.12),

$$a_{n,p}^0 \leq c_{12} + b_7 \sum_{i=0}^{p-1} \varepsilon_i + d_7 h \sum_{i=0}^{n-1} \sum_{l=0}^{m-1} a_{i,p}^l.$$

Using (3.3.16), we deduce that

$$
\begin{aligned}
a_{n,p}^{0} \leq & c_{12} + b_7 \sum_{i=0}^{p-1} \varepsilon_i + d_7 h \sum_{i=0}^{n-1} a_{i,p}^{0} + d_7 h \sum_{i=0}^{n-1} \left(c_{15} + b_{10} \sum_{i=0}^{p-1} \varepsilon_i + e_{10} a_{i,p}^{0} + h d_{10} \sum_{i=0}^{n-1} a_{i,p}^{0} \right) \\
& \leq \underbrace{(c_{12} + d_7 \tau_2 c_{15})}_{c_{16}} + \underbrace{(b_7 + \tau_2 b_{10} d_7)}_{b_{11}} \sum_{i=0}^{p-1} \varepsilon_i + h \underbrace{(d_7 + d_7 e_{10} + d_{10} d_7 \tau_2)}_{d_{11}} \sum_{i=0}^{n-1} a_{i,p}^{0} \\
& \leq c_{16} + b_{11} \sum_{i=0}^{p-1} \varepsilon_i + h d_{11} \sum_{i=0}^{n-1} a_{i,p}^{0}.
\end{aligned}
\tag{3.3.17}
$$

Using the notation of Lemma (1.7.1), for all $n = 0, 1, ..., N-1$, we put

$$
\varepsilon_n = a_{n,p}^{0}, \quad p_0 = c_{16} + b_{11} \sum_{i=0}^{p-1} \varepsilon_i, \quad k_i = h d_{11}.
$$

Then, for all $n = 0, 1, ..., N-1$ and $j = 0, 1, ..., m$, we obtain

$$
\begin{aligned}
a_{n,p}^{0} \leq & \left(c_{16} + b_{11} \sum_{i=0}^{p-1} \varepsilon_i \right) exp \left(\sum_{k=0}^{n-1} h d_{11} \right) \\
& \leq \underbrace{c_{16} exp\,(\tau_2 d_{11})}_{c_{17}} + \underbrace{b_{11} exp\,(\tau_2 d_{11})}_{b_{12}} \sum_{i=0}^{p-1} \varepsilon_i \\
& \leq c_{17} + b_{12} \sum_{i=0}^{p-1} \varepsilon_i.
\end{aligned}
$$

Then, by Lemma (1.7.1), we obtain for all $p = 0, ..., r-1$, $n = 0, ..., N-1$, and $j = 0, ..., m$,

$$\begin{aligned}
a^{0}_{n,p} \leq& c_{17} exp\left(\sum_{k=0}^{p-1} b_{12}\right)\\
\leq& c_{17} exp\left((r-1)b_{12}\right)\\
\leq& c_{18}.
\end{aligned}$$

Which implies, from(3.3.16), that for all $j = 1, .., m$ and $n = 0, 1, ..., N-1$

$$\begin{aligned}
a^{j}_{n,p} &\leq c_{15} + b_{10}\sum_{i=0}^{p-1}\varepsilon_i + e_{10}a^{0}_{n,p} + hd_{10}\sum_{i=0}^{n-1}a^{0}_{i,p}\\
&\leq c_{15} + b_{10}\sum_{i=0}^{p-1}\varepsilon_i + e_{10}c_{18} + \underbrace{\tau_2 d_{10} c_{18}}_{c_{19}}\\
&\leq c_{20} + b_{10}\sum_{i=0}^{p-1}\varepsilon_i.
\end{aligned}$$

Using Lemma (1.7.1), we obtain,

$$\begin{aligned}
a^{0}_{n,p} \leq& c_{20} exp\left(\sum_{k=0}^{p-1} b_{10}\right)\\
\leq& c_{20} exp\left((r-1)b_{10}\right)\\
\leq& c_{21}.
\end{aligned}$$

■

This completes the proof of Lemma (3.3.1).

3.3.2 Experimental order of convergence

The following theorem describes the order of convergence of the method.

Theorem 3.3.1 *Let g, k_1, k_2 and k_3 be m times continuously differentiable on their respective domains. Then, (3.2.2)-(3.2.16) define a unique approximation $u \in S_{m-1}^{(-1)}(\Pi_N)$, and the resulting error function $e = x - u$ satisfies*

$$\| e \|_{L(I)^\infty} \leq Ch^m$$

provided that h is sufficiently small, where C is a finite constant independent of h.

Proof. The proof is split into two steps.

claim 1. There exists a constant C_1 independent of h such that $\| e^0 \|_{L^\infty(\sigma^0)} \leq C_1 h^m$, where the error $e^0 = e \mid_{\sigma^0}$ that is defined on σ_n^0, by $e^0(t) = e_n^0(t) = \mid x(t) - u_n^0(t) \mid$ for all $n = 0, 1, ..., N-1$.

Let $t \in \sigma_0^0$. Then, we have from Theorem (3.3.1), for sufficient small h,

$$\mid e_0^0(t) \mid = \mid x(t) - u_0^0 \mid \leq \frac{\| x^m \|_{L^\infty(\sigma_0^0)}}{m!} h^m \leq \frac{\alpha(m)}{m!} h^m.$$

In general for $n = 1, 2, ..., N_1 - 1$ and $t \in \sigma_n^0$, we have from (3.2.4),

$$x(t) - \hat{u}_{n,0}(t) = \sum_{i=0}^{n-1} \int_{t_i^0}^{t_{i+1}^0} k_1(t,s) \left(x(s) - u_i^0(s) \right) ds + \int_{t_n^0}^{t} k_1(t,s) \left(x(s) - \hat{u}_{n,0}(s) \right) ds.$$

And for $n = N_1, ..., N-1$ and $t \in \sigma_n^0$, we have from (3.2.5),

$$\begin{aligned} x(t) - \hat{u}_{n,0}(t) = & \sum_{i=0}^{n-N_1-1} \int_{t_i^0}^{t_{i+1}^0} k_2(t,s) \left(x(s) - u_i^0(s) \right) ds + \int_{t_n^0 - \tau_1}^{t-\tau_1} k_2(t,s) \left(x(s) - u_{n-N_1}^0(s) \right) ds \\ & + \int_{t-\tau_1}^{t_{n+1}^0 - \tau_1} k_1(t,s) \left(x(s) - u_{n-N_1}^0(s) \right) ds + \sum_{i=n+1-N_1}^{n-1} \int_{t_i^0}^{t_{i+1}^0} k_1(t,s) \left(x(s) - u_i^0(s) \right) ds \\ & + \int_{t_n^0}^{t} k_1(t,s) \left(x(s) - \hat{u}_{n,0}(s) \right) ds. \end{aligned}$$

This implies that for $n = 1, 2, ..., N_1 - 1$,

$$\| x - \hat{u}_{n,0} \|_{L^\infty(\sigma_n^0)} \leq hk \sum_{i=0}^{n-1} \| e_i^0 \|_{L^\infty(\sigma_i^0)} + hk \| x - \hat{u}_{n,0} \|_{L^\infty(\sigma_n^0)} \,. \tag{3.3.18}$$

And for $n = N_1, ..., N - 1$,

$$\begin{aligned} \| x - \hat{u}_{n,0} \|_{L^\infty(\sigma_n^0)} \leq hk \sum_{i=0}^{n-N_1-1} \| e_i^0 \|_{L^\infty(\sigma_i^0)} + 2hk \| e_{n-N_1}^0 \|_{L^\infty(\sigma_i^0)} \\ + hk \sum_{i=n+1-N_1}^{n-1} \| e_i^0 \|_{L^\infty(\sigma_i^0)} + hk \| x - \hat{u}_{n,0} \|_{L^\infty(\sigma_n^0)} \,. \end{aligned} \tag{3.3.19}$$

From (3.3.18),(3.3.19), for $n = 1, ..., N - 1$, we have

$$\| x - \hat{u}_{n,0} \|_{L^\infty(\sigma_n^0)} \leq 2hk \sum_{i=0}^{n-1} \| e_i^0 \|_{L^\infty(\sigma_i^0)} + hk \| x - \hat{u}_{n,0} \|_{L^\infty(\sigma_n^0)},$$

where $k = max\Big(\| k_i \|_{L^\infty(\sigma_i^0)}, i = 1, 2, 3\Big)$,

hence,

$$(1 - hk) \| x - \hat{u}_{n,0} \|_{L^\infty(\sigma_n^0)} \leq 2hk \sum_{i=0}^{n-1} \| e_i^0 \|_{L^\infty(\sigma_i^0)} \,.$$

For $1 - hk > 0$

$$\| x - \hat{u}_{n,0} \|_{L^\infty(\sigma_n^0)} \leq \frac{2hk}{1 - hk} \sum_{i=0}^{n-1} \| e_i^0 \|_{L^\infty(\sigma_i^0)} \,.$$

Therefore, by theorem (3.3.1), for $n = 1, 2, ..., N - 1$,

$$\begin{aligned}
\| e_n^0 \|_{L^\infty(\sigma_n^0)} &\leq \| x - u_n^0 \|_{L^\infty(\sigma_n^0)} \\
&\leq \| x - \hat{u}_{n,0} \|_{L^\infty(\sigma_n^0)} + \| \hat{u}_{n,0} - u_n^0 \|_{L^\infty(\sigma_n^0)} \\
&\leq \| x - \hat{u}_{n,0} \|_{L^\infty(\sigma_n^0)} + \frac{\alpha(m)}{m!} h^m.
\end{aligned}$$

Then, for $n = 1, 2, ..., N - 1$,

$$\| e_n^0 \|_{L^\infty(\sigma_n^0)} \leq \frac{2hk}{1 - hk} \sum_{i=0}^{n-1} \| e_i^0 \|_{L^\infty(\sigma_i^0)} + \frac{\alpha(m)}{m!} h^m.$$

Hence, by Lemma (1.7.1), for all $n = 0, 1, ..., N_1 - 1$,

$$\begin{aligned}
\| e_n^0 \|_{L^\infty(\sigma_n^0)} &\leq \frac{\alpha(m) h^m}{m!} exp\left(\sum_{j=0}^{n-1} \frac{2hk}{1 - hk} \right) \\
&\leq \frac{\alpha(m) h^m}{m!} exp\left(\tau_1 \frac{2k}{1 - hk} \right) \\
&\leq C_1 h^m.
\end{aligned}$$

claim 2. There exists a constant C_2 independent of h such that $\| e \|_{L^\infty(I)} \leq C_2 h^m$. Define the error $e^p(t)$ on σ^p by $e^p(t) = x(t) - u^p(t)$ and on σ_n^p by $e^p(t) = e_n^p(t) = x(t) - u_n^p(t)$ for all $n = 0, ..., N - 1$ and $p = 0, ..., r - 1$.

First, let $t \in \sigma_0^p$. Then, we have from equation (3.2.9),

$$
\begin{aligned}
x(t) - \hat{u}_{0,p}(t) = & \sum_{i=0}^{p-1}\sum_{d=0}^{N-1}\int_{t_d^i}^{t_{d+1}^i} k_3(t,s)\left(x(s) - u_d^i(s)\right) ds + \int_{t_0^{p-1}}^{t-\tau_2} k_3(t,s)\left(x(s) - u_0^{p-1}(s)\right) ds \\
& + \int_{t-\tau_2}^{t_1^{p-1}} k_2(t,s)\left(x(s) - u_0^{p-1}(s)\right) ds + \sum_{d=1}^{N-N_1-1}\int_{t_d^{p-1}}^{t_{d+1}^{p-1}} k_2(t,s)\left(x(s) - u_d^{p-1}(s)\right) ds \\
& + \int_{t_0^p-\tau_1}^{t-\tau_1} k_2(t,s)\left(x(s) - u_{N-N_1}^{p-1}(s)\right) ds + \int_{t-\tau_1}^{t_1^p-\tau_1} k_1(t,s)\left(x(s) - u_{N-N_1}^{p-1}(s)\right) ds \\
& + \sum_{d=N-N_1+1}^{N-1}\int_{t_d^{P-1}}^{t_{d+1}^{P-1}} k_1(t,s)\left(x(s) - u_d^{P-1}(s)\right) ds + \int_{t_0^P}^{t} k_1(t,s)\left(x(s) - \hat{u}_{0,P}(s)\right) ds.
\end{aligned}
$$

This implies that for $n = 0, ..., N-1$ and $p = 0, ..., r-1$,

$$
\begin{aligned}
\| x - \hat{u}_{0,p} \|_{L^\infty(\sigma_0^p)} \le & \, hk\sum_{i=0}^{p-1}\sum_{d=0}^{N-1} \| e_d^i \|_{L^\infty(\sigma_d^i)} + 2hk \| e_0^{p-1} \|_{L^\infty(\sigma_0^{p-1})} \\
& + hk\sum_{d=1}^{N-N_1-1} \| e_d^{p-1} \|_{L^\infty(\sigma_d^{p-1})} + 2hk \| e_{N-N_1}^{p-1} \|_{L^\infty(\sigma_{N-N_1}^{p-1})} \\
& + hk\sum_{d=N-N_1+1}^{N-1} \| e_d^{p-1} \|_{L^\infty(\sigma_d^{p-1})} + hk \| x - \hat{u}_{0,p} \|_{L^\infty(\sigma_0^p)} .
\end{aligned}
\tag{3.3.20}
$$

From (3.3.20), for $n = 0, ..., N-1$, and $p = 0, ..., r-1$, we have

$$
\begin{aligned}
\| x - \hat{u}_{0,p} \|_{L^\infty(\sigma_0^p)} \le & \, hk\sum_{i=0}^{p-1}\sum_{d=0}^{N-1} \| e_d^i \|_{L^\infty(\sigma_d^i)} + 2hk \| e_0^{p-1} \|_{L^\infty(\sigma_0^{p-1})} \\
& + 2hk\sum_{d=1}^{N-1} \| e_d^{p-1} \|_{L^\infty(\sigma_d^{p-1})} + hk \| x - \hat{u}_{0,p} \|_{L^\infty(\sigma_0^p)},
\end{aligned}
$$

where $k = max\left(\| k_i \|_{L^\infty(\sigma_i^0)}, i = 1, 2, 3\right)$,

hence,

$$(1-hk)\parallel x-\hat{u}_{n,0}\parallel_{L^{\infty}(\sigma_n^0)} \leq hk\sum_{i=0}^{p-1}\sum_{d=0}^{N-1}\parallel e_d^i\parallel_{L^{\infty}(\sigma_d^i)} +2hk\parallel e_0^{p-1}\parallel_{L^{\infty}(\sigma_0^{p-1})}$$
$$+\,2hk\sum_{d=1}^{N-1}\parallel e_d^{p-1}\parallel_{L^{\infty}(\sigma_d^{p-1})}.$$

For $1-hk>0$

$$\parallel x-\hat{u}_{0,p}\parallel_{L^{\infty}(\sigma_0^p)} \leq \frac{hk}{1-hk}\sum_{i=0}^{p-1}\sum_{d=0}^{N-1}\parallel e_d^i\parallel_{L^{\infty}(\sigma_d^i)} +\frac{2hk}{1-hk}\parallel e_0^{p-1}\parallel_{L^{\infty}(\sigma_0^{p-1})}$$
$$+\,\frac{2hk}{1-hk}\sum_{d=1}^{N-1}\parallel e_d^{p-1}\parallel_{L^{\infty}(\sigma_d^{p-1})}.$$

Therefore, by theorem (3.3.1), for $n=1,2,...,N_1-1$,

$$\begin{aligned}\parallel e_0^p\parallel_{L^{\infty}(\sigma_0^p)} &\leq\parallel x-u_0^p\parallel_{L^{\infty}(\sigma_n^0)}\\ &\leq\parallel x-\hat{u}_{0,p}\parallel_{L^{\infty}(\sigma_0^p)}+\parallel \hat{u}_{n,0}-u_0^p\parallel_{L^{\infty}(\sigma_0^p)}\\ &\leq\parallel x-\hat{u}_{0,p}\parallel_{L^{\infty}(\sigma_0^p)}+\frac{\alpha(m)}{m!}h^m.\end{aligned}$$

Then, for $n=1,2,...,N_1-1$,

$$\parallel e_0^p\parallel_{L^{\infty}(\sigma_0^p)} \leq \frac{hk}{1-hk}\sum_{i=0}^{p-1}\sum_{d=0}^{N-1}\parallel e_d^i\parallel_{L^{\infty}(\sigma_d^i)} +\frac{2hk}{1-hk}\parallel e_0^{p-1}\parallel_{L^{\infty}(\sigma_0^{p-1})}$$
$$+\,\frac{2hk}{1-hk}\sum_{d=1}^{N-1}\parallel e_d^{p-1}\parallel_{L^{\infty}(\sigma_d^{p-1})}+\frac{\alpha(m)}{m!}h^m.$$

Then,

$$\| e_0^p \|_{L^\infty(\sigma_0^p)} \leq \frac{hk}{1-hk}\sum_{i=0}^{p-1}\sum_{d=0}^{N-1} \| e_d^i \|_{L^\infty(\sigma_d^i)} + \frac{2hk}{1-hk}\sum_{i=0}^{p-1}\sum_{d=0}^{N-1} \| e_d^i \|_{L^\infty(\sigma_0^{p-1})}$$
$$+ \frac{2hk}{1-hk}\sum_{i=0}^{p-1}\sum_{d=0}^{N-1} \| e_d^i \|_{L^\infty(\sigma_d^{p-1})} + \frac{\alpha(m)}{m!}h^m,$$

which implies,

$$\| e_0^p \|_{L^\infty(\sigma_0^p)} \leq \frac{5hk}{1-hk}\sum_{i=0}^{p-1}\sum_{d=0}^{N-1} \underbrace{\| e_d^i \|_{L^\infty(\sigma_d^i)}}_{\leq max\|e_d^i\| = \|e^i\|} + \frac{\alpha(m)}{m!}h^m,$$

which implies,

$$\begin{aligned}
\| e_0^p \|_{L^\infty(\sigma_0^p)} &\leq \frac{5hkN}{1-hk}\sum_{i=0}^{p-1} \| e_d^i \|_{L^\infty(\sigma_d^i)} + \frac{\alpha(m)}{m!}h^m \\
&\leq \frac{5k\tau_2}{1-hk}\sum_{i=0}^{p-1} \| e_d^i \|_{L^\infty(\sigma_d^i)} + \frac{\alpha(m)}{m!}h^m,
\end{aligned}$$

hence, by Lemma (1.7.1), for all $n = 0, 1, ..., N-1$,

$$\begin{aligned}
\| e_n^p \|_{L^\infty(\sigma_0^p)} &\leq \frac{\alpha(m)h^m}{m!} exp\left(\sum_{i=0}^{p-1}\frac{5k\tau_2}{1-hk}\right) \\
&\leq \frac{\alpha(m)h^m}{m!} exp\left(\frac{5Tk}{1-hk}\right) \\
&\leq \frac{\alpha(m)h^m}{m!} exp\left(c_{22}\right) \\
&\leq C_2^1 h^m.
\end{aligned}$$

Next, let $t \in \sigma_n^p$ for $n = 1, 2, ..., N_1 - 1$. Then, we have from (3.2.12),

$$\begin{aligned}
x(t) - \hat{u}_{n,p}(t) &= \sum_{i=0}^{p-2} \sum_{d=0}^{N-1} \int_{t_d^i}^{t_{d+1}^i} k_3(t,s)\left(x(s) - u_d^i(s)\right) ds + \sum_{i=0}^{n-1} \int_{t_i^{p-1}}^{t_{i+1}^{p-1}} k_3(t,s)\left(x(s) - u_i^{p-1}(s)\right) ds \\
&+ \int_{t_n^{p-1}}^{t-\tau_2} k_3(t,s)\left(x(s) - u_n^{p-1}(s)\right) ds + \int_{t-\tau_2}^{t_{n+1}^{p-1}} k_2(t,s)\left(x(s) - u_n^{p-1}(s)\right) ds \\
&+ \sum_{d=n+1}^{N-N_1+n-1} \int_{t_d^{p-1}}^{t_{d+1}^{p-1}} k_2(t,s)\left(x(s) - u_d^{p-1}(s)\right) ds + \int_{t_n^p-\tau_1}^{t-\tau_1} k_2(t,s)\left(x(s) - u_{N-N_1+n}^{p-1}(s)\right) ds \\
&+ \int_{t-\tau_1}^{t_{n+1}^p-\tau_1} k_1(t,s)\left(x(s) - u_{N-N_1+n}^{p-1}(s)\right) ds + \sum_{i=N-N_1+n+1}^{N-1} \int_{t_i^{P-1}}^{t_{i+1}^{P-1}} k_1(t,s)\left(x(s) - u_i^{p-1}(s)\right) ds \\
&+ \sum_{i=0}^{n-1} \int_{t_i^p}^{t_{i+1}^p} k_1(t,s)\left(x(s) - u_i^p(s)\right) ds + \int_{t_n^P}^{t} k_1(t,s)\left(x(s) - \hat{u}_{n,p}(s)\right) ds.
\end{aligned} \tag{3.3.21}$$

And for $n = N_1, ..., N-1$ and $t \in \sigma_n^P$, we have from (3.2.13),

$$\begin{aligned}
x(t) - \hat{u}_{n,p}(t) &= \sum_{i=0}^{p-2} \sum_{d=0}^{N-1} \int_{t_d^i}^{t_{d+1}^i} k_3(t,s)\left(x(s) - u_d^i(s)\right) ds + \sum_{i=0}^{n-1} \int_{t_i^{p-1}}^{t_{i+1}^{p-1}} k_3(t,s)\left(x(s) - u_i^{p-1}(s)\right) ds \\
&+ \int_{t_n^{p-1}}^{t-\tau_2} k_3(t,s)\left(x(s) - u_n^{p-1}(s)\right) ds + \int_{t-\tau_2}^{t_{n+1}^{p-1}} k_2(t,s)\left(x(s) - u_n^{p-1}(s)\right) ds \\
&+ \sum_{d=n+1}^{N-1} \int_{t_d^{p-1}}^{t_{d+1}^{p-1}} k_2(t,s)\left(x(s) - u_d^{p-1}(s)\right) ds + \sum_{i=0}^{n-N_1-1} \int_{t_i^p}^{t_{i+1}^p} k_2(t,s)\left(x(s) - u_i^p(s)\right) ds \\
&+ \int_{t_n^p-\tau_1}^{t-\tau_1} k_2(t,s)\left(x(s) - u_{n-N_1}^p(s)\right) ds + \int_{t-\tau_1}^{t_{n+1}^p-\tau_1} k_1(t,s)\left(x(s) - u_{n-N_1}^p(s)\right) ds \\
&+ \sum_{i=n+1-N_1}^{n-1} \int_{t_i^P}^{t_{i+1}^p} k_1(t,s)\left(x(s) - u_i^P(s)\right) ds + \int_{t_n^P}^{t} k_1(t,s)\left(x(s) - \hat{u}_{n,p}(s)\right) ds.
\end{aligned} \tag{3.3.22}$$

This implies from (3.3.21),(3.3.22), for $n = 1, ..., N-1$, we have

$$\begin{aligned}\| x - \hat{u}_{n,p} \|_{L^\infty(\sigma_n^p)} \leq{}& hk \sum_{i=0}^{p-1} \sum_{d=0}^{N-1} \| e_d^i \|_{L^\infty(\sigma_d^i)} + hk \sum_{i=0}^{n-1} \| e_i^{p-1} \|_{L^\infty(\sigma_i^{p-1})} \\ &+ 2hk \| e_n^{p-1} \|_{L^\infty(\sigma_n^{p-1})} + hk \sum_{d=n+1}^{N-1} \| e_d^{p-1} \|_{L^\infty(\sigma_d^{p-1})} \\ &+ hk \sum_{i=0}^{n-1} \| e_i^p \|_{L^\infty(\sigma_i^p)} + hk \| x - \hat{u}_{n,p} \|_{L^\infty(\sigma_n^p)},\end{aligned}$$

where $k = max\left(\| k_i \|_{L^\infty(\sigma_i^0)}, i = 1, 2, 3\right)$,

hence,

$$(1 - hk) \| x - \hat{u}_{n,p} \|_{L^\infty(\sigma_n^p)} \leq 2hk \sum_{i=0}^{p-1} \sum_{d=0}^{N-1} \| e_d^i \|_{L^\infty(\sigma_d^i)} + 2hk \sum_{i=0}^{n-1} \| e_i^{p-1} \|_{L^\infty(\sigma_i^{p-1})} .$$

For $1 - hk > 0$

$$\| x - \hat{u}_{n,p} \|_{L^\infty(\sigma_n^p)} \leq \frac{2hk}{1 - hk} \sum_{i=0}^{p-1} \sum_{d=0}^{N-1} \| e_d^i \|_{L^\infty(\sigma_d^i)} + \frac{2hk}{1 - hk} \sum_{i=0}^{n-1} \| e_i^{p-1} \|_{L^\infty(\sigma_i^{p-1})} .$$

Therefore, by theorem (3.3.1), for $n = 1, 2, ..., N - 1$,

$$\begin{aligned}\| e_n^p \|_{L^\infty(\sigma_n^p)} &\leq \| x - u_n^p \|_{L^\infty(\sigma_n^p)} \\ &\leq \| x - \hat{u}_{n,p} \|_{L^\infty(\sigma_n^p)} + \| \hat{u}_{n,p} - u_n^p \|_{L^\infty(\sigma_n^p)} \\ &\leq \| x - \hat{u}_{n,p} \|_{L^\infty(\sigma_n^p)} + \frac{\alpha(m)}{m!} h^m.\end{aligned}$$

Then, for $n = 1, 2, ..., N - 1$,

$$\begin{aligned}\| e_n^p \|_{L^\infty(\sigma_n^p)} \le & \frac{2hkN}{1-hk}\sum_{i=0}^{p-1} \| e^i \|_{L^\infty(\sigma^i)} + \frac{2hk}{1-hk}\sum_{i=0}^{n-1} \| e_i^{p-1} \|_{L^\infty(\sigma_i^{p-1})} \\ & + \frac{\alpha(m)}{m!}h^m,\end{aligned}$$

where $e^i = e|_{\sigma^i} = x - u^i$.

Hence, by Lemma (1.7.1), for all $n = 0, 1, ..., N-1$,

$$\begin{aligned}\| e_n^p \|_{L^\infty(\sigma_n^p)} &\le \left(\frac{\alpha(m)h^m}{m!} + \frac{2k\tau_2}{1-hk}\sum_{i=0}^{p-1} \| e^i \|_{L^\infty(\sigma^i)}\right) exp\left(\sum_{j=0}^{n-1} \frac{2hk}{1-hk}\right) \\ &\le \left(\frac{\alpha(m)h^m}{m!} + \frac{2k\tau_2}{1-hk}\sum_{i=0}^{p-1} \| e^i \|_{L^\infty(\sigma^i)}\right) \underbrace{exp\left(\frac{2k\tau_2}{1-hk}\right)}_{c_{23}} \\ &\le \frac{\alpha(m)c_{23}h^m}{m!} + \frac{2\tau_2 k c_{23}}{1-hk}\sum_{i=0}^{p-1} \| e^i \|_{L^\infty(\sigma^i)},\end{aligned}$$

using again Lemma (1.7.1), for all $n = 0, 1, ..., N-1$,

$$\begin{aligned}\| e_n^p \|_{L^\infty(\sigma_n^p)} &\le \frac{\alpha(m)c_{23}h^m}{m!} exp\left(\sum_{i=0}^{p-1} \frac{2\tau_2 k c_{23}}{1-hk}\right) \\ &\le \frac{\alpha(m)c_{23}h^m}{m!} exp\left(\frac{2Tkc_{23}}{1-hk}\right) \\ &\le \frac{\alpha(m)c_{23}h^m}{m!} exp\left(c_{24}\right) \\ &\le C_2^2 h^m.\end{aligned}$$

We taking $C_2 = max\{C_2^1, C_2^2\}$.

Thus, the proof is completed by taking $C = max\{C_1, C_2\}$. ∎

3.4 Numerical examples

In this section, we give two numerical examples to illustrate the theoretical results obtained in the second chapter. In each example, we calculate the error between x and the Taylor collocation solution u. We notice that the absolute error decreases with increasing value N and m.

Example 3.4.1 *Consider the double delay Volterra integral equation*

$$x(t) = g(t) + \int_{t-1}^{t-\frac{1}{2}} \frac{s}{1+t} x(s)ds, \tag{3.4.1}$$

for $t \in [1, 7]$ *and* g *is chosen so that* $x(t) = \frac{1+t}{e^t}$, $\Phi(t) = \frac{1+t}{e^t}$ *for* $t \in [0, 1]$.
The absolute errors for $(N, m) = \{(2, 2), (4, 4), (6, 6), (8, 8)\}$ *are presented in Table 3.1.*

Table 3.1: Absolute errors of Example 3.4.1

t	$(N, m) = (2, 2)$	$(N, m) = (4, 4)$	$(N, m) = (6, 6)$	$(N, m) = (8, 8)$
1	0.0	0.0	0.0	0.0
2	3.66×10^{-4}	4.12×10^{-6}	9.82×10^{-10}	9.52×10^{-11}
3	1.75×10^{-3}	1.90×10^{-6}	6.02×10^{-10}	7.28×10^{-11}
4	1.90×10^{-3}	4.32×10^{-7}	2.65×10^{-10}	4.20×10^{-11}
5	1.43×10^{-3}	1.15×10^{-7}	9.24×10^{-11}	2.20×10^{-11}
6	9.09×10^{-4}	2.14×10^{-7}	2.32×10^{-11}	1.09×10^{-11}
7	1.49×10^{-3}	8.70×10^{-7}	6.58×10^{-11}	5.32×10^{-12}

Example 3.4.2 *Consider the double delay Volterra integral equation*

$$\begin{aligned} x(t) = g(t) &+ \int_{0}^{t-\tau_2} \sin(s+t)x(s)ds + \int_{t-\tau_2}^{t-\tau_1} (ts + \cos(s+t))x(s)ds \\ &+ \int_{t-\tau_1}^{t} \cos(s+t)x(s)ds, \end{aligned}$$

for $t \in [1, 6]$ *and* g *is chosen so that* $x(t) = \sin(t) + 1$, $\Phi(t) = \sin(t) + 1$ *for* $t \in [0, 1]$.
The absolute errors for $(N, m) = \{(4, 6), (6, 6), (8, 6), (8, 8)\}$ *are presented in Table 3.2.*

Table 3.2: Absolute errors of Example 3.4.2

t	$(N,m)=(4,6)$	$(N,m)=(6,6)$	$(N,m)=(8,6)$	$(N,m)=(8,8)$
1	0.0	0.0	0.0	0.0
2	2.31×10^{-8}	6.17×10^{-9}	3.17×10^{-9}	2.60×10^{-10}
3	3.18×10^{-7}	4.69×10^{-8}	7.94×10^{-9}	2.53×10^{-9}
4	5.10×10^{-6}	6.34×10^{-7}	2.34×10^{-7}	3.65×10^{-8}
5	9.02×10^{-5}	1.05×10^{-5}	3.81×10^{-6}	6.41×10^{-7}
6	4.72×10^{-3}	5.75×10^{-4}	2.02×10^{-4}	3.37×10^{-5}

CONCLUSION

In conclusion, we have successfully studied the existence and uniqueness of theorems for nonlinear integral equations involving two constant delays. Furthermore, we employed the Taylor polynomials method to obtain a numerical solution for these double delay integral equations. We also investigated the convergence of the approximate solution and found that it is a reliable approach. The implementation of this method is straightforward, making it easily applicable in practice. Extending our research to encompass generalizations of the proposed approach for nonlinear double delay integral equations would be beneficial.

$$x(t) = g(t) + \int_0^{t-\tau_2} k_3(t,s,x(s))ds + \int_{t-\tau_2}^{t-\tau_1} k_2(t,s,x(s))ds + \int_{t-\tau_1}^{t} k_1(t,s,x(s))ds.$$

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