

People's Democratic Republic of Algeria
Ministry of Higher Education and scientific Research

University Center Abd Elhafid Boussouf MILA



Institute of Science and Technology

Department of Mathematics and Informatics

Thesis presented to obtain the degree of Master
Specialty: Fundamental Mathematics

**Ordinary Generating Functions for Second-Order Recurrence
Sequences**

Presented by : **Meissa Bahloul**

Cheima Ahmed Yahia

Board of Examiners:

Fahima Bouzekria	MAA	University Center Mila	President
khadidja Boubellouta	MCB	University Center Mila	Supervisor
Mohamed Kecies	MCB	University Center Mila	Examiner

Academic Year : 2022 – 2023

ACKNOWLEDGMENTS

We first thank **ALLAH** who helped us and who gave us the strength, the will and the courage to carry out this work

We would like to express our gratitude and our sincere thanks to our supervisor, professor **Boubellouta Khadidja** for her patience, the time she devoted to us, her availability and above all her judicious advice which contributed to fulfilling our reflection

We would also like to thank the members of the jury for agreeing to read and evaluate our work not forgetting the teachers, our colleagues and administrators of the mathematics and department

We reserve a special place here to warmly thank our families for their continued support, precisely our **father** and our **mother** and all those who in one way or another have helped and encouraged us in carrying out this work

Meissa, Cheima

DEDICATION

It is with great gratitude and sincere words that I dedicate this modest end-of-study work

To my Mother: Sabah , who showered me with her support and devoted unconditional love to the moon that lights up my nights.

To my Father: Abdel Malek, who always sacrificed himself to see me succeed that God grant him good health and long life. I hope that one day, I can give them back a little of what they do for me. I also dedicate this work

To my sister: Djouhaina

To my brothers: Borhan, Moussab and Mounder.

To my husband: Omar.

To my buddy: Cheima,

who shared with me the moments difficulties of this work;

to my family Bahloul and my husband's family Khabat To all the members of my promotion, to all my teachers since my first year of studies. Also many other people that I have not had the opportunity to mention them. What helped me directly or indirectly.

Meissa

DEDICATION

I dedicate this modest work as a testimony of affection respect and admiration to: **To my very dear mother: Loubna**, for all the love she gave me and for all the sacrifices. I will do my best to stay a subject of pride in these eyes with the hope never disappointing

To my father: Lakhder for the trust he gave me and for his continued support

To my sisters: Asma, Allaa and Issraa who have always encouraged me

To my brother: Ayoub

To my the companion of the path and the friend of the good and bad days.

My dear husband **Zaki**, for the moral and material support he gave me

To my buddy: Meissa who shared with me all the good and bad moments

To my all the members of my promotion, to all my teachers since my first year of studies.

Also many other people that I have not had the opportunity to mention them Which helped me more or less

Cheima

CONTENTS

Introduction	7
1 Notations and preliminaries	9
1.1 Linear recurrence relations	9
1.2 Linear homogeneous recurrence relations with constant coefficients . . .	11
1.2.1 Recurrence relations of some numbers and polynomials	14
1.3 Generating functions	19
1.3.1 Formal series	19
1.3.2 Ordinary generating functions (OGF)	21
2 Elementary and complete symmetric functions	29
2.1 Symmetric functions	29
2.1.1 Elementary symmetric functions	30
2.1.2 Complete symmetric functions	32

2.2	Some properties on symmetric functions	36
3	Ordinary generating functions of the products for some numbers an polynomials	40
3.1	Definitions and some properties	40
3.2	Principal formula	41

INTRODUCTION

The second-order linear recurrence sequences $(u_n(a, b; p, q))_{n \geq 0}$, or briefly $(u_n)_{n \geq 0}$ is defined as follow

$$u_n = pu_{n-1} + qu_{n-2}, \quad n \geq 2 \quad (1)$$

With the initials conditions $u_0 = a, u_1 = b$ [17, 21] . This sequence was introduced by Horadam in 1985 (see[17, 18, 24]). Which were generalized many sequences, mostly depending on p, q, a and b . Examples of such sequences include the Fibonacci numbers sequences and Lucas numbers sequences. Many authors have been interested in studying these sequences. For example Hoggatt in [16], Vorobiov in[30], and recently Marques in [22] and Shattuck in [27].

Generating functions are a particularly useful tool in the study of combinatorics. They allow us to apply analysis and algebraic techniques to combinatorial problems, particularly in the context of recurrence, many authors have calculated generating functions for the products of some numbers and polynomials , you can refer to [4, 7, 26].

The main purpose of this work is to obtain new results on generating functions using the concept of symmetric functions . Our thesis is organized into three chapters:

In the first chapter, we present the tools and preliminaries necessary for understand-

ing the following chapters. We first give definitions and properties of linear recurrence relations for some numbers and polynomials. Following that, we offer some reminders regarding formal series. At the end, we introduce the ordinary generating functions of some polynomials.

In the second chapter, we recall the elementary and completes symmetric functions, as well as their properties.

In the third chapter, by making use the symmetrizing operators $\delta_{a_1 a_2}^k$ we introduce new generating functions for the products of the Fibonacci number, Lucas numbers and Jacobsthal numbers with bivariate Fibonacci polynomials, bivariate Lucas polynomial, bivariate Pell Lucas polynomials, bivariate Jacobsthal polynomials and bivariate Jacobsthal Lucas polynomials.

CHAPTER 1

NOTATIONS AND PRELIMINARIES

In this chapter, we mention some definitions and fundamental theorems related to linear recurrence relations, formal series and generating functions that are utilized in our work, for more details, you can refer to [1, 2, 25].

1.1 Linear recurrence relations

Definition 1.1.1 [28] *A linear recurrence relation of degree k is a recurrence relation of the form*

$$u_n + f_1(n) u_{n-1} + f_2(n) u_{n-2} + \dots + f_k(n) u_{n-k} = g(n), \quad (1.1)$$

where $f_1(n), f_2(n), \dots, f_k(n)$ et $g(n)$ are functions of n and $f_k(n) \neq 0$.

Remark 1.1.1

1. If $g(n) = 0$, then (1.1) is said to be homogeneous recurrence relation, if $g(n) \neq 0$, then (1.1) is said to be non-homogeneous recurrence relation.

2. If $f_1(n), f_2(n), \dots, f_k(n)$ are constants, then the recurrence relation (1.1) is known as a linear recurrence relations of constant coefficients.

Theorem 1.1.1 [28] *The linear recurrence relation*

$$u_n + f_1(n) u_{n-1} + f_2(n) u_{n-2} + \dots + f_k(n) u_{n-k} = g(n),$$

with $u_0 = a_0, u_1 = 1, \dots, u_{k-1} = a_{k-1}$, are constants has a unique solution.

Lemma 1.1.1 [28] *Let $u_n^{(1)}$ be the solution of the relation:*

$$u_n + f_1(n) u_{n-1} + f_2(n) u_{n-2} + \dots + f_k(n) u_{n-k} = g_1(n),$$

and $u_n^{(2)}$ the solution of the relation:

$$u_n + f_1(n) u_{n-1} + f_2(n) u_{n-2} + \dots + f_k(n) u_{n-k} = g_2(n).$$

Then $c_1 u_n^{(1)} + c_2 u_n^{(2)}$ is the solution of

$$u_n + f_1(n) u_{n-1} + f_2(n) u_{n-2} + \dots + f_k(n) u_{n-k} = c_1 g_1(n) + c_2 g_2(n).$$

Proof. We have

$$\begin{aligned} & \left[c_1 u_n^{(1)} + c_2 u_n^{(2)} \right] + f_1(n) \left[c_1 u_{n-1}^{(1)} + c_2 u_{n-1}^{(2)} \right] + \dots + f_k(n) \left[c_1 u_{n-k}^{(1)} + c_2 u_{n-k}^{(2)} \right] \\ &= c_1 u_n^{(1)} + c_1 f_1(n) u_{n-1}^{(1)} + \dots + c_1 f_k(n) u_{n-k}^{(1)} + c_2 u_n^{(2)} + c_2 f_1(n) u_{n-1}^{(2)} + \dots + c_2 f_k(n) u_{n-k}^{(2)} \\ &= c_1 \left[u_n^{(1)} + f_1(n) u_{n-1}^{(1)} + \dots + f_k(n) u_{n-k}^{(1)} \right] + c_2 \left[u_n^{(2)} + f_1(n) u_{n-1}^{(2)} + \dots + f_k(n) u_{n-k}^{(2)} \right] \\ &= c_1 g_1(n) + c_2 g_2(n). \end{aligned}$$

This completes the proof ■

1.2 Linear homogeneous recurrence relations with constant coefficients

Definition 1.2.1 [24] A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$u_n = c_1 u_{n-1} + c_2 u_{n-2} + \dots + c_k u_{n-k}, \quad (1.2)$$

where $c_1, c_2, \dots, c_k$ are real numbers, and $c_k \neq 0$

Example 1.2.1 The recurrence relation $F_n = F_{n-1} + F_{n-2}$ is a linear homogeneous recurrence relation of degree two.

Example 1.2.2 The recurrence relation $u_n = u_{n-5}$ is a linear homogeneous recurrence relation of degree five.

Example 1.2.3 The recurrence relation $u_n = u_{n-1} + u_{n-2}^2$ is not linear. The recurrence relation $u_n = 3u_{n-1} + 5$ is not homogeneous. The recurrence relation $v_n = nv_{n-1}$ does not have constant coefficients.

Remark 1.2.1 [24] The basic approach for solving linear homogeneous recurrence relations is to look for solutions of the form $u_n = z^n$ where t is a constant. Note that $u_n = z^n$ is a solution of the recurrence relation $u_n = c_1 u_{n-1} + c_2 u_{n-2} + \dots + c_k u_{n-k}$ if and only if

$$z^n = c_1 z^{n-1} + \dots + c_k z^{n-k}$$

when both sides of this equation are divided by z^{n-k} and the right-hand side is subtracted from the left, we obtain the equation $z^k - c_1 z^{k-1} - \dots - c_k = 0$. Consequently, the sequence u_n with $u_n = z^n$ is a solution if and only if z is a solution of this last equation. We call this the **characteristic equation** of the recurrence relation. The solutions of this equation are called the **characteristic roots** of the recurrence relation. As we will see, these characteristic roots can be used to give an explicit formula for all the solutions of the recurrence relation.

Definition 1.2.2 [3] *The characteristic polynomial of the recurrence relation*

$$u_n = c_1 u_{n-1} + c_2 u_{n-2} + \dots + c_k u_{n-k}$$

is

$$P(z) = z^k - c_1 z^{k-1} - \dots - c_k.$$

Example 1.2.4 *The characteristic polynomial of the recurrence relation $u_n = 5u_{n-1} - 6u_{n-2}$ is $z^2 - 5z + 6$.*

Example 1.2.5 *The characteristic polynomial of the recurrence relation $P_n = P_{n-2} + P_{n-3}$ is $z^3 - z - 1$.*

Remark 1.2.2 *The characteristic equation associated with the recurrence relation is obtained by canceling the characteristic polynomial of the latter.*

Theorem 1.2.1 *If z is a root of multiplicity m of recurrence relation (1.2) then $u_n = n^r z^n, 0 \leq r < m$ is a solution of the relation (1.2)*

Theorem 1.2.2 [24] *Let $c_1, c_2, \dots, c_k$ be real number. Suppose that the characteristic equation*

$$z^k - c_1 z^{k-1} - \dots - c_k = 0,$$

has k distinct roots $z_1, z_2, \dots, z_k$. Then a sequence u_n is a solution of the recurrence relation

$$u_n = c_1 u_{n-1} + c_2 u_{n-2} + \dots + c_k u_{n-k},$$

if and only if

$$u_n = \alpha_1 z_1^n + \alpha_2 z_2^n + \dots + \alpha_k z_k^n.$$

for $n = 0, 1, 2, \dots$, where $\alpha_1, \alpha_2, \dots, \alpha_k$ are constants.

Example 1.2.6 Consider the following recurrence relation: $u_n = u_{n-1} + 6u_{n-2}$ with initial conditions $u_0 = 0, u_1 = 1$.

The characteristic equation is $z^2 - z - 6$. This can be factored as $(z + 1)(z - 3)$, hence there are two real roots: -1 and 3 , then the general solution is

$$u_n = c_1(-2)^n + c_23^n.$$

The initial conditions $u_0 = 0$ and $u_1 = 1$ implies that $c_1 = \frac{-1}{5}$ and $c_2 = \frac{1}{5}$. Thus the solution is

$$u_n = \frac{(3)^n}{5} - \frac{(-2)^n}{5}.$$

Theorem 1.2.3 [24] Let $c_1, c_2, \dots, c_k$ be real numbers. Suppose that the characteristic equation

$$z^k - c_1z^{k-1} - \dots - c_k = 0,$$

has r distinct roots $z_1, z_2, \dots, z_r$ with multiplicities $m_1, m_2, \dots, m_r$, respectively, so that $m_i \geq 1 \forall i = 1, 2, \dots, r$ and $\sum_{i=1}^r m_i = k$. Then a sequence $\{u_n\}$ is a solution of the recurrence relation

$$u_n = c_1u_{n-1} + c_2u_{n-2} + \dots + c_ku_{n-k}.$$

If and only if

$$\begin{aligned} u_n = & (\alpha_{1,0} + \alpha_{1,1}n + \dots + \alpha_{1,m_1-1}n^{m_1-1})z_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1}n + \dots + \alpha_{2,m_2-1}n^{m_2-1})z_2^n + \\ & \cdot \\ & \cdot \\ & \cdot \\ & + (\alpha_{r,0} + \alpha_{r,1}n + \dots + \alpha_{r,m_r-1}n^{m_r-1})z_r^n \end{aligned}$$

for $n = 0, 1, 2, \dots$, where α_{ij} are constants for $1 \leq i \leq r$ and $0 \leq j \leq m_i - 1$.

1.2.1 Recurrence relations of some numbers and polynomials

Definition 1.2.3 [3] Generalized Fibonacci sequence $(G_n)_{n \in \mathbb{N}}$ is defined by the following recurrence relation

$$\begin{cases} G_n = pG_{n-1} + qG_{n-2}, n \geq 2 \\ G_0 = \alpha, G_1 = \beta. \end{cases} \quad (1.3)$$

With $p, q \in \mathbb{R}_+$ and $\alpha, \beta \in \mathbb{C}$.

Lemma 1.2.1 [3] Let $z^2 - pz - q = 0$, the characteristic equation of the recurrence relation (1.3). Then

1. If the characteristic equation has two real solutions z_1 and z_2 , then the general solution for (1.3) is given by:

$$G_n = \frac{\lambda_1 z_1^n - \lambda_2 z_2^n}{z_1 - z_2},$$

with $\lambda_1 = \beta - \alpha z_2$ and $\lambda_2 = \beta - \alpha z_1$.

2. If the characteristic equation has only one real solution z , then the general solution for (1.3) is given by:

$$G_n = (c_1 + c_2 n)z^n,$$

with $c_1 = \alpha$ and $c_2 = \frac{\beta - \alpha z}{z}$.

Definition 1.2.4 [8] The k -Fibonacci numbers are defined by the following recurrence relation:

$$\begin{cases} F_{k,n} = kF_{k,n-1} + F_{k,n-2}, \forall n \geq 2 \\ F_{k,0} = 0, F_{k,1} = 1. \end{cases} \quad (1.4)$$

The first terms of the k -Fibonacci numbers are given by

n	0	1	2	3	4	5	6	7
$F_{k,n}$	0	1	k	$k^2 + 1$	$k^3 + 2k$	$k^4 + 3k^2 + 1$	$k^5 + 4k^3 + 3k$	$k^6 + 5k^4 + 6k^2 + 1$

The Binet formula is written:

$$F_{k,n} = \frac{1}{\sqrt{k^2 + 4}} \left(\left(\frac{k + \sqrt{k^2 + 4}}{2} \right)^n - \left(\frac{k - \sqrt{k^2 + 4}}{2} \right)^n \right). \quad (1.5)$$

Definition 1.2.5 [12] The k -Lucas numbers are defined by the following recurrence relation:

$$\begin{cases} L_{k,n} = kL_{k,n-1} + L_{k,n-2}, \forall n \geq 2 \\ L_{k,0} = 2, L_{k,1} = k. \end{cases} \quad (1.6)$$

The first terms of the k -Lucas numbers are given by

n	0	1	2	3	4	5	6
$L_{k,n}$	2	k	$k^2 + 2$	$k^3 + 4k$	$k^4 + 6k^2 + 4$	$k^5 + 8k^3 + 12k$	$k^6 + 10k^4 + 24k^2 + 8$

The Binet form is written:

$$L_{k,n} = \left(\frac{k + \sqrt{k^2 + 4}}{2} \right)^n + \left(\frac{-k - \sqrt{k^2 + 4}}{2} \right)^{-n}. \quad (1.7)$$

Definition 1.2.6 [9] The k -Pell numbers are defined by the following recurrence relation:

$$\begin{cases} P_{k,n} = 2P_{k,n-1} + kP_{k,n-2}, \forall n \geq 2 \\ P_{k,0} = 0, P_{k,1} = 1. \end{cases} \quad (1.8)$$

The first terms of the k -Pell numbers are given by

n	0	1	2	3	4	5	6	7
$P_{k,n}$	0	1	2	$k + 4$	$4k + 8$	$k^2 + 12k + 16$	$6k^2 + 32k + 32$	$k^3 + 24k^2 + 80k + 64$

The Binet formula is written:

$$P_{k,n} = \frac{1}{2\sqrt{1+k}} \left((1 + \sqrt{1+k})^n - (1 - \sqrt{1+k})^n \right). \quad (1.9)$$

Definition 1.2.7 [10] The k -Pell-Lucas numbers are defined by the following recurrence rela-

tion:

$$\begin{cases} Q_{k,n} = 2Q_{k,n-1} + kQ_{k,n-2}, \forall n \geq 2 \\ Q_{k,0} = 2, Q_{k,1} = 2. \end{cases} \quad (1.10)$$

The first terms of the k -Pell-Lucas numbers are given by

n	0	1	2	3	4	5	6
$Q_{k,n}$	2	2	$2k + 4$	$6k + 8$	$2k^2 + 16k + 16$	$10k^2 + 40k + 32$	$2k^3 + 36k^2 + 96k + 64$

The Binet formula is written:

$$Q_{k,n} = \left(1 + \sqrt{1+k}\right)^n + \left(1 - \sqrt{1+k}\right)^n. \quad (1.11)$$

Definition 1.2.8 [19] The k -Jacobsthal numbers are defined by the following recurrence relation:

$$\begin{cases} J_{k,n+1} = kJ_{k,n} + 2J_{k,n-1}, \forall n \geq 1 \\ J_{k,0} = 0, J_{k,1} = 1. \end{cases} \quad (1.12)$$

The first terms of the k -Jacobsthal numbers are given by

n	0	1	2	3	4	5	6	7
$J_{k,n}$	0	1	k	$k^2 + 2$	$k^3 + 4k$	$k^4 + 6k^2 + 4$	$k^5 + 8k^3 + 12k$	$k^6 + 10k^4 + 24k^2 + 8$

The Binet formula is written:

$$J_{k,n} = \frac{1}{\sqrt{k^2 + 8}} \left(\left(\frac{k + \sqrt{k^2 + 8}}{2} \right)^n - \left(\frac{k - \sqrt{k^2 + 8}}{2} \right)^n \right). \quad (1.13)$$

Definition 1.2.9 [11] The k -Jacobsthal-Lucas numbers are defined by the following recurrence relation:

$$\begin{cases} j_{k,n+1} = kj_{k,n} + 2j_{k,n-1}, \forall n \geq 1 \\ j_{k,0} = 2, j_{k,1} = k. \end{cases} \quad (1.14)$$

The first terms of the k -Jacobsthal-Lucas numbers are given by

n	0	1	2	3	4	5	6
$j_{k,n}$	2	k	$k^2 + 4$	$k^3 + 6k$	$k^4 + 8k^2 + 8$	$k^5 + 10k^3 + 20k$	$k^6 + 12k^4 + 36k^2 + 16$

The Binet formula is written:

$$j_{k,n} = \left(\frac{k + \sqrt{k^2 + 8}}{2} \right)^n + \left(\frac{k - \sqrt{k^2 + 8}}{2} \right)^n. \quad (1.15)$$

- By taking $k = 1$ in the identity (1.4)-(1.15), we obtain the following table.

Recurrence relations	Binet Formula
$F_n = F_{n-1} + F_{n-2}, \forall n \geq 2$ $F_0 = 0, F_1 = 1.$	$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right)$
$L_n = L_{n-1} + L_{n-2}, \forall n \geq 2$ $L_0 = 2, L_1 = 1$	$L_n = \left(\frac{1 + \sqrt{5}}{2} \right)^n + \left(\frac{-1 - \sqrt{5}}{2} \right)^n$
$P_n = 2P_{n-1} + P_{n-2}, \forall n \geq 2$ $P_0 = 0, P_1 = 1$	$P_n = \frac{1}{2\sqrt{2}} \left((1 + \sqrt{2})^n - (1 - \sqrt{2})^n \right)$
$Q_n = 2Q_{n-1} + Q_{n-2}, \forall n \geq 2$ $Q_0 = 2, Q_1 = 2.$	$Q_n = (1 + \sqrt{2})^n + (1 - \sqrt{2})^n$
$J_{n+1} = J_n + 2J_{n-1}, \forall n \geq 1$ $J_0 = 0, J_1 = 1.$	$J_n = \frac{1}{3} (2^n - (-1)^n)$
$j_{n+1} = j_n + 2j_{n-1}, \forall n \geq 1$ $j_0 = 2, j_1 = 1.$	$j_n = 2^n + (-1)^n.$

Tabel 1: Binet Formula of Recurrence relations

Definition 1.2.10 [25] For $n \in \mathbb{N}$, the bivariate Fibonacci polynomials are defined by

$$\begin{cases} F_n(x, y) = xF_{n-1}(x, y) + yF_{n-2}(x, y), n \geq 2 \\ F_0(x, y) = 0, F_1(x, y) = 1. \end{cases} \quad (1.16)$$

Definition 1.2.11 [25] For $n \in \mathbb{N}$, the bivariate Lucas polynomials are defined by

$$\begin{cases} L_n(x, y) = xL_{n-1}(x, y) + yL_{n-2}(x, y), n \geq 2 \\ L_0(x, y) = 2, L_1(x, y) = x. \end{cases} \quad (1.17)$$

Definition 1.2.12 [25] For $n \in \mathbb{N}$, the bivariate Pell polynomials are defined by

$$\begin{cases} P_n(x, y) = 2xyP_{n-1}(x, y) + yP_{n-2}(x, y), n \geq 2 \\ P_0(x, y) = 0, P_1(x, y) = 1. \end{cases} \quad (1.18)$$

Definition 1.2.13 [25] For $n \in \mathbb{N}$, the bivariate Pell Lucas polynomials are defined by

$$\begin{cases} Q_n(x, y) = 2xyQ_{n-1}(x, y) + yQ_{n-2}(x, y), n \geq 2 \\ Q_0(x, y) = 2, Q_1(x, y) = 2xy. \end{cases} \quad (1.19)$$

Definition 1.2.14 [25] For $n \in \mathbb{N}$, the bivariate Jacobsthal polynomials are defined by

$$\begin{cases} J_n(x, y) = xyJ_{n-1}(x, y) + 2yJ_{n-2}(x, y), n \geq 2 \\ J_0(x, y) = 0, J_1(x, y) = 1. \end{cases} \quad (1.20)$$

Definition 1.2.15 [25] For $n \in \mathbb{N}$, the bivariate Jacobsthal Lucas polynomials are defined by

$$\begin{cases} j_n(x, y) = xyj_{n-1}(x, y) + 2yj_{n-2}(x, y), n \geq 2 \\ j_0(x, y) = 2, j_1(x, y) = xy. \end{cases} \quad (1.21)$$

the following table gives the Binet formula:[25, 7, 29]

Polynomials	Values of α, β	Binet formula
$F_n(x, y)$	$\alpha(x, y) = \frac{x + \sqrt{x^2 + 4y}}{2}, \beta(x, y) = \frac{x - \sqrt{x^2 + 4y}}{2}$	$\frac{\alpha^n - \beta^n}{\alpha - \beta}$
$L_n(x, y)$	$\alpha(x, y) = \frac{x + \sqrt{x^2 + 4y}}{2}, \beta(x, y) = \frac{x - \sqrt{x^2 + 4y}}{2}$	$\alpha^n + \beta^n$
$P_n(x, y)$	$\alpha(x, y) = xy + \sqrt{x^2 y^2 + y}, \beta(x, y) = xy - \sqrt{x^2 y^2 + y}$	$\frac{\alpha^n - \beta^n}{\alpha - \beta}$
$Q_n(x, y)$	$\alpha(x, y) = xy + \sqrt{x^2 y^2 + y}, \beta(x, y) = xy - \sqrt{x^2 y^2 + y}$	$\alpha^n + \beta^n$
$J_n(x, y)$	$\alpha(x, y) = \frac{xy + \sqrt{x^2 y^2 + 8y}}{2}, \beta(x, y) = \frac{xy - \sqrt{x^2 y^2 + 8y}}{2}$	$\frac{\alpha^n - \beta^n}{\alpha - \beta}$
$j_n(x, y)$	$\alpha(x, y) = \frac{xy + \sqrt{x^2 y^2 + 8y}}{2}, \beta(x, y) = \frac{xy - \sqrt{x^2 y^2 + 8y}}{2}$	$\alpha^n + \beta^n$

Tabel 2: Binet formula of some polynomials.

1.3 Generating functions

1.3.1 Formal series

Let $\mathbb{K}$ be a commutative field ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).

Definition 1.3.1 [3] The elements of the set $\mathbb{K}[[z]] = \left\{ \sum_{n=0}^{\infty} a_n z^n, a_n \in \mathbb{K} \right\}$ are called formal series with coefficients in $\mathbb{K}$. for $n \in \mathbb{N}$, z^n called the monomial of degree n and a_n it's coefficient.

Definition 1.3.2 [3] Let $\alpha(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\beta(z) = \sum_{n=0}^{\infty} b_n z^n$ be two formal series. Then the sun of $\alpha(z)$ and $\beta(z)$ is given by

$$\alpha(z) + \beta(z) = \sum_{n=0}^{\infty} (a_n + b_n) z^n.$$

Definition 1.3.3 [3] Let $\alpha(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\beta(z) = \sum_{n=0}^{\infty} b_n z^n$ be two formal series. Then the

product of $\alpha(z)$ and $\beta(z)$ is given by

$$\alpha(z)\beta(z) = \sum_{n=0}^{\infty} c_n z^n,$$

with

$$c_n = \sum_{k=0}^{\infty} a_k b_{n-k}.$$

Definition 1.3.4 [3] Two formal series $\alpha(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\beta(z) = \sum_{n=0}^{\infty} b_n z^n$ are equal if and only if for all $n \neq 0$, $a_n = b_n$.

Definition 1.3.5 [3] We say that the series $\sum_{n=0}^{\infty} a_n z^n$ is the inverse of the series $\sum_{n=0}^{\infty} b_n z^n$ if:

$$\left(\sum_{n=0}^{\infty} a_n z^n \right) \left(\sum_{n=0}^{\infty} b_n z^n \right) = 1.$$

Example 1.3.1 The series $\alpha = \sum_{n=0}^{\infty} a_n z^n$ is invertible and its inverse is $(1 - z)$, indeed

$$\begin{aligned} (1 - z) \left(\sum_{n=0}^{\infty} z^n \right) &= \sum_{n=0}^{\infty} z^n - z \sum_{n=0}^{\infty} z^n \\ &= \sum_{n=0}^{\infty} z^n - \sum_{n=0}^{\infty} z^{n+1} \\ &= 1 + \sum_{n=0}^{\infty} z^n - \sum_{n=0}^{\infty} z^n \\ &= 1. \end{aligned}$$

Proposition 1.3.1 [3] A formal series $\sum_{n=0}^{\infty} a_n z^n$ is invertible if and only if $a_n \neq 0$.

Proof. We need to determine whether or not there exists a formal series $\beta(z) = \sum_{n=0}^{\infty} b_n z^n$, in $\mathbb{K}[[z]]$ such that $\alpha(z)\beta(z) = 1$. Expanding the product, we have

$$\begin{aligned}\alpha(z)\beta(z) &= \left(\sum_{n=0}^{\infty} a_n z^n\right) \left(\sum_{n=0}^{\infty} b_n z^n\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_k b_{n-k}\right) z^n.\end{aligned}$$

Comparing the coefficient of z^n on both sides of $\alpha(z)\beta(z) = 1$, we see that $\beta(z)$ satisfies the equation if and only if $a_0 b_0 = 1$ and $\sum_{k=0}^n a_k b_{n-k} = 0$ for all $n \geq 1$. If a_0 is not invertible in $\mathbb{K}$ then the equation $\alpha(z)\beta(z) = 1$ can not be solved for b_0 , so that $\beta(z)$ does not exist and $\alpha(z)$ is not invertible in $\mathbb{K}[[z]]$. If a_0 is invertible in $\mathbb{K}$, then $b_0 = a_0^{-1}$ exists. Each of the remaining equations (for $n \geq 1$) can be rewritten as $a_0 b_n = -\sum_{k=0}^n a_k b_{n-k}$, or upon multiplying by b_0 , $b_n = -b_0 \sum_{k=0}^n a_k b_{n-k}$. These equations can be solved by induction on $k \geq 1$, yielding a solution for $\beta(z)$ which gives the multiplicative inverse of $\alpha(z)$. Therefore, $\beta(z)$ is invertible in $\mathbb{K}[[z]]$. ■

Proposition 1.3.2 [3] *If $\alpha(z)$ and $\beta(z)$ are two nonzero formal series then $\alpha(z)\beta(z)$ is also nonzero.*

1.3.2 Ordinary generating functions (OGF)

Definition 1.3.6 [3] *The ordinary generating function (OGF) of the sequence $(a_n)_{n \in \mathbb{N}} = (a_0, a_1, a_2, \dots)$, is defined by:*

$$G(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Example 1.3.2 *The generating functions for the sequences $(a_n)_{n \in \mathbb{N}}$ with $a_n = 5$, $a_n = n + 1$ and $a_n = 2^n$ are $\sum_{n=0}^{\infty} 5z^n$, $\sum_{n=0}^{\infty} (n + 1)z^n$ and $\sum_{n=0}^{\infty} 2^n z^n$, respectively.*

Theorem 1.3.1 [13] *Let $A(z)$ the ordinary generating function of $(a_n)_{n \in \mathbb{N}}$ and $B(z)$ the ordinary generating function of $(b_n)_{n \in \mathbb{N}}$, so*

1. $A(z)+B(z)$ is the ordinary generating function of $(a_n + b_n)_{n \geq 0}$.
2. $zA(z)$ is the ordinary generating function of $(0, a_0, a_1, a_2, \dots, a_{n-1})$.
3. $A(z)B(z)$ is the ordinary generating function of $(a_0, a_0b_1 + a_1b_0 + a_1b_1 + a_2b_0, \dots)$
4. $(1-z)A(z)$ is the ordinary generating function of $(a_0, a_1 - a_0, a_2 - a_1, \dots, a_n - a_{n-1}, \dots)$.

Theorem 1.3.2 [3] Let the sequence $(G_n)_{n \in \mathbb{N}}$ defined by the following recurrence relation

$$\begin{cases} G_n = pG_{n-1} + qG_{n-2} \\ G_0 = \alpha, G_1 = \beta \end{cases} \quad (1.22)$$

with $p, q \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{C}$. So the generating function of $(G_n)_{n \geq 0}$ is given by

$$G(z) = \frac{\alpha + (\beta - p\alpha)z}{1 - pz - qz^2}.$$

Proof. We have:

$$\begin{aligned} G(z) = \sum_{n=0}^{\infty} G_n z^n &= G_0 + G_1 z + \sum_{n=2}^{\infty} G_n z^n \\ &= \alpha + \beta z + \sum_{n=2}^{\infty} (pG_{n-1} + qG_{n-2}) z^n \\ &= \alpha + \beta z + pz \sum_{n=2}^{\infty} G_{n-1} z^{n-1} + qz^2 \sum_{n=2}^{\infty} G_{n-2} z^{n-2} \\ &= \alpha + \beta z + pz \sum_{n=1}^{\infty} G_n z^n + qz^2 \sum_{n=0}^{\infty} G_n z^n \\ &= \alpha + \beta z + pz \left(\sum_{n=0}^{\infty} G_n z^n - \alpha \right) + qz^2 \sum_{n=0}^{\infty} G_n z^n \\ &= \alpha + (\beta - p\alpha)z + pzG(z) + qz^2G(z). \end{aligned}$$

So,

$$G(z) = \frac{\alpha + (\beta - p\alpha)z}{1 - pz - qz^2}.$$

This completes the proof. ■

From the previous theorems we deduce the following table [3]:

values of p, q, α, β	coefficient of z^n	generating function
$p = k, \beta = 1, q = 1, \alpha = 0$	$F_{k,n}$	$\frac{1}{1-kz-z^2}$
$p = \beta = k, q = 1, \alpha = 2$	$L_{k,n}$	$\frac{2-kz}{1-kz-z^2}$
$\alpha = 0, p = 2, q = k, \beta = 1$	$P_{k,n}$	$\frac{z}{1-2z-kz^2}$
$\alpha = \beta = p = 2, q = k$	$Q_{k,n}$	$\frac{2-2z}{1-2z-kz^2}$
$p = k, q = 2, \alpha = 0, \beta = 1$	$J_{k,n}$	$\frac{z}{1-kz-2z^2}$
$p = \beta = k, \alpha = q = 2$	$j_{k,n}$	$\frac{2-kz}{1-kz-2z^2}$

Table 1: Generating function of some k numbers .

For $k = 1$ in the table 1 we obtain the following table [3].

values of p, q, α, β	coefficient of z^n	generating function
$p = 1, \beta = 1, q = 1, \alpha = 0$	F_n	$\frac{1}{1-z-z^2}$
$p = \beta = 1, q = 1, \alpha = 2$	L_n	$\frac{2-z}{1-z-z^2}$
$\alpha = 0, p = 2, q = 1, \beta = 1$	P_n	$\frac{z}{1-2z-z^2}$
$\alpha = \beta = p = 2, q = 1$	Q_n	$\frac{2-2z}{1-2z-z^2}$
$p = 1, q = 2, \alpha = 0, \beta = 1$	J_n	$\frac{z}{1-z-2z^2}$
$p = \beta = 1, \alpha = q = 2$	j_n	$\frac{2-z}{1-z-2z^2}$

Table 2:Generating function of some numbers .

Theorem 1.3.3 [25] *The generating function of the bivariate Fibonacci polynomials is given by*

$$\sum_{n=1}^{\infty} F_n(x, y)z^n = \frac{z}{1 - xz - yz^2}. \quad (1.23)$$

Proof. We have:

$$\begin{aligned}
 G(z) = \sum_{n=0}^{\infty} F_n(x, y)z^n &= F_0(x, y) + F_1(x, y)z + \sum_{n=2}^{\infty} F_n(x, y)z^n \\
 &= z + \sum_{n=2}^{\infty} (xF_{n-1}(x, y) + yF_{n-2}(x, y))z^n \\
 &= z + xz \sum_{n=2}^{\infty} F_{n-1}(x, y)z^{n-1} + yz^2 \sum_{n=2}^{\infty} F_{n-2}(x, y)z^{n-2} \\
 &= z + xz \sum_{n=1}^{\infty} F_n(x, y)z^n + yz^2 \sum_{n=0}^{\infty} F_n(x, y)z^n \\
 &= z + xz \left(\sum_{n=0}^{\infty} F_n(x, y)z^n - 0 \right) + yz^2 \sum_{n=0}^{\infty} F_n(x, y)z^n \\
 &= z + xzG(z) + yz^2G(z).
 \end{aligned}$$

Then,

$$G(z) = \frac{z}{1 - xz - yz^2}.$$

This completes the proof. ■

Theorem 1.3.4 [25] *The generating function of the bivariate Lucas polynomials is given by*

$$\sum_{n=2}^{\infty} L_n(x, y)z^n = \frac{2 - xz}{1 - xz - yz^2}.$$

Proof. By definition, we have

$$\begin{aligned}
 G(z) &= \sum_{n=2}^{\infty} L_n(x, y)z^n \\
 &= L_0(x, y) + L_1(x, y)z + \sum_{n=2}^{\infty} L_n(x, y)z^n \\
 &= 2 + xz + \sum_{n=2}^{\infty} (xL_{n-1}(x, y) + yL_{n-2}(x, y))z^n \\
 &= 2 + xz + xz \sum_{n=2}^{\infty} L_{n-1}(x, y)z^{n-1} + yz^2 \sum_{n=2}^{\infty} L_{n-2}(x, y)z^{n-2}
 \end{aligned}$$

$$\begin{aligned}
&= 2 + xz + xz \sum_{n=1}^{\infty} L_n(x, y)z^n + yz^2 \sum_{n=0}^{\infty} L_n(x, y)z^n \\
&= 2 + xz + xz \left(\sum_{n=0}^{\infty} L_n(x, y)z^n - 2 \right) + yz^2 \sum_{n=0}^{\infty} L_n(x, y)z^n \\
&= 2 - xz + xzG(z) + yz^2G(z)
\end{aligned}$$

Then,

$$G(z)(1 - xz - yz^2) = 2 - xz.$$

So,

$$G(z) = \frac{2 - xz}{1 - xz - yz^2}.$$

This completes the proof. ■

Theorem 1.3.5 [25] *The generating function of the bivariate Pell polynomials is given by*

$$\sum_{n=2}^{\infty} P_n(x, y)z^n = \frac{z}{1 - 2xyz - yz^2}$$

Proof. We have

$$\begin{aligned}
G(z) &= \sum_{n=0}^{\infty} P_n(x, y)z^n \\
&= P_0(x, y) + P_1(x, y)z + \sum_{n=2}^{\infty} P_n(x, y)z^n \\
&= z + \sum_{n=2}^{\infty} (2xyP_{n-1}(x, y) + yP_{n-2}(x, y))z^n \\
&= z + 2xyz \sum_{n=2}^{\infty} P_{n-1}(x, y)z^{n-1} + yz^2 \sum_{n=2}^{\infty} P_{n-2}(x, y)z^{n-2} \\
&= z + 2xyz \sum_{n=1}^{\infty} P_n(x, y)z^n + yz^2 \sum_{n=0}^{\infty} P_n(x, y)z^n \\
&= z + 2xyz \left(\sum_{n=0}^{\infty} P_n(x, y)z^n - 0 \right) + yz^2 \sum_{n=0}^{\infty} P_n(x, y)z^n \\
&= z + 2xyzG(z) + yz^2G(z).
\end{aligned}$$

Then

$$G(z)(1 - 2xyz - yz^2) = z.$$

So

$$G(z) = \frac{z}{1 - 2xyz - yz^2}.$$

This completes the proof. ■

Theorem 1.3.6 [25] *The generating function of the bivariate Pell-Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} Q_n(x, y)z^n = \frac{2 - 2xyz}{1 - 2xyz - yz^2}$$

Proof. We have

$$\begin{aligned} G(z) &= \sum_{n=0}^{\infty} Q_n(x, y)z^n \\ &= Q_0(x, y) + Q_1(x, y)z + \sum_{n=2}^{\infty} Q_n(x, y)z^n \\ &= 2 + 2xyz + \sum_{n=2}^{\infty} (2xyQ_{n-1}(x, y) + yQ_{n-2}(x, y))z^n \\ &= 2 + 2xyz + \sum_{n=2}^{\infty} Q_{n-1}(x, y)z^{n-1} + yz^2 \sum_{n=2}^{\infty} Q_{n-2}(x, y)z^{n-2} \\ &= 2 + 2xyz + \sum_{n=1}^{\infty} Q_n(x, y)z^n + yz^2 \sum_{n=0}^{\infty} Q_n(x, y)z^n \\ &= 2 + xyz + 2xyz \left(\sum_{n=0}^{\infty} Q_n(x, y)z^n - 2 \right) + yz^2 \sum_{n=0}^{\infty} Q_n(x, y)z^n \\ &= 2 - 2xyz + 2xyzG(z) + yz^2G(z). \end{aligned}$$

Then

$$G(z)(1 - 2xyz - yz^2) = 2 - 2xyz.$$

So

$$G(z) = \frac{2 - 2xyz}{1 - 2xyz - yz^2}.$$

This completes the proof. ■

Theorem 1.3.7 [25] *The generating function of the bivariate Jacobsthal polynomials is given by*

$$\sum_{n=0}^{\infty} J_n(x, y)z^n = \frac{z}{1 - xyz - 2yz^2}$$

Proof. We have

$$\begin{aligned} G(z) &= \sum_{n=0}^{\infty} J_n(x, y)z^n \\ &= J_0(x, y) + J_1(x, y)z + \sum_{n=2}^{\infty} J_n(x, y)z^n \\ &= z + \sum_{n=2}^{\infty} (xyJ_{n-1}(x, y) + 2yJ_{n-2}(x, y))z^n \\ &= z + xyz \sum_{n=2}^{\infty} J_{n-1}(x, y)z^{n-1} + 2yz^2 \sum_{n=2}^{\infty} J_{n-2}(x, y)z^{n-2} \\ &= z + \sum_{n=1}^{\infty} J_n(x, y)z^n + 2yz^2 \sum_{n=0}^{\infty} J_n(x, y)z^n \\ &= z + xyz \left(\sum_{n=0}^{\infty} J_n(x, y)z^n - 0 \right) + 2yz^2 \sum_{n=0}^{\infty} J_n(x, y)z^n \\ &= z + xyzG(z) + 2yz^2G(z). \end{aligned}$$

Then

$$G(z)(1 - xyz - 2yz^2) = z.$$

So

$$G(z) = \frac{z}{1 - xyz - 2yz^2}.$$

This completes the proof. ■

Theorem 1.3.8 [25] *The generating function of the bivariate Jacobsthal-Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} j_n(x, y)z^n = \frac{2 - xyz}{1 - xyz - 2yz^2}$$

Proof. We have

$$\begin{aligned}
 G(z) &= \sum_{n=0}^{\infty} j_n(x, y) z^n \\
 &= j_0(x, y) + j_1(x, y)z + \sum_{n=2}^{\infty} j_n(x, y) z^n \\
 &= 2 + xyz + \sum_{n=2}^{\infty} (xy j_{n-1}(x, y) + 2y j_{n-2}(x, y)) z^n \\
 &= 2 + xyz + xyz \sum_{n=2}^{\infty} j_{n-1}(x, y) z^{n-1} + 2yz^2 \sum_{n=2}^{\infty} j_{n-2}(x, y) z^{n-2} \\
 &= 2 + xyz + xyz \sum_{n=1}^{\infty} j_n(x, y) z^n + 2yz^2 \sum_{n=0}^{\infty} j_n(x, y) z^n \\
 &= 2 + xyz + xyz \left(\sum_{n=0}^{\infty} j_n(x, y) z^n - 2 \right) + 2yz^2 \sum_{n=0}^{\infty} j_n(x, y) z^n \\
 &= 2 - xyz + xyz G(z) + 2yz^2 G(z).
 \end{aligned}$$

Then

$$G(z) (1 - xyz - 2yz^2) = 2 - xyz.$$

So

$$G(z) = \frac{2 - xyz}{1 - xyz - 2yz^2}.$$

This completes the proof ■

CHAPTER 2

ELEMENTARY AND COMPLETE SYMMETRIC FUNCTIONS

Symmetric functions are important in many areas of mathematics, with a wide range of applications in algebra, combinatorics, number theory and physics. For example, in combinatorics they are used to solve problems involving permutations and partitions. In physics, they have applications in the study of quantum field theory. In algebra, they play a key role in the study of polynomials and their properties. In this chapter, we set some definitions and properties of elementary and complete symmetric function, for more details you can refer to [3, 1, 6].

2.1 Symmetric functions

Definition 2.1.1 [26] *A function $f(x_1, x_2, \dots, x_n)$ in n variables is symmetric if for all permutations of the index set $(1, 2, \dots, n)$ the following equality holds:*

$$f(x_1, x_2, \dots, x_n) = f(x_{s(1)}, x_{s(2)}, \dots, x_{s(n)}).$$

Which means, a function of several variables is symmetric if its values does not change when we swap variables.

2.1.1 Elementary symmetric functions

Definition 2.1.2 [3] Let k and n be tow positive integer and $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ are the set of given variables. Then, the elementary symmetric function $e_k(\lambda_1, \lambda_2, \dots, \lambda_n)$ is defined by

$$e_k^{(n)} = e_k(\lambda_1, \lambda_2, \dots, \lambda_n) = \sum_{i_1+i_2+\dots+i_n=k} \lambda_1^{i_1} \lambda_2^{i_2} \dots \lambda_n^{i_n}, \quad 0 \leq k \leq n, \quad (2.1)$$

with $i_1, i_2, \dots, i_n = 0 \vee 1$.

Remark 2.1.1 [26] Set $e_0(\lambda_1, \lambda_2, \dots, \lambda_n) = 1$, by usual convention, for $k < 0$, we set $e_k(\lambda_1, \lambda_2, \dots, \lambda_n) = 0$.

Example 2.1.1 For an equation of degree 2, we have :

$$e_0(\lambda_1, \lambda_2) = 1, \quad e_1(\lambda_1, \lambda_2) = \lambda_1 + \lambda_2, \quad e_2(\lambda_1, \lambda_2) = \lambda_1 \lambda_2.$$

Example 2.1.2 For an equation of degree 3, we have:

$$e_0(\lambda_1, \lambda_2, \lambda_3) = 1, \quad e_1(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 + \lambda_2 + \lambda_3, \quad e_2(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3, \\ e_3(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 \lambda_2 \lambda_3.$$

Proposition 2.1.1 [14] The generating function of the elementary symmetric functions is given by:

$$E(z) = \sum_{k \geq 0} e_k z^k = \prod_{i=1}^n (1 + \lambda_i z).$$

Proof. We have

$$e_k^{(n)} = e_k(\lambda_1, \lambda_2, \dots, \lambda_n) = \sum_{i_1+i_2+\dots+i_n=k} \lambda_1^{i_1} \lambda_2^{i_2} \dots \lambda_n^{i_n} \text{ with } e_k^{(n)} = 0 \text{ if } k > n.$$

For $n = 2$, we have

$$\begin{aligned} \prod_{i=1}^2 (1 + \lambda_i z) &= (1 + \lambda_1 z)(1 + \lambda_2 z) \\ &= 1 + (\lambda_1 + \lambda_2)z + \lambda_1 \lambda_2 z^2 \\ &= e_0 + e_1 z + e_2 z^2 \\ &= \sum_{k=0}^2 e_k z^k. \end{aligned}$$

So the assertion is true for $n = 2$. Assume the proposition is true for n , i.e that

$$\sum_{n \geq 0} e_k z^k = \prod_{i=1}^n (1 + \lambda_i z)$$

and we want prove that the proposition is true for $n + 1$, i.e that

$$\sum_{k=0}^{n+1} e_k z^k = \prod_{i=1}^{n+1} (1 + \lambda_i z)$$

we have,

$$\begin{aligned} \prod_{i=1}^{n+1} (1 + \lambda_i z) &= \prod_{i=1}^n (1 + \lambda_i z)(1 + \lambda_{n+1} z) \\ &= \left(\sum_{k=0}^n e_k z^k \right) (1 + \lambda_{n+1} z) \\ &= \sum_{k=0}^n e_k z^k + \lambda_{n+1} \sum_{k=0}^n e_k z^{k+1} \\ &= \sum_{k=0}^n e_k z^k + \lambda_{n+1} \sum_{k=1}^{n+1} e_{k-1} z^k \\ &= \sum_{k=0}^n e_k z^k + \lambda_{n+1} \sum_{k=0}^n e_{k-1} z^k \\ &= \sum_{k \geq 0} (e_k^{(n)} + \lambda_{n+1} e_{k-1}^{(n)}) z^k \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k \geq 0} e_k^{(n+1)} z^k \\
 &= \sum_{k=0}^{n+1} e_k z^k.
 \end{aligned}$$

Thus the proposition is true for all $n \geq 0$ ■

2.1.2 Complete symmetric functions

Definition 2.1.3 [3] Let k and n be two positive integer and $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ are the set of given variables. Then, the complete symmetric functions $h_k(\lambda_1, \lambda_2, \dots, \lambda_n)$ is defined by

$$h_k^{(n)} = h_k(\lambda_1, \lambda_2, \dots, \lambda_n) = \sum_{i_1+i_2+\dots+i_n=k} \lambda_1^{i_1} \lambda_2^{i_2} \dots \lambda_n^{i_n}, \quad (2.2)$$

with $i_1, i_2, \dots, i_n \geq 0$.

Remark 2.1.2 [26] Set $h_0(\lambda_1, \lambda_2, \dots, \lambda_n) = 1$, by usual convention, for $k < 0$, we set $h_k(\lambda_1, \lambda_2, \dots, \lambda_n) = 0$.

Example 2.1.3 For an equation of degree 2 we have

$$h_0(\lambda_1, \lambda_2) = 1, \quad h_1(\lambda_1, \lambda_2) = \lambda_1 + \lambda_2, \quad h_2(\lambda_1, \lambda_2) = \lambda_1^2 + \lambda_2^2 + \lambda_1 \lambda_2, \quad h_3(\lambda_1, \lambda_2) = \lambda_1^3 + \lambda_2^3 + \lambda_1^2 \lambda_2 + \lambda_1 \lambda_2^2, \dots$$

Example 2.1.4 For an equation of degree 3, we have

$$\begin{aligned}
 h_0(\lambda_1, \lambda_2, \lambda_3) &= 1, \quad h_1(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 + \lambda_2 + \lambda_3, \quad h_2(\lambda_1, \lambda_2, \lambda_3) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3, \\
 h_3(\lambda_1, \lambda_2, \lambda_3) &= \lambda_1^3 + \lambda_2^3 + \lambda_3^3 + \lambda_1^2 \lambda_2 + \lambda_1^2 \lambda_3 + \lambda_2^2 \lambda_1 + \lambda_2^2 \lambda_3 + \lambda_3^2 \lambda_1 + \lambda_3^2 \lambda_2 + \lambda_1 \lambda_2 \lambda_3, \dots
 \end{aligned}$$

Proposition 2.1.2 [14] The generating function of the complete symmetric function is given by:

$$H(z) = \sum_{k \geq 0} h_k z^k = \frac{1}{\prod_{i=1}^n (1 - \lambda_i z)}.$$

Proof. We have

$$h_k^n = h_k(\lambda_1, \lambda_2, \dots, \lambda_n) = \sum_{i_1+i_2+\dots+i_n=k} \lambda_1^{i_1} \lambda_2^{i_2} \dots \lambda_n^{i_n},$$

for $n=2$, we have:

$$\begin{aligned} \sum_{k \geq 0} h_k^2 z^k &= h_0^2 + h_1^2 z + h_2^2 z^2 + \dots \\ &= 1 + (\lambda_1 + \lambda_2)z + (\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2)z^2 + \dots \\ &= (1 + \lambda_1 z + \lambda_1^2 z^2 + \dots)(1 + \lambda_2 z + \lambda_2^2 z^2 + \dots) \\ &= \left(\sum_{k \geq 0} (\lambda_1 z)^k \right) \left(\sum_{k \geq 0} (\lambda_2 z)^k \right) \\ &= \frac{1}{(1 - \lambda_1 z)(1 - \lambda_2 z)} \\ &= \frac{1}{\prod_{i=1}^2 (1 - \lambda_i z)}. \end{aligned}$$

So the assertion is true for $n = 2$. Assume the proposition is true for n , i.e that

$$\sum_{k \geq 0} h_k^{(n)} z^k = \frac{1}{\prod_{i=1}^n (1 - \lambda_i z)},$$

and we want prove that the proposition is true for $n + 1$, i.e that

$$\sum_{k \geq 0} h_k^{n+1} z^k = \frac{1}{\prod_{i=1}^{n+1} (1 - \lambda_i z)},$$

we have

$$h_k^{(n+1)} = \lambda_{n+1} h_{k-1}^{(n+1)} + h_k^{(n)}.$$

Thus

$$\sum_{k \geq 0} h_k^{n+1} z^k = \sum_{k \geq 0} (\lambda_{n+1} h_{k-1}^{(n+1)} + h_k^{(n)}) z^k$$

$$\begin{aligned}
&= \lambda_{n+1} \sum_{k \geq 0} h_{k-1}^{(n+1)} z^k + \sum_{k \geq 0} h_k^n z^k \\
&= \lambda_{n+1} \sum_{k=1}^{\infty} h_{k-1}^{(n+1)} z^k + \sum_{k \geq 0} h_k^n z^k \\
&= \lambda_{n+1} z \sum_{k \geq 0} h_k^{(n+1)} z^k + \sum_{k \geq 0} h_k^n z^k,
\end{aligned}$$

which gives

$$\sum_{k \geq 0} h_k^{(n+1)} z^k - \lambda_{n+1} z \sum_{k \geq 0} h_k^{(n+1)} z^k = \sum_{k \geq 0} h_k^{(n)} z^k.$$

Thus

$$\sum_{k \geq 0} h_k^{(n+1)} z^k (1 - \lambda_{n+1} z) = \sum_{k \geq 0} h_k^{(n)} z^k = \frac{1}{\prod_{i=1}^n (1 - \lambda_i z)},$$

then

$$\begin{aligned}
\sum_{k \geq 0} h_k^{(n+1)} z^k &= \frac{(1 - \lambda_{n+1} z)^{-1}}{\prod_{i=1}^n (1 - \lambda_i z)} \\
&= \frac{1}{\prod_{i=1}^{n+1} (1 - \lambda_i z)}.
\end{aligned}$$

Thus the proposition is true for all $n \geq 0$ ■

Proposition 2.1.3 For all $m \in \mathbb{N}$, we have

$$\sum_{i=0}^m (-1)^i e_i h_{m-1} = \delta_{m,0}$$

Such that δ is the Kronecker symbol

$$\delta_{i,j} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \quad (2.3)$$

Proof. Let $f(z)$ and $g(z)$ be two functions such that

$$f(z) = \prod_{i=1}^n (1 - \lambda_i z) \quad \text{and} \quad g(z) = \frac{1}{\prod_{i=1}^n (1 - \lambda_i z)}$$

According to the above, we obtain

$$f(z) = \sum_{i=0}^{\infty} e_i (-z)^i = \sum_{i=0}^{\infty} (-1)^i e_i z^i \quad \text{and} \quad g(z) = \sum_{j=0}^{\infty} h_j z^j$$

These function being inverse:

$$\left(\sum_{i=0}^{\infty} (-1)^i e_i z^i \right) \left(\sum_{j=0}^{\infty} h_j z^j \right) = 1$$

thus,

$$\sum_{m=0}^{\infty} \left(\sum_{i=0}^m (-1)^i e_i h_{m-i} \right) z^m = 1.$$

So all the coefficients of the series are zero, except the first so:

$$\sum_{i=0}^{\infty} (-1)^i e_i h_{m-i} = \delta_{m,0}$$

This completes the proof. ■

Proposition 2.1.4 [3] *For all $n \geq 0$, we have*

$$1. \ H(z).E(-z) = 1.$$

$$2. \ H(-z).E(z) = 1.$$

Proof. 1. We have

$$E(z) = \sum_{k \geq 0} e_k z^k = \prod_{i \geq 1} (1 + \lambda_i z)$$

$$E(-z) = \sum_{k \geq 0} e_k (-z)^k = \prod_{i \geq 1} (1 - \lambda_i z)$$

$$H(z) = \sum_{k \geq 0} h_k(-z)^k = \prod_{i \geq 1} (1 - \lambda_i z)^{-1}.$$

Then,

$$H(z).E(-z) = \left(\prod_{i \geq 1} (1 - \lambda_i z)^{-1} \right) \left(\prod_{i \geq 1} (1 - \lambda_i z) \right) = 1$$

2. The proof is similar to the proof of 1.

This completes the proof. ■

2.2 Some properties on symmetric functions

Definition 2.2.1 [3] Let n be positive integer and $A = \{a_1, a_2\}$ are set of given variables, then the symmetric function S_n is defined by

$$S_n(A) = S_n(a_1 + a_2) = \frac{a_1^{n+1} - a_2^{n+1}}{a_1 - a_2},$$

with

$$S_0(a_1 + a_2) = 1, \quad S_1(a_1 + a_2) = a_1 + a_2, \quad S_2(a_1 + a_2) = a_1^2 + a_1 a_2 + a_2^2.$$

Remark 2.2.1 [3] We have $S_n(a_1 + a_2) = 0$, for $n < 0$.

Remark 2.2.2 Let $A = \{a_1, a_2\}$ an alphabet, we have

$$S_n(a_1 + a_2,) = h_n(a_1, a_2).$$

Definition 2.2.2 [1] Let A and B be any two alphabets. We define $S_n(A - B)$ by the following form

$$\sum_{j=0}^{\infty} S_j(A - B) z^j = E(-z) H(z). \quad (2.4)$$

With

$$H(z) = \prod_{b \in B} (1 - bz)^{-1} \quad \text{and} \quad E(-z) = \prod_{a \in A} (1 - az).$$

Proposition 2.2.1 [26] *By taking $A = \phi$ in (2.4), we obtain*

$$\sum_{j=0}^{\infty} S_j(-B)z^j = \prod_{b \in B} (1 - bz), \quad (2.5)$$

Proposition 2.2.2 [26] *By taking $B = \phi$ in (2.4), we obtain*

$$\sum_{j=0}^{\infty} S_j(A)z^j = \frac{1}{\prod_{a \in A} (1 - az)}. \quad (2.6)$$

Lemma 2.2.1 [3] *Given two alphabet $A = \{x\}$ and $B = \{b_1, b_2, \dots, b_n\}$, we have*

$$S_{n+k}(x - B) = x^k S_n(x - B),$$

for all $k \geq 0$.

Proposition 2.2.3 [3] *If A is of cardinal 1 (i.e. $A = x$), so*

$$\frac{\prod_{b \in B} (1 - bz)}{(1 - xz)} = 1 + \dots + z^{j-1} S_{j-1}(x - B) + z^j \frac{S_j(x - B)}{(1 - xz)}. \quad (2.7)$$

Proof. According to (2.4) we have:

$$\frac{\prod_{b \in B} (1 - bz)}{(1 - xz)} = \sum_{j=0}^{\infty} S_j(x - B)z^j. \quad (2.8)$$

Thus

$$\begin{aligned} \sum_{j=0}^{\infty} S_j(x - B)z^j &= 1 + \dots + S_{j-1}(x - B)z^{j-1} + S_{j+1}(x - B)z^{j+1} + \dots \\ &= 1 + \dots + S_{j-1}(x - B)z^{j-1} + z^j(S_j(x - B) + S_{j+1}(x - B)z + \dots) \\ &= 1 + \dots + S_{j-1}(x - B)z^{j-1} + z^j(S_j(x - B) + xS_j(x - B)z + \dots) \end{aligned}$$

$$\begin{aligned}
 &= 1 + \dots + S_{j-1}(x-B)z^{j-1} + z^j S_j(x-B)(1+xz+x^2z^2+\dots) \\
 &= 1 + \dots + z^{j-1} S_{j-1}(x-B) + z^j \frac{S_j(x-B)}{(1-xz)}.
 \end{aligned}$$

Then

$$\frac{\prod_{b \in B} (1-bz)}{(1-xz)} = 1 + \dots + z^{j-1} S_{j-1}(x-B) + z^j \frac{S_j(x-B)}{(1-xz)}.$$

This completes the proof. ■

Proposition 2.2.4 [2] *Considering successively the case $A = \phi$, $B = \phi$, we get the following factorization*

$$\sum_{j=0}^{\infty} S_j(A-B)z^j = \sum_{j=0}^{\infty} S_j(A)z^j \sum_{j=0}^{\infty} S_j(-B)z^j.$$

Thus,

$$S_j(A-B)z^j = \sum_{k=0}^j S_{j-k}(A)S_k(-B).$$

Corollary 2.2.1 [25] *For $n \in \mathbb{N}$, the symmetric function of bivariate Fibonacci polynomials is given by*

$$F(x, y) = S_{n-1}(a_1 + [-a_2]).$$

Corollary 2.2.2 [25] *For $n \in \mathbb{N}$, the symmetric function of bivariate Lucas polynomials is given by*

$$L(x, y) = S_n(a_1 + [-a_2]) - xS_{n-1}(a_1 + [-a_2]).$$

Corollary 2.2.3 [25] *For $n \in \mathbb{N}$, the symmetric function of bivariate Pell polynomials is given by*

$$P(x, y) = S_{n-1}(a_1 + [-a_2]).$$

Corollary 2.2.4 [25] *For $n \in \mathbb{N}$, the symmetric function of bivariate Pell-Lucas polynomials is given by*

$$Q(x, y) = 2S_n(a_1 + [-a_2]) - 2xyS_{n-1}(a_1 + [-a_2]).$$

Corollary 2.2.5 [25] For $n \in \mathbb{N}$, the symmetric function of bivariate Jacobsthal polynomials is given by

$$J(x, y) = S_{n-1}(a_1 + [-a_2]).$$

Corollary 2.2.6 [25] For $n \in \mathbb{N}$, the symmetric function of bivariate Jacobsthal-Lucas polynomials is given by

$$j(x, y) = 2S_n(a_1 + [-a_2]) - xS_{n-1}(a_1 + [-a_2]).$$

In the following table we give symmetric functions of some numbers[25]

Sequences	Symmetric functions
F_n	$S_{n-1}(a_1 + [-a_2])$
L_n	$2S_n(a_1 + [-a_2]) - S_{n-1}(a_1 + [-a_2])$
P_n	$S_{n-1}(a_1 + [-a_2])$
Q_n	$2S_n(a_1 + [-a_2]) - 2S_{n-1}(a_1 + [-a_2])$
J_n	$S_{n-1}(a_1 + [-a_2])$
j_n	$2S_n(a_1 + [-a_2]) - S_{n-1}(a_1 + [-a_2])$

Table 1 : Symmetric functions of some numbers.

In the following table we give symmetric functions of some polynomials[25]

Sequences	Symmetric functions
$F_n(x)$	$S_{n-1}(a_1 + [-a_2])$
$L_n(x)$	$2S_n(a_1 + [-a_2]) - xS_{n-1}(a_1 + [-a_2])$
$P_n(x)$	$S_{n-1}(a_1 + [-a_2])$
$Q_n(x)$	$2S_n(a_1 + [-a_2]) - 2xS_{n-1}(a_1 + [-a_2])$
$J_n(x)$	$S_{n-1}(a_1 + [-a_2])$
$j_n(x)$	$2S_n(a_1 + [-a_2]) - xS_{n-1}(a_1 + [-a_2])$

Table 2 : Symmetric functions of some polynomials.

CHAPTER 3

ORDINARY GENERATING FUNCTIONS OF THE PRODUCTS FOR SOME NUMBERS AND POLYNOMIALS

In this chapter, we introduce some new generating functions for the products of Fibonacci numbers, Pell numbers, Jacobsthal numbers with bivariate Fibonacci polynomials, bivariate Pell polynomials, bivariate Jacobsthal polynomials, bivariate Lucas polynomials, bivariate Pell-Lucas polynomials and bivariate Jacobsthal-Lucas polynomials.

3.1 Definitions and some properties

Definition 3.1.1 [3] Let f be a function on $\mathbb{R}^n$, the divided difference $\partial_{a_i, a_{i+1}}$ is defined by

$$\partial_{a_i, a_{i+1}}(f) = \frac{f(a_1, \dots, a_i, a_{i+1}, \dots, a_n) - f(a_1, \dots, a_{i-1}, a_{i+1}, a_i, \dots, a_n)}{a_i - a_{i+1}}. \quad (3.1)$$

Definition 3.1.2 [3] The symmetrizing operator $\delta_{a_1 a_2}^k$ is defined by

$$\delta_{a_1 a_2}^k f(a_1) = \frac{a_1^k f(a_1) - a_2^k f(a_2)}{a_1 - a_2}, \quad \forall k \in \mathbb{N}. \quad (3.2)$$

Remark 3.1.1 [3] If $f(a_i) = a_i$, in the formula (3.2), we obtain

$$\delta_{a_1 a_2}^k f(a_1) = \frac{a_1^{k+1} - a_2^{k+1}}{a_1 - a_2}.$$

Remark 3.1.2 [3] Let $A = \{a_1, a_2\}$ an alphabet, we have

$$\delta_{a_1 a_2}^k(a_1) = S_k(a_1 + a_2).$$

3.2 Principal formula

In this section, we present a theorem in order to derive new generating functions of the products of some well-known numbers and polynomials. The presented theorem is based on symmetric functions.

Theorem 3.2.1 [4] Let A, B and C be three alphabets, respectively, $\{a_1, a_2\}$, $\{b_1, b_2\}$ and $\{c_1, c_2\}$ then we have

$$\sum_{n=0}^{\infty} S_n(A) S_{n+k-1}(B) S_{n+k-1}(C) z^n = \frac{b_1^k b_2^k \left(\left(\sum_{n=0}^{\infty} S_n(-A) b_2^n c_1^n z^n \right) \left(\sum_{n=0}^{\infty} S_n(-A) b_1^n c_1^n z^n \right) \sum_{n=0}^{\infty} S_n(-A) S_{n-k-1}(B) C_2^{n+k} z^n \right.}{(c_1 - c_2) \prod_{a \in A} (1 - ab_1 c_1 z) \prod_{a \in A} (1 - ab_2 c_1 z) \prod_{a \in A} (1 - ab_1 c_2 z) \prod_{a \in A} (1 - ab_2 c_2 z)}, \quad (3.3)$$

for all $k \in \mathbb{N}_0$.

Proof. By applying the operator $\delta_{c_1 c_2}^k \delta_{b_1 b_2}^k$ to the series $f(b_1 c_1 z) = \sum_{n=0}^{\infty} S_n(A) b_1^n c_1^n z^n$, we have

$$\begin{aligned}
 \delta_{c_1 c_2}^k \delta_{b_1 b_2}^k (f(b_1 c_1 z)) &= \delta_{c_1 c_2}^k \delta_{b_1 b_2}^k \left(\sum_{n=0}^{\infty} S_n(A) b_1^n c_1^n z^n \right) \\
 &= \delta_{c_1 c_2}^k \left(\frac{b_1^k \sum_{n=0}^{\infty} S_n(A) b_1^n c_1^n z^n - b_2^k \sum_{n=0}^{\infty} S_n(A) b_2^n c_1^n z^n}{b_1 - b_2} \right) \\
 &= \delta_{c_1 c_2}^k \left(\sum_{n=0}^{\infty} S_n(A) \frac{b_1^{n+1} - b_2^{n+k}}{b_1 - b_2} c_1^n z^n \right) \\
 &= \delta_{c_1 c_2}^k \left(\sum_{n=0}^{\infty} S_n(A) S_{n+k-1}(B) c_1^n z^n \right) \\
 &= \frac{c_1^k \sum_{n=0}^{\infty} S_n(A) S_{n+k-1}(B) c_1^n z^n - c_2^k \sum_{n=0}^{\infty} S_n(A) S_{n+k-1}(B) c_2^n z^n}{c_1 - c_2} \\
 &= \frac{\sum_{n=0}^{\infty} S_n(A) S_{n+k-1}(B) c_1^{n+k} z^n - \sum_{n=0}^{\infty} S_n(A) S_{n+k-1}(B) c_2^{n+k} z^n}{c_1 - c_2} \\
 &= \sum_{n=0}^{\infty} S_n(A) S_{n+k-1}(B) \left(\frac{c_1^{n+k} - c_2^{n+k}}{c_1 - c_2} \right) z^n \\
 &= \sum_{n=0}^{\infty} S_n(A) S_{n+k-1}(B) S_{n+k-1}(C) z^n.
 \end{aligned}$$

On the other hand, since

$$f(b_1 c_1 z) = \prod_{a \in A} \frac{1}{(1 - a b_1 c_1 z)},$$

$$\begin{aligned}
 \delta_{c_1 c_2}^k \delta_{b_1 b_2}^k \left(\frac{1}{\prod_{a \in A} (1 - ab_1 c_1 z)} \right) &= \delta_{c_1 c_2}^k \left(\frac{\frac{b_1^k}{\prod_{a \in A} (1 - ab_1 c_1 z)} - \frac{b_2^k}{\prod_{a \in A} (1 - ab_2 c_1 z)}}{b_1 - b_2} \right) \\
 &= \delta_{c_1 c_2}^k \left(\frac{b_1^k \prod_{a \in A} (1 - ab_2 c_1 z) - b_2^k \prod_{a \in A} (1 - ab_1 c_1 z)}{(b_1 - b_2) \prod_{a \in A} (1 - ab_1 c_1 z) \prod_{a \in A} (1 - ab_2 c_1 z)} \right).
 \end{aligned}$$

Using the fact that:

$$\sum_{n=0}^{\infty} S_n(-A) b_1^n b_2^n z^n = \prod_{a \in A} (1 - ab_1 c_1 z),$$

then

$$\begin{aligned}
 \delta_{c_1 c_2}^k \delta_{b_1 b_2}^k \left(\frac{1}{\prod_{a \in A} (1 - ab_1 c_1 z)} \right) &= \delta_{c_1 c_2}^k \left(\frac{\sum_{n=0}^{\infty} S_n(-A) b_1^k b_2^n c_1 z^n - \sum_{n=0}^{\infty} S_n(-A) b_2^k b_1^n c_1 z^n}{(b_1 - b_2) \prod_{a \in A} (1 - ab_1 c_1 z) \prod_{a \in A} (1 - ab_2 c_1 z)} \right) \\
 &= \delta_{c_1 c_2}^k \left(\frac{-b_1^k b_2^k \sum_{n=0}^{\infty} S_n(-A) \left(\frac{b_1^{n-k} - b_2^{n-k}}{b_1 - b_2} \right) c_1^n z^n}{\prod_{a \in A} (1 - ab_1 c_1 z) \prod_{a \in A} (1 - ab_2 c_1 z)} \right) \\
 &= \delta_{c_1 c_2}^k \left(\frac{-b_1^k b_2^k \sum_{n=0}^{\infty} S_n(-A) S_{n-k-1}(B) c_1^n z^n}{\prod_{a \in A} (1 - ab_1 c_1 z) \prod_{a \in A} (1 - ab_2 c_1 z)} \right) \\
 &= \frac{1}{c_1 - c_2} \left(\frac{-c_1^k b_1^k b_2^k \sum_{n=0}^{\infty} S_n(-A) S_{n-k-1}(B) c_1^n z^n}{\prod_{a \in A} (1 - ab_1 c_1 z) \prod_{a \in A} (1 - ab_2 c_1 z)} + \frac{c_1^k b_1^k b_2^k \sum_{n=0}^{\infty} S_n(-A) S_{n-k-1}(B) c_2^n z^n}{\prod_{a \in A} (1 - ab_1 c_1 z) \prod_{a \in A} (1 - ab_2 c_1 z)} \right)
 \end{aligned}$$

$$= \frac{b_1^k b_2^k \left[\left(\sum_{n=0}^{\infty} S_n(-A) b_2^n c_1^n z^n \right) \left(\sum_{n=0}^{\infty} S_n(-A) b_1^n c_1^n z^n \right) \sum_{n=0}^{\infty} S_n(-A) S_{n-k-1}(B) C_2^{n+k} z^n \right]}{(c_1 - c_2) \prod_{a \in A} (1 - ab_1 c_1 z) \prod_{a \in A} (1 - ab_2 c_1 z) \prod_{a \in A} (1 - ab_1 c_2 z) \prod_{a \in A} (1 - ab_2 c_2 z)}.$$

This completes the proof. ■

If $k = 0, 1, 2$ in the theorem (3.2.1), we deduce the following lemmas.

Lemma 3.2.1 [5] Let A, B and C be three alphabets, respectively, $\{a_1, a_2\}$, $\{b_1, b_2\}$ and $\{c_1, c_2\}$ then we have

$$\sum_{n=0}^{\infty} S_n(A) S_{n-1}(B) S_{n-1}(C) z^n = \frac{M}{D}, \quad n \in \mathbb{N}, \quad (3.4)$$

with

$$\begin{aligned} M = & (a_1 + a_2)z - a_1 a_2 (b_1 + b_2)(c_1 + c_2)z^2 + b_1 b_2 c_1 c_2 (a_1 + a_2)(2a_1 a_2 - (a_1 + a_2)^2)z^3 + a_1 a_2 b_1 b_2 c_1 c_2 \\ & (b_1 + b_2)(c_1 + c_2)(a_1 + a_2)^2 z^4 - b_1 b_2 c_1 c_2 a_1^2 a_2^2 (a_1 + a_2)(b_1 b_2 (c_1 + c_2)^2 + c_1 c_2 (b_1 + b_2)^2 - c_1 c_2 b_1 b_2)z^5 \\ & + a_1^3 a_2^3 b_1^2 b_2^2 c_1^2 c_2^2 (b_1 + b_2)(c_1 + c_2)z^6. \\ D = & 1 - (a_1 + a_2)(b_1 + b_2)(c_1 + c_2)z + (b_1 b_2 (a_1 + a_2)^2 (c_1 + c_2)^2 + ((b_1 + b_2)^2 - 2b_1 b_2)((a_1 + a_2)^2 c_1 c_2 \\ & - 2a_1 a_2 c_1 c_2 + a_1 a_2 (c_1 + c_2)^2))z^2 - (a_1 + a_2)(b_1 + b_2)(c_1 + c_2)(b_1 b_2 c_1 c_2 (a_1 + a_2)^2 + b_1 b_2 a_1 a_2 (c_1 + c_2)^2 \\ & + a_1 a_2 c_1 c_2 (b_1 + b_2)^2 - 5a_1 a_2 c_1 c_2 b_1 b_2)z^3 + (a_1^2 a_2^2 c_1^2 c_2^2 (b_1 + b_2)^4 + c_1^2 c_2^2 b_1^2 b_2^2 (a_1 + a_2)^4 + a_1^2 a_2^2 b_1^2 b_2^2 (c_1 + c_2)^4 \\ & - a_1 a_2 b_1 b_2 c_1 c_2 (4b_1 b_2 c_1 c_2 (a_1 + a_2)^2 + 4a_1 a_2 c_1 c_2 (b_1 + b_2)^2 + 4a_1 a_2 b_1 b_2 (c_1 + c_2)^2 - (a_1 + a_2)^2 (b_1 + b_2)^2 \\ & (c_1 + c_2)^2) + 6a_1^2 a_2^2 b_1^2 b_2^2 c_1^2 c_2^2)z^4 - a_1 a_2 b_1 b_2 c_1 c_2 (a_1 + a_2)(b_1 + b_2)(c_1 + c_2)(b_1 b_2 c_1 c_2 (a_1 + a_2)^2 + a_1 a_2 c_1 c_2 \\ & (b_1 + b_2)^2 - 5a_1 a_2 b_1 b_2 c_1 c_2)z^5 + (a_1^2 a_2^2 b_1^3 b_2^3 c_1^2 c_2^2 (a_1 + a_2)^2 (c_1 + c_2)^2 + a_1^2 a_2^2 b_1^2 b_2^2 c_1^2 c_2^2 ((b_1 + b_2)^2 - 2b_1 b_2) \\ & ((a_1 + a_2)^2 c_1 c_2 - 2a_1 a_2 c_1 c_2 + a_1 a_2 (b_1 + b_2)^2))z^6 - a_1^3 a_2^3 b_1^3 b_2^3 c_1^3 c_2^3 (a_1 + a_2)(b_1 + b_2)(c_1 + c_2)z^7 \\ & + a_1^4 a_2^4 b_1^4 b_2^4 c_1^4 c_2^4 z^8. \end{aligned}$$

Lemma 3.2.2 [5] *Let A , B and C be three alphabets, respectively, $\{a_1, a_2\}$, $\{b_1, b_2\}$ and $\{c_1, c_2\}$ then we have*

$$\sum_{n=0}^{\infty} S_n(A)S_{n+1}(B)S_{n+1}(C)z^n = \frac{M_1}{D}, \quad n \in \mathbb{N}, \quad (3.5)$$

with

$$\begin{aligned} M_1 = & (c_1 + c_2)(b_1 + b_2) - (a_1 + a_2)(c_1c_2(b_1 + b_2)^2 + b_1b_2(c_1 + c_2)^2 - c_1c_2b_1b_2)z + c_1c_2b_1b_2(a_1 + a_1)^2 \\ & (b_1 + b_1)(c_1 + c_1)z^2 - c_1^2c_2^2b_1^2b_2^2(a_1 + a_1)((a_1 + a_1)^2 - 2a_1a_2)z^3 - a_1^2a_2^2b_1^2b_2^2c_1^2c_2^2(b_1 + b_2)(c_1 + c_2)z^4 \\ & + a_1^2a_2^2b_1^3b_2^3c_1^3c_2^3(a_1 + a_2)z^5. \end{aligned}$$

From the previous lemma we deduce the following relationship

$$\sum_{n=0}^{\infty} S_{n-1}(A)S_n(B)S_n(C)z^n = \frac{M_2}{D}, \quad n \in \mathbb{N}, \quad (3.6)$$

with

$$\begin{aligned} M_2 = & (c_1 + c_2)(b_1 + b_2)z - (a_1 + a_2)(c_1c_2(b_1 + b_2)^2 + b_1b_2(c_1 + c_2)^2 - c_1c_2b_1b_2)z^2 + c_1c_2b_1b_2(a_1 + a_1)^2 \\ & (b_1 + b_1)(c_1 + c_1)z^3 - c_1^2c_2^2b_1^2b_2^2(a_1 + a_1)((a_1 + a_1)^2 - 2a_1a_2)z^4 - a_1^2a_2^2b_1^2b_2^2c_1^2c_2^2(b_1 + b_2)(c_1 + c_2)z^5 \\ & + a_1^2a_2^2b_1^3b_2^3c_1^3c_2^3(a_1 + a_2)z^6. \end{aligned}$$

Lemma 3.2.3 [5] *Let A , B and C be three alphabets, respectively, $\{a_1, a_2\}$, $\{b_1, b_2\}$ and $\{c_1, c_2\}$ then we have*

$$\sum_{n=0}^{\infty} S_n(A)S_n(B)S_n(C)z^n = \frac{M_3}{D}, \quad n \in \mathbb{N}, \quad (3.7)$$

with

$$\begin{aligned} M_3 = & 1 - (a_1a_2c_1c_2(b_1 + b_2)^2 + a_1a_2b_1b_2(c_1 + c_2)^2 + b_1b_2c_1c_2(a_1 + a_2)^2 - 3a_1a_2c_1c_2b_1b_2)z^2 + 2a_1a_2b_1b_2c_1c_2 \\ & (a_1 + a_2)(b_1 + b_2)(c_1 + c_2)z^3 - (b_1b_2a_1^2a_2^2c_1^2c_2^2(b_1 + b_2)^2 + c_1c_2a_1^2a_2^2b_1^2b_2^2(c_1 + c_2)^2 + a_1a_2b_1^2b_2^2c_1^2c_2^2 \\ & (a_1 + a_2)^2 - 3a_1^2a_2^2b_1^2b_2^2c_1^2c_2^2)z^4 + a_1^3a_2^3b_1^3b_2^3c_1^3c_2^3z^6. \end{aligned}$$

From the previous lemma we deduce the following relationship

$$\sum_{n=0}^{\infty} S_{n-1}(A)S_{n-1}(B)S_{n-1}(C)z^n = \frac{M_4}{D}, \quad n \in \mathbb{N}, \quad (3.8)$$

with

$$\begin{aligned} M_4 = & z - (a_1a_2c_1c_2(b_1 + b_2)^2 + a_1a_2b_1b_2(c_1 + c_2)^2 + b_1b_2c_1c_2(a_1 + a_2)^2 - 3a_1a_2c_1c_2b_1b_2)z^3 + 2a_1a_2b_1b_2c_1c_2 \\ & (a_1 + a_2)(b_1 + b_2)(c_1 + c_2)z^4 - (b_1b_2a_1^2a_2^2c_1^2c_2^2(b_1 + b_2)^2 + c_1c_2a_1^2a_2^2b_1^2b_2^2(c_1 + c_2)^2 + a_1a_2b_1^2b_2^2c_1^2c_2^2 \\ & (a_1 + a_2)^2 - 3a_1^2a_2^2b_1^2b_2^2c_1^2c_2^2)z^5 + a_1^3a_2^3b_1^3b_2^3c_1^3c_2^3z^7. \end{aligned}$$

By making the following restriction: $c_1 - c_2 = 1$, $b_1 - b_2 = 1$, $a_1 - a_2 = x$, $c_1c_2 = 1$, $b_1b_2 = 1$, $a_1a_2 = y$, and by replacing a_2 by $[-a_2]$ and b_2 by $[-b_2]$ and c_2 by $[-c_2]$ in (3.8), we get a new generating function, involving the product of Fibonacci numbers with bivariate Fibonacci polynomials as follows.

$$\sum_{n=0}^{\infty} S_{n-1}(c_1 + [-c_2])S_{n-1}(b_1 + [-b_2])S_{n-1}(a_1 + [-a_2]) = \frac{N_{FFF(x,y)}}{D_{FFF(x,y)}}, \quad (3.9)$$

with

$$\begin{aligned} N_{FFF(x,y)} &= z - (x^2 + 5y)z^3 - 2xyz^4 + (x^2y + 5y^2)z^5 - y^3z^7. \\ D_{FFF(x,y)} &= 1 - xz - (4x^2 + 9y)z^2 - (x^3 + 7xy)z^3 + (16y^2 + x^4 + 3x^2y)z^4 + (7xy^2 + x^3y)z^5 - \\ & (4y^2x^2 + 9y^3)z^6 + xy^3z^7 + y^4z^8. \end{aligned}$$

Corollary 3.2.1 $\forall n \in \mathbb{N}$ we have

$$F_n F_n F_n(x, y) = S_{n-1}(c_1 + [-c_2])S_{n-1}(b_1 + [-b_2])S_{n-1}(a_1 + [-a_2]).$$

Put $y = 1$ and $x = y = 1$ in the relation (3.9) we get the following corollaries

Proposition 3.2.1 For $n \in \mathbb{N}$, a generating Function of the product of Fibonacci numbers with Fibonacci polynomials is given by:

$$\sum_{n=0}^{\infty} F_n F_n F_n(x) = \frac{z - (x^2 + 5)z^3 - 2xz^4 + (x^2 + 5)t^5 - t^7}{D_{FFF(x)}}, \quad (3.10)$$

with

$$D_{FFF(x)} = 1 - xz - (4x^2 + 9)z^2 - (x^3 + 7x)z^3 + (16 + x^4 + 3x^2)z^4 + (7x + x^3)z^5 - (4x^2 + 9)z^6 + xz^7 + z^8.$$

Then

$$F_n F_n F_n(x) = S_{n-1}(c_1 + [-c_2])S_{n-1}(b_1 + [-b_2])S_{n-1}(a_1 + [-a_2]).$$

Corollary 3.2.2 For $n \in \mathbb{N}$, a generating Function for the cubes Fibonacci numbers is given by:

$$\sum_{n=0}^{\infty} F_n F_n F_n = \frac{z - 6z^3 - 2z^4 + 6z^5 - z^7}{1 - z - 13z^2 - 8z^3 + 20z^4 + 8z^5 - 13z^6 + z^7 + z^8}. \quad (3.11)$$

With

$$F_n F_n F_n = S_{n-1}(c_1 + [-c_2])S_{n-1}(b_1 + [-b_2])S_{n-1}(a_1 + [-a_2]).$$

By replacing a_2 by $[-a_2]$, b_2 by $[-b_2]$ and c_2 by $[-c_2]$, taking $c_1 - c_2 = 1$, $b_1 - b_2 = 1$, $a_1 - a_2 = x$, $c_1 c_2 = 1$, $b_1 b_2 = 1$, $a_1 a_2 = y$ in (3.4) we obtain:

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_{n-1}(c_1 + [-c_2])S_{n-1}(b_1 + [-b_2])z^n = \frac{xz + yz^2 - x(x^2 + 2y)z^3 - yx^2z^4 + 3xy^2z^5 - y^3z^6}{D_{FFF(x,y)}}. \quad (3.12)$$

Theorem 3.2.2 For $n \in \mathbb{N}$, a generating Function of the product of Fibonacci numbers with bivariate Lucas polynomials is given by:

$$\sum_{n=0}^{\infty} F_n F_n L_n(x, y) z^n = \frac{N_{FFL(x,y)}}{D_{FFL(x,y)}}. \quad (3.13)$$

With

$$\begin{aligned} N_{FFL(x,y)} &= -2y^3t^7 + t^6xy^3 + (-3x^2y^2 + 2x^2y + 10y^2)t^5 + (x^3y - 4xy)t^4 + (x^4 + 2x^2y - 2x^2 - 10y)t^3 \\ &- t^2xy + (-x^2 + 2)t, \end{aligned}$$

and

$$D_{FFL(x,y)} = D_{FFF(x,y)}.$$

Proof. We have

$$\begin{aligned} \sum_{n=0}^{\infty} F_n F_n L_n(x, y) z^n &= \sum_{n=0}^{\infty} S_{n-1}(c_1 + [-c_2]) S_{n-1}(b_1 + [-b_2]) (2S_n(a_1 + [-a_2]) - x S_{n-1}(a_1 + [-a_2])) z^n \\ &= 2 \sum_{n=0}^{\infty} S_{n-1}(c_1 + [-c_2]) S_{n-1}(b_1 + [-b_2]) S_n(a_1 + [-a_2]) z^n \\ &\quad - x \sum_{n=0}^{\infty} S_{n-1}(c_1 + [-c_2]) S_{n-1}(b_1 + [-b_2]) S_{n-1}(a_1 + [-a_2]) z^n \\ &= 2 \frac{z + -(x^2 + 5y)z^3 - 2xyz^4 + (x^2y + 5y^2)z^5 - y^3z^7}{D_{FFL(x,y)}} \\ &= -x \frac{xz + yz^2 - x(x^2 + 2y)z^3 - yx^2z^4 + 3xy^2z^5 - y^3z^6}{D_{FFL(x,y)}} \end{aligned}$$

so,

$$\sum_{n=0}^{\infty} F_n F_n L_n(x, y) z^n = \frac{N_{FFL(x,y)}}{D_{FFL(x,y)}},$$

with

$$\begin{aligned} N_{FFL(x,y)} &= -2y^3t^7 + t^6xy^3 + (-3x^2y^2 + 2x^2y + 10y^2)t^5 + (x^3y - 4xy)t^4 + (x^4 + 2x^2y - 2x^2 - 10y)t^3 \\ &\quad - t^2xy + (-x^2 + 2)t. \end{aligned}$$

This completes the proof ■

Corollary 3.2.3 For $n \in \mathbb{N}$, we have

$$F_n F_n L_n(x, y) = S_{n-1}(c_1 + [-c_2]) S_{n-1}(b_1 + [-b_2]) (2S_n(a_1 + [-a_2]) - x S_{n-1}(a_1 + [-a_2])).$$

Put $y = 1$ and $x = y = 1$ in the relation (3.13) we get the following corollaries.

Corollary 3.2.4 For $n \in \mathbb{N}$, a generating Function of the product of Fibonacci numbers with Lucas polynomials is given by

$$\sum_{n=0}^{\infty} F_n F_n L_n(x) = \frac{(2-x^2)z - xz^2 + (x^4-10)z^3 + (x^3-4x)z^4 + (10-x^2)z^5 + xz^6 - 2z^7}{D_{FFL(x)}},$$

with

$$D_{FFL(x)} = D_{FFF(x)}.$$

Then

$$F_n F_n L_n(x) = S_{n-1}(c_1 + [-c_2])S_{n-1}(b_1 + [-b_2])(2S_n(a_1 + [-a_2]) - S_{n-1}(a_1 + [-a_2])).$$

Corollary 3.2.5 For $n \in \mathbb{N}$, a generating Function for the product of Fibonacci numbers with Lucas numbers is given by:

$$\sum_{n=0}^{\infty} F_n F_n L_n = \frac{z - z^2 - 9z^3 - 3z^4 + 9z^5 + z^6 - 2z^7}{1 - z - 13z^2 - 8z^3 + 20z^4 + 8z^5 - 13z^6 + z^7 + z^8}, \quad (3.14)$$

with

$$F_n F_n L_n = S_{n-1}(c_1 + [-c_2])S_{n-1}(b_1 + [-b_2])(2S_n(a_1 + [-a_2]) - S_{n-1}(a_1 + [-a_2])).$$

By making the following restriction: $c_1 - c_2 = 2xy$, $b_1 - b_2 = a_1 - a_2 = 2$ and $c_1 c_2 = y$, $b_1 b_2 = a_1 a_2 = 1$ and by replacing a_2 by $[-a_2]$ and b_2 by $[-b_2]$ in (3.8), we get a new generating function, involving the product of Pell numbers with bivariate Pell polynomials as follows.

$$\sum_{n=0}^{\infty} S_{n-1}(c_1 + [-c_2])S_{n-1}(b_1 + [-b_2])S_{n-1}(a_1 + [-a_2]) = \frac{N_{PPP(x,y)}}{D_{PPP(x,y)}}, \quad (3.15)$$

with

$$N_{PPP(x,y)} = z - (4x^2y^2 + 11)z^3 - 16xyz^4 + (4x^2y^2 + 11)z^5 - z^7.$$

$$D_{PPP(x,y)} = 1 - 8xyz - (40x^2y^2 + 36y)z^2 - (104xy^2 + 32x^3y^3)z^3 + (64y^2 + 16x^2y^2 - 48x^2y^3)z^4 + (104xy^3 + 32x^3y^4)z^5 - (40y^4x^2 + 36y^3)z^6 + 8xy^4z^7 + y^4z^8.$$

Corollary 3.2.6 For $n \in \mathbb{N}$, we have

$$P_n P_n P_n(x, y) = S_{n-1}(c_1 + [-c_2]) S_{n-1}(b_1 + [-b_2]) S_{n-1}(a_1 + [-a_2]).$$

Put $y = 1$ in the relation (3.15) we get the following corollary.

Corollary 3.2.7 For $n \in \mathbb{N}$, a generating Function of the product of Pell numbers with pell polynomials is given by:

$$\sum_{n=0}^{\infty} P_n P_n P_n(x) = \frac{z - (11 + 4x^2)z^3 - 16xz^4 + (11 + 4x^2)z^5 - z^7}{D_{PPP(x)}}, \quad (3.16)$$

with

$$D_{PPP(x)} = 1 - 8xz - (40x^2 + 36)z^2 - (104x + 32x^3)z^3 + (70 + 16x^4 - 48x^2)z^4 + (104x + 32x^3)z^5 - (40x^2 + 36)z^6 + 8xz^7 + z^8.$$

Then

$$P_n P_n P_n(x) = S_{n-1}(c_1 + [-c_2]) S_{n-1}(b_1 + [-b_2]) S_{n-1}(a_1 + [-a_2]).$$

Monsour in [20] give the formula(3.17) which is obtained by taking $x = y = 1$ in the relation (3.15)

$$\sum_{n=0}^{\infty} P_n P_n P_n = \frac{z(1 - 4z - z^2)}{(1 - 2z - z^2)(-1 - 14z - z^2)}. \quad (3.17)$$

representing a generating function of the cubes of Pell numbers.

We deduce the following corollary.

Corollary 3.2.8 For $n \in \mathbb{N}$, we have

$$P_n P_n P_n = S_{n-1}(c_1 + [-c_2]) S_{n-1}(b_1 + [-b_2]) S_{n-1}(a_1 + [-a_2]).$$

By replacing a_2 by $[-a_2]$, b_2 by $[-b_2]$ and c_2 by $[-c_2]$, taking $c_1 - c_2 = b_1 - b_2 = 2$, $a_1 - a_2 = 2xy$, $c_1 c_2 = b_1 b_2 = 1$, $a_1 a_2 = y$ in (3.4) we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} S_{n-1}(c_1 + [-c_2])S_{n-1}(b_1 + [-b_2])S_n(a_1 + [-a_2])z^n \\ &= \frac{2xyz + 4yz^2 - (4xy^2 + 8x^3y^3)z^3 - 16x^2y^3z^4 + 18xy^3z^5 - 4y^3z^6}{D_{PPP(x,y)}}. \end{aligned} \quad (3.18)$$

Theorem 3.2.3 For $n \in \mathbb{N}$, a generating Function for the product of Pell numbers with bivariate Pell-Lucas polynomials is given by

$$\sum_{n=0}^{\infty} P_n P_n Q_n(x, y) z^n = \frac{N_{PPQ(x,Y)}}{D_{PPQ(x,Y)}}, \quad (3.19)$$

with

$$\begin{aligned} N_{PPQ(x,Y)} &= 2xyz + 8yz^2 + (14xy^2 - 8x^3y^3)z^3 + (14xy^3 - 8y^4x^3)z^5 - 8y^3z^6 + 2xy^4z^7. \\ D_{PPQ(x,y)} &= D_{PPP(x,y)}. \end{aligned}$$

Proof. We have

$$\begin{aligned} \sum_{n=0}^{\infty} P_n P_n Q_n(x, y) z^n &= \sum_{n=0}^{\infty} S_{n-1}(c_1 + [-c_2])S_{n-1}(b_1 + [-b_2])(2S_n(a_1 + [-a_2]) - 2yxS_{n-1}(a_1 + [-a_2]))z^n \\ &= 2 \sum_{n=0}^{\infty} S_{n-1}(c_1 + [-c_2])S_{n-1}(b_1 + [-b_2])S_n(a_1 + [-a_2])z^n \\ &\quad - 2xy \sum_{n=0}^{\infty} S_{n-1}(c_1 + [-c_2])S_{n-1}(b_1 + [-b_2])S_n(a_1 + [-a_2])z^n \\ &= \frac{2(2xyz + 4yz^2 - (4xy^2 + 8x^3y^3)z^3 - 16x^2y^3z^4 + 18xy^3z^5 - 4y^3z^6)}{D_2} \\ &\quad - \frac{2xy(z - (11y + 4x^2y^2)z^3 - 16xy^2z^4 + (11y^2 + 4x^2y^3)z^5 - y^3z^7)}{D_2} \\ &= \frac{2xyz + 8yz^2 + (14xy^2 - 8x^3y^3)z^3 + (14xy^3 - 8y^4x^3)z^5 - 8y^3z^6 + 2xy^4z^7}{D_{PPQ(x,Y)}} \end{aligned}$$

This completes the proof. ■

Corollary 3.2.9 For $n \in \mathbb{N}$, we have

$$P_n P_n Q_n(x, y) = S_{n-1}(c_1 + [-c_2])S_{n-1}(b_1 + [-b_2])(2S_n(a_1 + [-a_2]) - xS_{n-1}(a_1 + [-a_2])).$$

Put $y = 1$ and $x = y = 1$ in the relation (3.19) we get the following corollaries

Corollary 3.2.10 *For $n \in \mathbb{N}$, a generating Function of the product of Pell numbers with pell-Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} P_n P_n Q_n(x) = \frac{2xz + 8z^2 + (14x - 8x^3)z^3 + (14x - 8x^3)z^5 - 8z^6 + 2xz^7}{D_{PPQ(x)}}, \quad (3.20)$$

with

$$P_n P_n Q_n(x) = S_{n-1}(c_1 + [-c_2])S_{n-1}(b_1 + [-b_2])(2S_n(a_1 + [-a_2]) - 2xS_{n-1}(a_1 + [-a_2])).$$

Corollary 3.2.11 *For $n \in \mathbb{N}$, a generating Function for the cubes Pell-Lucas numbers is given by*

$$\sum_{n=0}^{\infty} P_n P_n Q_n = \frac{2z + 8z^2 + 6z^3 + 6z^5 - 8z^6 + 2z^7}{1 - 8z - 76z^2 - 136z^3 + 32z^4 + 136z^5 - 76z^6 + 8z^7 + z^8}, \quad (3.21)$$

with

$$P_n P_n Q_n = S_{n-1}(c_1 + [-c_2])S_{n-1}(b_1 + [-b_2])(2S_n(a_1 + [-a_2]) - xS_{n-1}(a_1 + [-a_2])).$$

By making the following restriction: $c_1 - c_2 = xy$, $b_1 - b_2 = a_1 - a_2 = 1$, $c_1 c_2 = 2y$, $b_1 b_2 = a_1 a_2 = 2$ and by replacing a_2 by $[-a_2]$ and b_2 by $[-b_2]$ in (3.8), we get a new generating function, involving the product of Jacobsthal numbers with bivariate Jacobsthal polynomials as follows.

$$\sum_{n=0}^{\infty} S_{n-1}(c_1 + [-c_2])S_{n-1}(b_1 + [-b_2])S_{n-1}(a_1 + [-a_2]) = \frac{N_{JJJ(x,y)}}{D_{JJJ(x,y)}},$$

with

$$N_{JJJ(x,y)} = z - (32y + 4x^2y^2)z^3 - 16xy^2z^4 + (256y^2 + 32y^3x^2)z^5 - 512y^3z^7.$$

$$D_{JJJ(x,y)} = 1 - xyz - (12x^2y^2 + 50y)z^2 - xy(48y + 4x^2y^2)z^3 + (672y^2 + 16x^4y^4 + 120x^2y^3)z^4 + 8xy^2(48y + 4x^2y^2)z^5 - (128xy^4 + 320y^2(10y + 2x^2y^2))z^6 + 512xy^4z^7 + 4096y^4z^8.$$

Corollary 3.2.12 $\forall n \in \mathbb{N}$, we have

$$J_n J_n J_n(x, y) = S_{n-1}(c_1 + [-c_2]) S_{n-1}(b_1 + [-b_2]) S_{n-1}(a_1 + [-a_2]).$$

By replacing a_2 by $[-a_2]$, b_2 by $[-b_2]$ and c_2 by $[-c_2]$, taking $c_1 - c_2 = b_1 - b_2 = 1$, $a_1 - a_2 = xy$ and $c_1 c_2 = b_1 b_2 = 2$, $a_1 a_2 = 2y$ in (3.4) we obtain:

$$\begin{aligned} & \sum_{n=0}^{\infty} S_{n-1}(c_1 + [-c_2]) S_{n-1}(b_1 + [-b_2]) S_n(a_1 + [-a_2]) z^n \\ &= \frac{z - (32y + 4x^2 y^2) z^2 - 16xy^2 z^4 + (256y^2 + 32x^2 y^3) z^5 - 512y^3 z^7}{D_{JJJ}(x, y)}. \end{aligned} \quad (3.22)$$

Theorem 3.2.4 For $n \in \mathbb{N}$, a generating Function for the product of Jacobsthal numbers with bivariate Jacobsthal-Lucas polynomials is given by

$$\sum_{n=0}^{\infty} J_n J_n j_n(x, y) z^n = \frac{N_{JJj}(x, y)}{D_{JJj}(x, y)}. \quad (3.23)$$

With

$$N_{JJj}(x, y) = xyz + 4z^2 - 4x^2 y^2 z^3 - 32x^2 y^4 z^5 - 128y^3 z^6 + 512xy^4 z^7$$

$$D_{JJj}(x, y) = D_{JJJ}(x, y).$$

Proof. We have

$$\begin{aligned} \sum_{n=0}^{\infty} J_n J_n j_n(x, y) z^n &= \sum_{n=0}^{\infty} S_{n-1}(c_1 + [-c_2]) S_{n-1}(b_1 + [-b_2]) (2S_n(a_1 + [-a_2]) - xy S_{n-1}(a_1 + [-a_2])) z^n \\ &= 2 \sum_{n=0}^{\infty} S_{n-1}(c_1 + [-c_2]) S_{n-1}(b_1 + [-b_2]) S_n(a_1 + [-a_2]) z^n \\ &\quad - xy \sum_{n=0}^{\infty} S_{n-1}(c_1 + [-c_2]) S_{n-1}(b_1 + [-b_2]) S_{n-1}(a_1 + [-a_2]) z^n \\ &= \frac{2(xy z + 2y z^2 - (16xy^2 + 4x^2 y^2) - 8x^2 y^3 z^4 + 128xy^3 z^5 - 64y^3 z^6)}{D_{JJJ}(x, y)}. \end{aligned}$$

$$\begin{aligned}
 & - \frac{xy(z - (32y + 4x^2y^2)z^2 - 16xy^2z^4 + (256y^2 + 32x^2y^3)z^5 - 512y^3z^7)}{D_{JJJ(x,y)}} \\
 & = \frac{xyz + 4z^2 - 4x^2y^2z^3 - 32x^2y^4z^5 - 128y^3z^6 + 512xy^4z^7}{D_{JJj(x,y)}}.
 \end{aligned}$$

This completes the proof. ■

Corollary 3.2.13 For $n \in \mathbb{N}$, we have

$$J_n J_n j_n(x, y) = S_{n-1}(c_1 + [-c_2]) S_{n-1}(b_1 + [-b_2]) (2S_n(a_1 + [-a_2]) - xy S_{n-1}(a_1 + [-a_2])).$$

Put $y = 1$ and $x = y = 1$ in the relation (3.23) we get the following corollaries

Corollary 3.2.14 For $n \in \mathbb{N}$, a generating Function of the product of Jacobsthal numbers with Jacobsthal-Lucas polynomials is given by

$$\sum_{n=0}^{\infty} J_n J_n j_n(x) = \frac{z + (32 + 4x^2)z^3 - 16xz^4 + (256 + 32x^2)z^5 - 512z^7}{D_{JJj(x)}}, \quad (3.24)$$

with

$$D_{JJj(x)} = 1 - xz - (12x^2 + 50)z^2 - x(48 + 4x^2)z^3 + (672 + 16x^4 + 120x^2)z^4 + 8x(48 + 4x^2)z^5 - (128x + 320(10 + 2x^2))z^6 + 512xz^7 + 4096z^8.$$

Then

$$J_n J_n j_n(x) = S_{n-1}(c_1 + [-c_2]) S_{n-1}(b_1 + [-b_2]) (2S_n(a_1 + [-a_2]) - x S_{n-1}(a_1 + [-a_2])).$$

Corollary 3.2.15 For $n \in \mathbb{N}$, a generating Function for the product of Jacobsthal numbers with Jacobsthal-Lucas numbers is given by

$$\sum_{n=0}^{\infty} J_n J_n j_n = \frac{z - 36z^3 - 16z^4 + 288z^5 - 512z^7}{1 - z - 62z^2 - 52z^3 + 808z^4 + 416z^5 - 3968z^6 + 512z^7 + 4096z^8}, \quad (3.25)$$

with

$$J_n J_n j_n = S_{n-1}(c_1 + [-c_2]) S_{n-1}(b_1 + [-b_2]) (2S_n(a_1 + [-a_2]) - S_{n-1}(a_1 + [-a_2])).$$

CONCLUSION

In this dissertation, by making use the concept of symmetric functions we have derived some new ordinary generating functions for the products of Fibonacci numbers, Pell numbers, Pell-Lucas numbers and Jacobsthal et Jacobsthal-Lucas numbers with bivariate Fibonacci polynomials, bivariate Lucas polynomials, bivariate Pell polynomials, bivariate Pell-Lucas polynomials, bivariate Jacobsthal polynomials and bivariate Jacobstha-Lucas polynomials.

ABSTRACT. In this dissertation, we present a theorem in order to calculate new generating functions for second-order recurrences relations. The presented theorem is based on symmetric functions. Where by making use the symmetrizing operator $\delta_{c_1 c_2}^k \delta_{b_1 b_2}^k$ on the series $\sum_{n=0}^{\infty} S_n(A) b_1^n c_1^n z^n$ we gived the generating functions of the products of Fibonacci numbers, Pell numbers and Jacobsthal numbers with the bivariate Fibonacci polynomials, bivariate Lucas polynomial, bivariate Pell Lucas polynomials, bivariate Jacobsthal and Jacobsthal Lucas polynomials.

Key Words: Symmetric functions, generating functions, recurrence relations.

RÉSUMÉ. Dans ce mémoire, nous présentons un théorème afin de calculer des nouvelles fonctions génératrices pour les relations de récurrences du second ordre, Le théorème présenté est basé sur les fonctions symétriques, où l'utilisation de symétriseur $\delta_{c_1 c_2}^k \delta_{b_1 b_2}^k$ sur la série $\sum_{n=0}^{\infty} S_n(A) b_1^n c_1^n z^n$ nous permet d'obtenir les fonctions génératrices des produits de nombre de Fibonacci, les nombres de Pell et les nombres de Jacobsthal avec les polynômes de Fibonacci bivariés, les polynômes de Lucas bivariés, les polynômes de Pell-Lucas bivariés et les polynômes de Jacobsthal et de Jacobsthal-Lucas bivariés.

Mots-clés: Fonctions génératrices, fonctions symétriques, relations de récurrences.

ملخص . في هذه المدكرة قمنا بعرض نظرية تعتمد على التوابع التناظرية وذلك لحساب الدوال المولدة للعلاقات التراجعية الخطية حيث ان استعمال المؤثر التناظري $\delta_{c_1 c_2}^k \delta_{b_1 b_2}^k$ على السلسلة $\sum_{n=0}^{\infty} S_n(A) b_1^n c_1^n z^n$ يسمح لنا بحساب الدوال المولدة لجداءات اعداد فيبوناتشي ، اعداد بال واعداد جاكوبستال مع كثيرات حدود فيبوناتشي بمتغيرين ، كثيرات حدود بال بمتغيرين ، كثيرات حدود جاكوبستال و جاكوبستال لوكاس بمتغيرين.

الكلمات المفتاحية: التوابع التناظرية، الدوال المولدة، العلاقات التراجعية.

BIBLIOGRAPHY

- [1] A. Abderrezzak, Généralisation de la transformation d'Euler d'une série formelle, Adv. Math., 103 , 180-195, 1994.
- [2] A. Abderrezzak, Généralisation d'identités de Carlitz, Howard et Lehmer, Aequationes Math., 49 , 36-46, 1995.
- [3] Kh. Boubellouta , Fonctions symétriques et leurs applications à certains nombres et polynômes, (Doctoral dissertaton). Mohamed Seddik Ben Yahia University , Jijel, Algeria. 2020.
- [4] kh. Boubellouta, A. Boussayoud and M. Kerada, Symmretric functions for second-order recurrence sequences, 2020.
- [5] Kh. Boubellouta, A. Boussayoud, S. Araci and M. Kerada, Some Theorems On Generating Functions and Their Applications, Advanced studies in contemporary mathematics. 30 (3), 307 - 324, 2020.
- [6] A. Boussayoud, L'action de l'opérateur δ_{e_1, e_2}^k sur la Série $\sum_{n=0}^{\infty} S_n(A) e_1^n z^n$ (Doctoral dissertation), Mohamed Seddik Ben Yahia University, Jijel, Algeria, 2017.

- [7] S. Boughaba, A. Boussayoud¹ and N. Saba, Generating functions of the products of Bivariate complex Fibonacci polynomials with Gaussian numbers and polynomials, *Discussiones Mathematicae*, 2020.
- [8] P. Catarino, On Some Identities for k -Fibonacci Sequence, *Int. J. Contemp. Math. Sciences*. 9 (1), 37 - 42, 2014.
- [9]] P. Catarino, On Some Identities and Generating Functions for k - Pell Numbers, *Int. Journal of Math. Analysis*. 7(38), 2013, 1877 - 1884.
- [10] P. Catarino and H. Campos, Incomplete k -Pell, k -Pell-Lucas and modified k -Pell numbers, *Hacettepe Journal of Mathematics and Statistics*, 46 (3), 361-372, 2017.
- [11] H. Campos, P. Catarino, A. P. Aires, P. Vasco and A. Borges, On Some Identities of k -Jacobsthal-Lucas Numbers, *Int. Journal of Math. Analysis*. 8 (10), 489 - 494, 2014.
- [12] S. Falcon, On the Lucas Triangle and its Relationship with the k -Lucas Numbers , *J. Math. Comput. Sci*. 2 (3), 425-434, 2012.
- [13] S. Fiorini, MATH-F-307 Mathématiques discrètes. Version 2012.
- [14] D. Foata et GN. Han, Principe de combinatoire classique, Université Louis Pasteur, Strasbourg Département de mathématique, 2008.
- [15] A. F Horadam, Generating functions for powers of a certain generalized sequence of numbers, *Duke Math. J*. 32, 437– 446, 1965.
- [16] V. E. Hoggatt, Fibonacci and Lucas Numbers. A Publication of the Fibonacci Association. University of Santa Clara, Santa Clara. Houghton Mifflin Company. 1969.
- [17] A. F Horadam, Basic properties of a certain generalized sequence of numbers, *The Fibonacci Quarterly* 3 ,161–176, 1965.
- [18] A. F Horadam, Generating functions for powers of a certain generalized sequence of numbers, *Duke Math. J*. 32, 1965.

- [19] D. Jhala, K. Sisodiya and G. P. S. Rathore, On Some Identities for k -Jacobsthal Numbers, *Int. Journal of Math. Analysis.* 7 (12), 551 - 556, 2013.
- [20] T. Mansour A Formula for the Generating Functions of Powers of Horadam's Sequence, *Australas. J. Comb.* 30, 207-212, 2004.
- [21] T. Mansour. Squaring the terms of an l th order linear recurrence, *Australas. J. Comb.* 31, 15–20, 2005.
- [22] D. Marques, The Order of Appearance of the Product of Consecutive Lucas Numbers, *the Fibonacci Quarterly.* 51 (1), 38-43, 2013.
- [23] M. Merca, A Generalization of the Symmetry Between Complete and Elementary Symmetric Functions, *Indian J. Pure Appl. Math.* 45, 75-89, 2014.
- [24] K. H. Rosen, *Discrete Mathematics and its Applications*, Monmouth University (and formerly AT et T Laboratories).
- [25] N. Saba and A. Boussayoud¹, Complete homogeneous symmetric functions of Gauss Fibonacci polynomials and bivariate Pell polynomials, *Open Journal of Mathematical Sciences*, 2020.
- [26] N. Saba and A. Boussayoud¹, Ordinary generating functions of binary products of (p,q) -Modified-Pell numbers and k -numbers at positive and negative indices ,627-648, 2020.
- [27] M. Shattuck, Combinatorial Proofs of Determinant Formulas for the Fibonacci and Lucas Polynomials, *the Fibonacci Quarterly.* 51 (1), 63-71, 2013.
- [28] A. M. Sharari, *Introduction to Combinatorial Theory*, Saudi University Publications, Kingdom of Saudi Arabia.
- [29] S. Uygun, Bivariate Jacobsthal and Jacobsthal Lucas polynomial sequences, *J. Math. Computer Sci.*, 21, 176–185, 2020.
- [30] N. N. Vorobiov, *Números de Fibonacci*, Editora MIR, URSS. 1974.