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Some countably recognizable classes of groups

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DEDICATION

Yousra

I take this opportunity to dedicate this modest work: To my very dear parents; my father ″***Abdelouahab*** ″ and my mother″***Chahrazed***″; no dedication smiles to express the love, esteem, and respect that i have always had for you to my father, who is my best example and the source of my confidence throughout my life.

-To my mother, this angel fallen from heaven, to her who accompanies me in each of my steps; this work is the fruit of your sacrifices that you made for my education and training.

I dedicate this work to you with all my wishes for happiness, health and success: to my very dear brothers ″***Abdelmoain*** ″ and ″***Mohamed Amine*** ″ and to my very dear sisters ″***Zouleykha*** ″and ″***Hibet Elraheman*** ″.I wish you a future full of joy, happiness, success and serenity. To all the ″***BOUKEZOULA*** ″ family without exception.

-To my dear friends each of their name and in particular. -To my colleague ***Abla*** . -To my supervisor ***Dr.Daoui Amina*** .

DEDICATION

Abla

I thanks ***God*** first and foremost the the great grace that he has destowex upon me. To you my father and mother i dedicate the fruit of my effort and my dissertation. My ***dear mother*** you are my support and the spriny of tendernes . . . . My ***dear father***, i tell you thank you you are a hero and a symbol of patience and you made fun of all my needs without delay . . . , I remember very well the first day and my first steps towards school with you, you were my support, and now that little girl has grown up and will leave the classroom.

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ABSTRACT

This work aims to study some groups that can be identified countably recognizable.
We begin with mentioning some of the basic and preliminary concepts used in this work.
In the first chapter, we studied the basic concepts of group theory: groups, subgroups, isomorphisms, group actions and permutations.

In the second chapter, we found many properties of infinite groups. The aim of this chapter is to discuss the main types of generalized nilpotent and soluble groups, conjugacy classes and groups with finite conjugacy classes.

In the third chapter we studied the class of minimax groups in particular. We studied the paper of F.de Giovanni and M.Trombetti who shows that the classes of minimax groups is countably recognizable and they proved also that the properties of weak maximal and minimal conditions are countably recognizable.
Keywords : Groups, conjugacy classes, countably recognizable class, minimax group, maximal condition, minimal condition.

RÉSUMÉ

Ce travail vise à étudier quelques groupes qui peuvent être reconnaissable de façon dénombrable.
Nous commençons par mentionner quelques concepts de base et préliminaires utilisés dans ce travail.

Dans le premier chapitre nous avons étudié les concepts de base de la théorie des groupes: les groupes, sous-groupes, isomorphismes, actions de groupe et permutations.

Dans le deuxième chapitre nous avons trouvé de nombreuses propriétés des groupes infinis. le but de ce chapitre est de discuter des principaux types de groupes nilpotents et résolubles généralisés, des classes de conjugaison, et des groupes à classes de conjugaison finies.

Dans le troisième chapitre, nous avons étudié la classe des groupes minimax est reconnaissable de façon dénombrable.
Mote clés : Groupes, classes de conjugaison, classe reconnissable de façon dénombrable, groupe minimax, condition maximale, condition minimale.

ملخص

يهدف هذا العمل إلى دراسة بعض الزمر التي يمكن عدها.

نبدأ بذكر بعض المفاهيم الأساسية والأولية المستخدمة في هذا العمل.

درسنا في الفصل الأول المفاهيم الأساسية لنظرية الزمر: الزمر، الزمر الفرعية، التماثل، أفعال الزمر والتبديلات.

في الفصل الثاني وضعنا العديد من خصائص الزمر اللانهائية. الهدف من هذا الفصل هو التذكير بالانواع الرئيسية لزمر المعممة ذات قوة عادمة والقابلة للحلحلة، زمر الاقتران والزمر ذات زمر الاقتران المحدودة.

في الفصل الثالث درسنا تصنيف الزمرة ذات الحد الأدنى و الأقصى على وجه الخصوص درسنا مقال فرانشيسكو دي جيرفاني وماركو تروميتي بحيث قاما بتبيين أن الزمرة ذات الحد الأذى والأقصى يمكن عدها عليها و أيضا قاما برهان أن الشروط القصوى والدنيا الضعيفة يمكن عدها.

الكلمات المفتاحية: الزمر، طبقات الاقتران، طبقات يمكن عدها، الزمر ذات الحد الأقصى، الزمر ذات الحد الأدنى.

CONTENTS

NOTATIONS

GL(W)	group of nonsingular linear transformations of a vector space W.
GL(n,E)	general linear groups.
S_n	symmetric group of degree n.
$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$	sets of integers, rational numbers, real numbers, complex numbers.
G, H, . . .	group, subgroup.
$H \leqslant G$, $H < G$	H is a subgroup, a proper subgroup of the group G.
xH, Hx	the left and right coset H in G respectively.
|G|	the order of group G.
|G : H|	the index of a subgroup H in a group G.
$H \lhd G$	H is normal subgroup of G.
Hom(G,G′)	the sete of all homomorphisms frome G to G′.
$\frac{G}{H}$	the quotient of G over N.
$H \simeq G$	H is isomorphic with G.
$C_G(x)$, $N_G(x)$	the centralizers, normalizers of x in G.
$Orb_G(x)$	the orbit of x in G.
$St_G(x)$	the stabilizer of x in G.
$[x, y]$	the commutator of x and y.
G′, $G^{(1)}$	the derived subgroup of G.

Z(G)	the center of a group G.
$\mathfrak{X}$	a classe of groups.
rank(G)	the rank of G.
α, β, f, ...	functions.

INTRODUCTION

Group theory is a branch of modern algebra that deals with the study of groups. The origins of group theory back to the early 19^{th} century when mathematicians began to study the symmetry of geometric figures. The theory of groups and their properties in the context of algebraic equations was first developed by the French mathematician Evariste Galois in the 1830s. Another important figure in the development of group theory was the German mathematician Felix Klein who introduced the concept of a group in a more general setting in the 1870s. In the early 20^{th} century, the study of group theory was greatly expanded by the work of several mathematicians. Today, group theory is a major area of research in mathematics with applications in many areas of science and engineering, including quantum mechanics, chemistry, and cryptography. Group theory is also used in the study of symmetry in music, art, and architecture.

In this dissertation we study the countably recognizable classes of groups. A group class $\mathfrak{X}$ is said to be countably recognizable if, whenever all countable subgroups of a group G belong of X, then G itself is an $\mathfrak{X}$-group.

We give examples of countably recognizable groups classes: the class of minimax groups is countably recognizable. A group G is called minimax if it has a series of finite length :

$$\{1\} = G_0 < G_1 < \cdots < G_n = G$$

each of whose factors satisfies either the minimal or the maximal condition on subgroups. In this paper, it is shown that the class of minimax groups is countably recognizable.

Also, the class of soluble groups of finite rank is countably recognizable. There are many examples of countably recognizable classes of groups such as class of groups with finite conjugacy classes(FC-groups).

This dissertation is composed of three chapters presented as follows :

In chapter 1, we study the fundamental concepts of group theory including groups, subgroups, normal subgroups, costs, isomorphisms, isomorphisms theorem, groups action and permutation. We give examples of groups; groups of numbers, groups of linear transformations, groups of permutation groups of matrices. We prove ($\mathbb{N}$,+) and ($\mathbb{Z}$,$\times$) are not groups.

In chapter 2, notions needed for our study, namely; soluble and nilpotent groups, conjugacy classes and centralizers and groups with finite conjugacy classes which has a role in our study of group theory.

This chapter consists of four sections. The first section, which is dedicated to soluble group, will be further explained by : compositions series, Jordan Hölder theorem, Derived subgroup. In the second section, we are going to study the nilpotent groups going through some notions : central series (descending and ascending central series). In the third section, we will study conjugacy classes and centralizers. We will further explain conjugacy classes, centralizers and normalizers. In section four we are going to study the groups with finite conjugacy classes going through some notions : FC-element and FC-group.

In the last chapter, we study the article of F. de Giovanniand Marco Trombetti "The class of minimax groups is countably recognizable".

This chapter consists of two sections. In the first section, we talk about free groups and reduced words. Then, we talk about rank of groups. We also give an example of rank of groups. In the second section, we study that the class of minimax groups is countably recognizable taking into consideration weak minimal and maximal conditions.

CHAPTER 1

FUNDAMENTAL CONCEPTS OF GROUP THEORY

This chapter is devoted to studying the essential concepts of group theory, which will be used all throughout the next chapters, in terms of definitions, exemples, theorems, propositions and remarks [12].

1.1 Groups

1.1.1 Binary operations

Definition 1.1.1 *Let S is a nonempty set, a binary operation $''o''$ on S is a function :*

$$o : S \times S \longrightarrow S$$
$$(x, y) \longrightarrow xoy$$

1.1.2 Groups

Definition 1.1.2 *A group is a set S together with a binary operation satisfying the following conditions :*

i) Associativity *: For all $x, y, z \in S$*

$$(xoy)oz = xo(yoz).$$

ii) Existence an indentity element *: There exists an element $e \in S$, Such that*

$$xoe = x = eox.$$

For all $x \in S$

iii) Existence of inverses *: For each element $x \in S$, there existe $x^{-1} \in S$, Such that*

$$xox^{-1} = e = x^{-1}ox.$$

We write x.y for xoy and 1 *For e, also we write $x + y$ for xoy and 0 for e. In the first case, the groups is said to be multiplicative, and in the second it is said to be additive.*
We say that the group S is abelian if : for all x,y $\in$ S, we have :

$$xoy = yox.$$

1.2 Examples of groups

1.2.1 Groups of numbers

$(\mathbb{Z}, +), (\mathbb{Q}, +), (\mathbb{R}, +)$ and $(\mathbb{C}, +)$ are abelian groups. Infact 0 being an identity and $-x$ is the inverse of x .
$\mathbb{Z}$ is the set of all integers, $\mathbb{Q}$: rational numbers , $\mathbb{R}$: real numbers, $\mathbb{C}$: complex numbers.
Let $(\mathbb{Q} - \{0\}, \times)$, $(\mathbb{R} - \{0\}, \times)$, $(\mathbb{C} - \{0\}, \times)$, are abelian groups, 1 being the identity and x^{-1} is the inverse of x.

1.2.2 Groups of linear transformations

If W is an n-dimensional vector space over a field E, let GL(W) denote the set of all bijective linear transformation of W.
Then GL(V) is a group with functional composition is the group operation :

$$\alpha o\beta(w) = \alpha(\beta(w)) \text{ where } w \in W \text{ and } \alpha, \beta \in GL(W).$$

There is a close connection between the groups GL(W) and GL(n,E) . For if a fixed ordered basis for W is chosen, each bijective linear transformation of W is associated with a nonsingular $n * n$ matrix over E. This correspondence is :

$$GL(W) \backsimeq GL(n, E).$$

Infact when two linear transformations are composed, the product of the corresponding matrices represents the composite.

1.2.3 Groups of permutation

Let A is a nonempty set, a bijection $\phi : A \longrightarrow A$ is called a permutation of A.
If $A = \{0, 1, 2, ..., n\}$ then Sym(A) is denoted by S_n , and called the symmetric group of degree n.

$$S_n = \{\phi : \{0, 1, ...n\} \longrightarrow \{0, 1, ...n\}\}.$$

$|S_n| = n!$
an element $\sigma \in S_n$ is written : $\begin{pmatrix} 1 & 2 & .....n \\ \sigma(1) & \sigma(2) & .....\sigma(n) \end{pmatrix}$ The signature of permutation $\phi \in S_n$ is defined to be :

$$sign\phi = \prod_{1 \leq i < j \leq n} \frac{\phi(i) - \phi(j)}{i - j}.$$

Which equal +1 or −1. Recall that ϕ is even if sign $\phi = +1$ and odd if sign $\phi = -1$ From the definition it is easy to check the formulas :

$$sign(\phi_1\phi_2) = (sign\phi_1)(sign\phi_2) \text{ and } sign(\phi^{-1}) = sign\phi.$$

1.2.4 Groups of matrices

Let V be a ring with an identity element and let GL(n,V) denote the set of all $n \times n$ matrices with coefficients in V which have inverses. Taking matrix multiplication as the group operation, we see from elementary properties of matrices that GL(n,V) is a group whose identity element is 1_n, the $n \times n$ identity matrix. This group is called the general linear group of degree n over V, it is nonabelian .
If $n > 1$ in particular, if E is a field, GL(n,E) is the group of all nonsingular $n \times n$ matrices over E.

1.3 Counter-example of groups

1.3.1 ($\mathbb{N}$,+)

is not a group because :

i) For all $a, b, c \in N$:

$$(a + b) + c = a + (b + c).$$

So + is associative.

ii) For all $a \in \mathbb{N}$:

$$a + 0 = 0 + a = a.$$

0 is indentity element of $\mathbb{N}$.

iii) For all $a \in \mathbb{N}$:

$$\nexists\, a^{-1} \in \mathbb{N} : a + a^{-1} = 0.$$

So there is no inverse .

1.3.2 $(\mathbb{Z},\times)$

is not a group because :

i) For all $a, b, c \in \mathbb{Z}$:

$$(a \times b) \times c = a \times (b \times c).$$

So $\times$ is associative.

ii) For all $a \in \mathbb{Z}$:

$$a \times 1 = 1 \times a = a.$$

0 is indentity element of $\mathbb{Z}$.

iii) For all $a \in \mathbb{Z}$:

$$\nexists\, \frac{1}{a} \in \mathbb{Z}.$$

So there is no inverse.

1.4 Subgroups

Definition 1.4.1 *Let G be a group and let H be a subset of G, we say that H is a subgroup of G (G and H nonempty) if :*

i) (H,.) is closed under the groupe operation of G, ie : $\forall a, b \in H\ a.b \in H$

ii) **Inverses** *: if $a \in H$, then $a^{-1} \in H$.*
We shal write :

$$H \leq G.$$

Remark 1.4.1 *If $H \leq G$ and $H \neq G$, we say H is subgroup propre of G. We shall write :*

$$H < G.$$

1.4.1 Examples of subgroups

1) $\mathbb{R}$, $\mathbb{Q}$ *and* $\mathbb{Z}$ *are subgroups of* $\mathbb{C}$.

2) $G = (\mathbb{Z}, +)$ *and* $H = ((n\mathbb{Z}), +)$.
So H is a subgroup of G.

1.4.2 Left and right cosets

Definition 1.4.2 *If H is subgroup of a group G, then for an element $x \in G$, the left coset of H in G is subset defined by :*

$$xH = \{xh/h \in H\}.$$

in a precisely similar way the right coset :

$$Hx = \{hx/h \in H\}.$$

is called right coset H in G.

Definition 1.4.3 *The order of a group G is the cardinality of that group or the order of a group G is the number of its elements.*

Notation 1.4.1 *The order of a group G is denoted by :* $|G|$.

Definition 1.4.4 *The index of a subgroup H in a group G is the number of right and left cosets xH and Hx (resp). We denote it :*

$$|G : H|$$

1.4.3 Normal subgroup

Definition 1.4.5 *Let G be a group and H be a subgroup of G. A subgroup H is said to be normal subgroup if for every* $a \in G$ *and* $h \in H$*; We have* $aha^{-1} \in H$
H is normal subgroup of G is denoted by : $H \lhd G$.

Proposition 1.4.1 *A subgroup of index 2 is a normal subgroup.*

Proof. *Let G be a group* $H \leq G$ *an index* $= 2$ *then :*
$[G : H] = 2 \Longleftrightarrow |xH| = |Hx| = 2$
$Hx = \{H, \bar{x}\} = \{H,\ Hx\} = G$
$xH = \{H, \bar{x}\} = \{H,\ xH\} = G$
$Hx = xH$
$H \lhd G$ ■

1.5 Isomorphisms

1.5.1 Homomorphisms

Definition 1.5.1 *Let G and* G' *two groups. A function* $\varphi : G \longrightarrow G'$ *is a homomorphisms if :*

$$\varphi(x, y) = \varphi(x).\varphi(y).$$

For all $x, y \in G$.

The set of all homomorphisms from G to $G^{'}$ is denoted by : Hom (G, $G^{'}$).

Remark 1.5.1

1) *Hom (G, $G^{'}$) is always nonempty, it has at least the 0 homomorphisms :*

$$\begin{aligned} 0 : G &\longrightarrow G^{'} \\ x &\longrightarrow eH \end{aligned}$$

2) *A homomorphisms $\varphi : G \longrightarrow G$ is called an endomorphism of G . The identity function $1 : G \longrightarrow G$ is clearly an endomorphism.*

1.5.2 Kernel of homomorphisms

The kernel ker φ of a homomorphism $\varphi : G \longrightarrow G^{'}$ is subsets defined as follows :

$$\ker \varphi = \{x \in G, \varphi(x) = 1_{G'}\}.$$

1.5.3 Examples of Homomorphisms

i) $\varphi : GL(n, F) \longrightarrow F^{\star}$ *where* $\varphi(A) = \det A$ *and* $F^{\star} = F \setminus \{0\}$. *Here F is a field.*

ii) $\varphi : \mathbb{R} \longrightarrow \mathbb{R}^{+}$ $(\mathbb{R}, +), (\mathbb{R}^{+}, \times)$
$x \longrightarrow e^{X}$
$\varphi(X + Y) = \varphi(X) \times \varphi(Y) \Longleftrightarrow e^{X+Y} = e^{X} \times e^{Y}$.

Proposition 1.5.1

1) $\varphi : G \longrightarrow G^{'}$ *homomorphisms, so* $\varphi(x^{n}) = (\varphi(x))^{n}$*, for all* $x \in G$.

2) $\varphi(x^{-1}) = (\varphi(x))^{-1}$, *for all* $x \in G$.

3) $Im(\varphi) \leqslant G^{'}$, $\ker\varphi \lhd G$.

Proposition 1.5.2 *Let* $\varphi : G \longrightarrow G^{'}$ *be a homomorphisms , monomorphisms , epimorphisms , and isomorphisms :*

1. φ *is an monomorphisms* $\Longleftrightarrow \begin{cases} \varphi \text{ is homomorphisms} \\ \text{and} \\ \varphi \text{ injective} \end{cases}$

2. φ *is an epimorphisms* $\Longleftrightarrow \begin{cases} \varphi \text{ is homomorphisms} \\ \text{and} \\ \varphi \text{ surjective} \end{cases}$

3. φ *is an isomorphisms* $\Longleftrightarrow \begin{cases} \varphi \text{ is homomorphisms} \\ \text{and} \\ \varphi \text{ bijective} \end{cases}$

Remark 1.5.2 *Let* $\varphi \in$ *Hom(G,G′) :*

i) φ *is monomorphism* $\Longleftrightarrow \ker\varphi = 1_G$.

ii) φ *is epimorphism* $\Longleftrightarrow Im\ \varphi = G'$.

iii) φ *is isomorphism* $\Longleftrightarrow \ker\varphi = 1_G$ *and* $Im\ \varphi = G'$.

Definition 1.5.2 *If N is a normal subgroup of a group G* ($N \lhd G$); $\dfrac{G}{H}$ *is the quotient group of N in G, is the set of all cosete of N in G equipped with the group operation :*

$$(Nx).(Ny)=N(x.y).$$

1.5.4 Isomorphisme theorems

[12]

1) ***First isomorphism theorem***

i) If $\varphi : G \longrightarrow G'$ is a homomorphisme of groups, the mapping :

$$\psi : \frac{G}{\ker \varphi} \longrightarrow Im\varphi.$$

is an isomorphism, that is :

$$\frac{G}{\ker \varphi} \simeq Im\psi.$$

ii) If G is a groups and $N \lhd G$, the mapping $s : G \longrightarrow \frac{G}{N}$ is an epimorphism frome G to $\frac{G}{N}$ with kernel N
this s is called natural or canonical homomorphisme.

2) ***Second isomorphism theorem***

Let G be a groups and $H \leq G$, $H \lhd G$, then $N \cap H \lhd H$ and :
$\varphi : \frac{H}{N \cap H} \longrightarrow \frac{NH}{N}$ is an isomorphisms

that is:

$$\frac{H}{N \cap H} \simeq \frac{NH}{N}$$

3) ***Third isomorphism theorem***

Let G be a group , $N \lhd G$ and $M \lhd G$ and let $N \leq M$. Then $\frac{M}{N} \lhd \frac{G}{N}$ and we have:

$$\frac{\frac{G}{N}}{\frac{M}{N}} \simeq \frac{G}{M}$$

1.6 Group actions and permutations

1.6.1 Left group action

We say that a group G oparates on the set X on the left if there exists a function :

$$f : G \times X \longrightarrow X$$
$$(a, x) \longrightarrow ax$$

Such that :

i) ***f associativ*** : $\forall a, b \in G, \forall x \in X$

$$a(bx) = (ab)x.$$

ii) ***f hav an indentity element*** : $\forall x \in X$

$$1x = x.$$

f is called left action.

1.6.2 Right group action

We say that a group G oparates on the set X on the right if there exists a function :

$$f : X \times G \longrightarrow X$$
$$(x, a) \longrightarrow xa$$

Such that :

i) ***f associativ*** : $\forall a, b \in G, \forall x \in X$

$$(xa)b = x(ab).$$

ii) ***f hav an indentity element*** : $\forall x \in X$

$$x1 = x.$$

f is called right action.

1.6.3 Permutation groups

Definition 1.6.1 *If X a nonempty set, a subgroup G of the symmetric group Sym X is called a permutation group on X.*
The degree of the permutation group is the cardinality of X, if the group G operates on X the relation is defined by :

$$X\mathfrak{R}_G Y \Longleftrightarrow (\text{ there exists } a \in G \text{ such that : } y = a.x \text{ }).$$

is an equivalence relation, that is :

i) $x\mathfrak{R}_G x \Longrightarrow x = e.x$

ii) $x\mathfrak{R}_G y \Longrightarrow y = a.x \Longrightarrow a^{-1}y = a^{-1}(ax) = ex = x \Longrightarrow y\mathfrak{R}_G x$

iii) $x\mathfrak{R}_G y$ *and* $x\mathfrak{R}_G z \Longrightarrow y = a.x$ *and* $z = b.y \Longrightarrow z = b.a.x = (b.a)x \Longrightarrow x\mathfrak{R}_G z$
The equivalence class of $x \in X$ *for* $\mathfrak{R}_G$ *is :*

$$G_x = \{a.x/a \in G\}$$

1.6.4 Orbits

Definition 1.6.2 *Let G be a group operates on a set X. For each $x \in G$:*

$$Orb_G(x) = \{ax/a \in G$$

the orbit of x under the action of G.

1.6.5 Stabilizer

Definition 1.6.3 *Let G be a group operates on a set X. For each $a \in X$:*

$$St_G(x) = \{a \in G/ax = x\}.$$

The stabilizer of $x \in G$.

Remark 1.6.1 *The stabilizer St_x of $x \in X$ is a subgroup of G.*

CHAPTER 2

SOLUBLE AND NILPOTENT GROUPS

In chapter 2, notions needed for our study, namely; soluble and niplotent groups, Conjugacy classes and centralizers and Groups with finite conjugacy classes which has a role in studies of group theory will be discussed [12].

2.1 Soluble groups

2.1.1 Composition series

Definition 2.1.1 *Let G be a group, a composition series for G of length n is a chain of subgroups H_i :*

$$G = H_1 > H_2 > \ldots > H_i = e.$$

Such that : $H_i \lhd H_{i-1}, \forall i = 1, \ldots, n.$

Definition 2.1.2 *Let Σ and Σ' be two compositions series of G:*

$$\Sigma : G = G_0 \geqslant G_1 \geqslant \cdots \geqslant G_n = (e)$$
$$\Sigma^{'} : G = K_0 \geqslant K_1 \geqslant \cdots \geqslant K_p = (e).$$

1) *We say that $\Sigma^{'}$ is a refinement of Σ, if $p \geqslant n$ and if the chain Σ is extracted from $\Sigma^{'}$ i.e: If there exists n positive integer:*

$$j_0 < j_1 < \cdots < j_n \leqslant p.$$

Such that, for all i $(0 \leqslant i \leqslant n)$ $G_i = K_{ji}$ we can then write : $\Sigma \subseteq \Sigma^{'}$.
If there exists an integer $j \in \{0, 1, \cdots\cdots, p\}$ such that : $K_j \neq G_i$, $\forall\ i\ (0 \leqslant i \leqslant n)$, we say that $\Sigma^{'}$ is a proper refinement of Σ in this case, we necessarily have $p > n$ and we will write: $\Sigma \subset \Sigma^{'}$.

2) *We say that the compositions series Σ and $\Sigma^{'}$ are equivalent if $n = p$ and if there exists a permutation σ of the integers $0, 1, 2, \cdots\cdots\cdots, n-1$, such that for all i $(0 \leqslant i \leqslant n-1)$:*

$$\frac{G_i}{G_{i+1}} \simeq \frac{K_{\sigma(i)}}{K_{\sigma(i)+1}}.$$

We will express the equivalence of the two chain of composition by the notation: $\Sigma \sim \Sigma^{'}$

Remark 2.1.1 *Any chain extracted from a composition series is not in general a composition series in fact the normality condition does not still hold.*

Definition 2.1.3 *A group is said to be simple if it does not admit any normal subgroup other than itself and the identity element.*

2.1.2 Jordan Hölder theorem

Definition 2.1.4 *A composition series of a group G will be called a Jordan-Höder chain if all the quotients of the chain are simple groups.*
This name will be justified by the Jordan-Höder theorem.

Proposition 2.1.1 *A composition series of G is a chain of Jordan-Höder iff it is strictly decreasing and does not admit any proper refinement.*

Proof. *Let* $\Sigma : G = G_0 \geqslant G_1 \geqslant \cdots \geqslant G_i \geqslant G_{i+1} \geqslant \cdots \geqslant G_n = (e)$
$\dfrac{G_i}{G_{i+1}}$ *simple* $\Longleftrightarrow G_{i+1} \neq G_i$ *and* G_{i+1} *normal and maximal in* $G_i \Longleftrightarrow \forall i \in \overline{0, n-1}, \nexists$ *the subgroup H such that:*

$$G_i > H > G_{i-1} \text{ and } H \lhd G_i.$$

■

Remark 2.1.2

1) *Any simple group G admits a unique chain of strictly decreasing composition, which is a chain of Jordan-Höder:*

$$G = G_0 > G_1 = (e).$$

2) *Any chain of strictly decreasing composition of G admits a refinement which is a chain of Jordan-Höder.*

3) *Any two Jordan-Höder chain of G are equivalent.*

2.1.3 Derived subgroup

Definition 2.1.5 *Let G be a group, the commutator of two elements* $x, y \in G$ *is defined by :*

$$[x, y] = x^{-1}y^{-1}xy.$$

Remark 2.1.3 *G is abelian group if only if every commutator equals to 1.*
Infact :
$[x, y] = x^{-1}y^{-1}xy = 1 \Leftrightarrow yxx^{-1}y^{-1}xy = yx \Leftrightarrow xy = yx.$

Definition 2.1.6 *Let G be a group, we call a derived subgroup of G the subgroup generated by all commutators in G.*
We denote : G' or $G^{(1)}$.
so :

$$G^{(1)} =< [x, y]; x, y \in G >$$

Theorem 2.1.4 *Let G be a group; we have :*

1) $G' \lhd G$.
2) $N \lhd G$, *so* $\dfrac{G}{N}$ *is ablelian group if only if* $G' \subseteq N$ *;*

Proof. *:*

1) *Let* $x, y, z \in G$ *:*
   we have :

$$z^{-1}[x, y]z = z^{-1}x^{-1}y^{-1}zxy$$

   Then :

$$[z^{-1}xz, z^{-1}yz] = z^{-1}x^{-1}zz^{-1}y^{-1}zz^{-1}xzz^{-1}yz = z^{-1}x^{-1}y^{-1}xyz$$

   hence :

$$z^{-1}[x, y]z = [z^{-1}xz, z^{-1}yz]$$

   So G' is normal in G ($G' \lhd G$).

2) *Let* $x, y \in G$ *and* $\bar{x}, \bar{y} \in \dfrac{G}{N}$ *:*
   $\dfrac{G}{N}$ *abelian* $\Leftrightarrow \bar{x}\bar{y} = \bar{y}\bar{x}$,
   $\Leftrightarrow (\overline{yx})^{-1}\overline{xy} = N$
   $\Leftrightarrow (yx)^{-1}xy \in N$
   $\Leftrightarrow x^{-1}y^{-1}xy \in N$
   $\Leftrightarrow$ *[x,y]* $\in N$ *hence* $\dfrac{G}{N}$ *abelian* $\Leftrightarrow G' \subseteq N$.

■

Definition 2.1.7 *A group G is said to be soluble if there exists an integer $n \geq 0$, such that :*

$$G^{(n)} = (e).$$

Remark 2.1.5 *If $G \neq (e)$ is a soluble group, and if n is the smallest positive integer such that $G^{(n)} = (e)$, so $\forall\ i\ (0 \leq i \leq n-1)$, we have : $G^{(i+1)} < G^{(i)}$.*
and by theorem (2.1.4) $G^{(i+1)} \lhd G^{(i)}$, we have :

$$G = G^{(0)} > G^{(1)} > \cdots > G^{(n)} = (e).$$

is a composition series of G.

Theorem 2.1.6 *If G is a soluble group, so every subgroup of G and every quotient of G is soluble.*

Proof. *Suppose that : $\exists\ n \in \mathbb{N}$, such that $G^{(n)} = (e)$.*
$*$ Let H be a subgroup of G.

$$H \leq G \Rightarrow H' \leq G'$$
$$H' \leq G' \Rightarrow H^{(2)} \leq G^{(2)}$$

Then we obtain $H^{(n)} \leq G^{(n)} = (e)$, hence $H^{(n)} = (e)$ is soluble.
$*$ Let $N \lhd G$, and let $\pi : G \longrightarrow \dfrac{N}{G}$ an canonical epimorphisms such that $\pi(x) = \bar{x}$, $\forall x \in G$.
as all commutator of $\dfrac{N}{G}$ is an image by π of a commutator in G.
Obvirously that $\left(\dfrac{G}{N}\right)' = \pi(G')$.
Like wise, on has $\left(\dfrac{G}{N}\right)^{(2)} = \pi(G^{(2)})$ and step by step we obtain :

$$\left(\frac{G}{N}\right)^{(n)} = \pi((G)^n) = (\bar{e}).$$

hence $\dfrac{G}{N}$ is soluble. ■

Definition 2.1.8 *G is said to be soluble iff G admits an abelian series (ie all quotients are abelian).*

2.2 Nilpotent groups

2.2.1 Central series

Definition 2.2.1 *The center of a group G, is denoted by Z(G) such that :*

$$Z(G) = \{a \in G/a.b = b.a, \forall b \in G\}.$$

Definition 2.2.2 *For a group G, a composition series :*

$$\Sigma : G = G_1 \geq G_2 \geq \ldots > G_{n-1} \geq G_n = (e).$$

is a central series, if it is a normal series and :

$$\frac{G_i}{G_{i+1}} \leqslant Z\left(\frac{G}{G_{i+1}}\right)$$

for all $(0 \leqslant i \leqslant n-1)$

Proposition 2.2.1 *The series Σ is cental iff $\forall i \in \overline{0,1}$:* $[G_i, G] \leq G_{i+1}$

Proof.
• $\Rightarrow$) *Suppose Σ is central $\Rightarrow G_{i+1} \lhd G$ and $\dfrac{G_i}{G_{i+1}} \leqslant Z\left(\dfrac{G}{G_{i+1}}\right)$*
Let $x \in G_i$, $g \in G \Rightarrow \bar{x} \in \dfrac{G_i}{G_{i+1}}$ and $\bar{y} \in \dfrac{G}{G_{i+1}}$ while $\Longleftrightarrow [\bar{x}, \bar{y}] = G_{i+1}$
hence : $[G_i, G] \leq G_{i+1}$.
• $\Leftarrow$) *suppose :*

$$[G_i, G] \leq G_{i+1} \tag{2.1}$$

Let $x \in G_{i+1}$ and $g \in G$, we have :

$G_{i+1} \leqslant G_i \Rightarrow x \in G_i$

moreover; according to (2.1) $\Rightarrow [x,g] = x^{-1}g^{-1}xg \in G_{i+1} \Rightarrow g^{-1}xg \in G_{i+1}$

hence $G_{i+1} \lhd G$

And hence $[x,g] \in G_{i+1} \Longleftrightarrow [\bar{x}, \bar{g}] = G_{i+1} \Longleftrightarrow \bar{x}\bar{g} = \bar{g}\bar{x}$
hence $\dfrac{G_i}{G_{i+1}} \leqslant Z\left(\dfrac{G}{G_{i+1}}\right)$ ■

2.2.2 Nilpotence

Definition 2.2.3 *A group G is nilpotent if G have a central series.*

Proposition 2.2.2 *Every nilpotent group is soluble.*

Proof. *lf G is nilpotent group $\Rightarrow$ G have a central series is normal series in all quotients are :*

$$\frac{G_i}{G_{i+1}} \leqslant Z\left(\frac{G}{G_{i+1}}\right).$$

While all quotients abelien hence G is soluble. ■

2.2.3 Descending central series

G being a group, let :

$\gamma_1(G) = G$, $\gamma_2(G) = [G, G] = G'$, $\gamma_3(G) = [\gamma_2(G), G]$

In general :

$$\gamma_{k+1}(G) = [\gamma_k(G), G]\ \forall k \in N^*.$$

By recurrence on k, we prove that :

$$\gamma_{k+1}(G) \leq \gamma_k(G),\ \forall k \in N^*$$

For k = 1, we have $\Rightarrow \gamma_2(G) = [G, G] = G' \Rightarrow G' \lhd G$, for $k \geq 1$ we have γ_{k+1}(G) is engender the elements of the form [x,y];(such that : $y \in \gamma_k(G)$, $x \in G$)

It is enough to show :

$$z^{-1}[x, y]z \in \gamma_{k+1}(G), \textit{for } z \in G$$

Then :

$z^{-1}[x,y]z = [z^{-1}yz, z^{-1}xz] \in \gamma_{k+1}(G)$

Hence : $\gamma_k(G) \lhd G$, *for* $k1$

Moreover the series $(\gamma_k(G))_{k\geq 1}$ *is descending for inclusion, infact for* $x \in G$, $y \in \gamma_k(G)$ *we have :*

$$[x,y] = y^{-1}(x^{-1}yx) \in \gamma_k(G),$$

hence $\gamma_k(G) \leq \gamma_{k+1}(G)\ \forall k \in N^*$

Thus, for all group G, we define the descending chain of subgroups :

$$G = \gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_k \geq \gamma_{k+1} \geq \ldots. \tag{2.2}$$

Definition 2.2.4 *A group G has a descending central serie of length n ($n \geq 1$ in $\mathbb{N}$), if the descending chain (2.2) is written :*

$$G = \gamma_1 > \gamma_2 > \ldots > \gamma_n > \gamma_{n+1} >= (e). \tag{2.3}$$

In this case, the group G is nilpotent.

Definition 2.2.5 *Let G be a group* $H \leqslant G$, *H is said to be characteristic if :* $\forall f \in Aut(G)$: $f(H) = H$ *and we denoted* $H \sqsubset G$.

($f \in Aut(G)$*ie :* $f : G \longrightarrow G$ *isomorphism).*

2.2.4 Ascending central series

G being a group, let $Z_0 = (e)$, $Z_1 = (G)$.

We know that Z_1 *is characteristic in G, hence the unique subgroup* Z_2 *of G, containing* Z_1*, such that* $\dfrac{Z_2}{Z_1} = Z\left(\dfrac{G}{Z_1}\right)$*, is characteristic in G.*

Step by step, we define for all $i \in N$*, the subgroup* Z_{i+1} *such that:*

$$\frac{Z_{i+1}}{Z_i} = Z\left(\frac{G}{Z_i}\right).$$

And Z_{i+1} is characteristic in G.
Thus we determine an ascending chain of subgroups of G :

$$(e) = Z_0 \leqslant Z_1 \leqslant \cdots \leqslant Z_i \leqslant Z_{i+1} \leqslant \cdots$$

In which, for all $i \in N$, we have $Z_i \sqsubset G$, so $Z_i \lhd G$ and Z_{i+1} is the largest subgroup of G containing Z, such that:

$$[Z_{i+1}, G] \leqslant Z_i.$$

If there is an integer $s \geqslant 0$ such that $Z_s = G$ the series is written:

$$(e) = Z_0 \leqslant Z_1 \leqslant \cdots \leqslant Z_{s-1} \leqslant Z_s = G.$$

and from above, the chain is a central sequence of G.
Moreover, if $G \neq (e)$ and if s is the smallest positive integer such that $Z_s = G$, then for all $i(0 \leqslant i \leqslant s-1)$ we have $Z_i < Z_{i+1}$, infact $Z_i = Z_{i+1}$ implies $Z_j = Z_i$ whatever $j \geqslant i$.

Definition 2.2.6 *A group G is said to have an ascending central series of length s $(s \geqslant 1)$, if the ascending chain is written:*

$$(e) = Z_0 < Z_1 < \cdots < Z_{s-1} < Z_s = G.$$

Then the group G is nilpotent.

Remark 2.2.1 *The group $G = (e)$ will be considered as having a descending (respectively, ascending) central series of length 0.*

2.3 Conjugacy classes and centralizers

2.3.1 Conjugacy classes

Definition 2.3.1 *Let G be a group, two elements x and h in G are conjugate when :*

$$h = gxg^{-1}.$$

For all $g \in G$

If X is a nonempty set and g is an element of group G, the conjugacy class of X by G, denoted X^g is the set :

$$X^g = gXg^{-1} = \{gxg^{-1}, x \in X\}.$$

Lemma 2.3.1 *Conjugacy is an equivalence relation.*

Proof.

i) **Reflexive***:Let $h \in G$, $h = ehe^{-1}$, for all $e \in G$.*

ii) **Symetrice***:*
Suppose $\exists\, x \in G$ such that : $h = gxg^{-1} \Rightarrow g^{-1}h = xg^{-1} \Rightarrow g^{-1}hg = xg^{-1}g \Rightarrow g^{-1}hg = x$
So $x = g^{-1}hg$; $\forall g, h \in G$.

iii) **Transitive***:*
Suppose $\exists\, x, y \in G$ such that : $h = gxg^{-1}$ and $x = lyl^{-1}$; $\forall h, l, y, x \in G$
$h = gxg^{-1} \Rightarrow h = glyl^{-1}g^{-1} \Rightarrow h = gly(gl)^{-1}$

■

Remark 2.3.1 *If H is a subgroup of G, all conjugacy classes of H; $H = xHx^{-1}$ is a subgroup of G.*

2.3.2 Centralizers

The centralizer of x in G, for all $x \in G$:

$$C_G(x) = \{g \in G, gx = xg\}.$$

Hence $|G : C_G(x)|$ = the cardinlety of the conjgacy class of x. Also $\{x\}$ is a conjugcy class if and only if x belongs in the center of G.

2.3.3 Normalizers

$$N_G(x) = \{g \in G, gxg^{-1} = x\}.$$

Is the normalizers of x in G.

Theorem 2.3.2 *Let H be a subgroup of group G, we have :*

$$H \lhd G \Leftrightarrow N_G(H) = G.$$

Proposition 2.3.1 *Let G be a group :*

1) *For all subgroup H of G, we have $H \lhd N_G(H)$.*
2) *If H and K are two subgroup of G and hence $H \leq K$, then :*
   $H \lhd K \Longrightarrow K \leq N_G(H)$.
3) $K \leq N_G(H) \Longrightarrow HK \leq G$ *and* $H \lhd HK$.

Proof.

1) *We remark that the definition of $N_G(H)$ implies $H \subseteq N_G(H)$, then :*

$$x \in N_G(H) \Longleftrightarrow xHx^{-1} = H$$

hence $H \lhd N_G(H)$.

2) *If $H \leq k \leq G$, then :*

$$H \rhd K \Longrightarrow kHk^{-1} = H, \forall k \in K.$$

hence $k \leq N_G(H)$.

3) *We have $k \leq N_G(H)$; consider $h \in H$ and $k \in K$, we have $k \in N_G(H)$ and hence $k^{-1} \in N_G(H)$, then $k^{-1}Hk = H$, we deduce that there exists $h' \in H$ such that $k^{-1}Hk = h'$, hence $hk = kh'$, which implies $HK \subseteq KH$.*
In a smilar way, we show that $HK \subseteq KH$, so Hk is subgroup of G
$H \subseteq N_G(H)$ and $K \subseteq N_G(H)$ implies $HK \subseteq N_G(H)$, Then by 1) H is normal in $N_G(H)$, which is a subgroup of HK, is normal in HK.

■

Notation 2.3.1 *For all $H \leq G$, $N_G(H)$ is the largest subgroup of G in which H is normal.*

2.4 Groups with finite conjugacy classes

Definition 2.4.1 *A torsion group or a periodic group is a group in which every element has a finite order. The exponent of such a group, if there exists is the least common multiple of the orders of the elements. The exponent exists for any finite group and it divides the order of the group.*

Remark 2.4.1 *A torsion-free group is a group whose the only element of finite order is the identity.*

2.4.1 FC-element

Definition 2.4.2 *An element g of a group G is called an FC-element if it has only a finite number of conjugates in G, that is to say if $|G\colon C_G(g)|$ is finite. It is a basic fact that the FC-element always form a subgroup.*

Proposition 2.4.1 *In any group G the FC-elements form a characteristic subgroup.*

Proof. *Let g and h be FC-elements of G.*
Then $C_G(g)$ and $C_G(h)$ have finite index, which implies that $C_G(g) \frown C_G(h)$ has finite index. But obviously $C_G(gh^{-1}) \geq C_G(g) \frown C_G(h)$, so $C_G(gh^{-1})$ has finite index and (gh^{-1}) is an FC-element. Thus the FC-elements form a subgroup. If $\beta \in AutG$, then $C_G(g^\beta) = C_G(g)^\beta$, from which it follows that $C_G(g)^\beta$ has finite index. Hence g^β is an FC-element. ∎

2.4.2 FC-group

Definition 2.4.3 *A group G is called an FC-group if it equals its FC-center, that is every conjugacy class of G is finite.*
In particular all abelian groups and all finite groups are FC-groups.

Proposition 2.4.2

1) *In any group G a finite normal subset consisting of element of finite order generates a finite normal subgroup.*

2) *A torsion group G is an FC-group if and only if each finite subset is contained in a finite normal subgroup.*

Proof.

1) *Let $X = \{x_1, x_2, \cdots, x_n\}$ be the normal subset and let $H = \langle X \rangle$. Obviously H is normal in G: we have to prove that it is finite.*

If $1 \neq h \in H$, then $h = x_{\alpha_1}^{m_1} \cdots x_{\alpha_r}^{m_r}$ where $1 \leq \alpha_i \leq n$. In general there will be many such expressions for h, among them some of shortest length, say r.
Furthermore among these expressions of shortest length there is one which appears first in the lexicographic ordering of r-tuples: this is the ordering in which $(\alpha_1, \ldots, \alpha_r)$ precedes $(\alpha_1^{'}, \ldots, \alpha_r^{'})$ if $\alpha_i = \alpha_i^{'}$ for $i < s$ and $\alpha_s < \alpha_s^{'}$ for some $s \leq r$. Denote this first expression by $h = y_1 y_2 \cdots y_r$ where $y_i = x_{\alpha_i}^{m_i}$.
Suppose that $\alpha_i = \alpha_j$ where $i < j$. Moving y_j to the left we obtain

$$h = y_1 \cdots y_{i-1}(y_i y_j) y_{j+1}^{y_j} \cdots y_{j-1}^{y_j} y_{j+1} \cdots y_r,$$

an expression of length less than r. Consequently the α_i are all different.
Now assume that $\alpha_i > \alpha_{i+1}$; then

$$h = y_1 \cdots y_{i-1} y_{i+1} y_i^{y_{i+1}} y_{i+2} \cdots y_r.$$

But this expression of length r precedes $y_1 y_2 \cdots y_r$ in the ordering of r-tuples.
Hence $\alpha_1 < \alpha_2 < \cdots < \alpha_r$. It follows that there are at most $\prod_{i=1}^{n} |x_i|$ possibilities for h.

2) *Let G be an FC-group and let F be a finite subset of G. The set of conjugates of elements of F in G is a finite normal subset. By proposition (2.4.2) it generates a finite normal subgroup. Conversely, if G has the property in question and $x \in G$, then $x \in F \lhd G$ for some finite F. All conjugates of x belong to F, so there are only finitely many of them.*

■

CHAPTER 3

THE CLASS OF MINIMAX GROUPS IS COUNTABLY RECOGNIZABLE

Countably recognizable classes of groups were introduced by Baer [1]. Baer produced many interesting examples of countably recognizable group classes, and later many other discovered countably recognizable group classes see for instance [3, 9, 10, 13, 4, 5] . On the other hand, certain important group classes are not countably recognizable; for example the class of free groups (see [8]) and of the class of groups which are embeddable into direct products of finite groups [4].

3.1 Free groups

Definition 3.1.1 *Let F be a group, X a nonempty set, and $\sigma : X \to F$ a function. Then F, or more exactly (F,σ), is said to be free on X if to each function σ from X to a group G there corresponds a unique homomorphism $\beta : F \to G$ such that $\alpha = \beta o \sigma$.*

A group which is free on some set is called a free group.
The function $\sigma : X \to F$ is necessarily injective. For suppose that $\sigma(x_1) = \sigma(x_2)$ and $x_1 \neq x_2$: let G be a group with at least two distinct elements g_1 and g_2 and choose a function $\alpha : X \to G$ such that $\alpha(x_1) = g_1$ and $\alpha(x_2) = g_2$. Then $\beta(\sigma(x_1)) = \beta(\sigma(x_2))$, whence $\alpha(x_1) = \alpha(x_2)$ and $g_1 = g_2$, a contradiction. Clearly F is also free on $Im\sigma$, the inclusion map $Im\ \sigma \to F$ taking the place of σ. Hence a free group is always free on a subset: in this case the commutativity of the diagram says that the restriction of β to X is α, so that β is the unique extension of α to F.
Another consequence of the definition is that Im σ generates F. Since this will follow from our construction of free groups, we need not prove it now [12].

3.1.1 Constructing free groups

Proposition 3.1.1 *If X is a nonempty set, there exists a group F and a function $\sigma : X \to F$ such that (F, σ) is free on X and $F = < Im\ \sigma >$.*

Proof. *Choose a set disjoint from X with the same cardinality: for notational reasons we shall denote this by $X^{-1} = \{x^{-1} | x \in X\}$ where of course X^{-1} is merely a symbol. By a word in X is meant a finite sequence of symbols from $X \cup X^{-1}$, written for convenience in the form :*

$$w = x_1^{\varepsilon_1} \cdots x_r^{\varepsilon_r}$$

$x_i \in X$, $\varepsilon_i = 1$ or $\varepsilon_i = -1$, $r \geq 0$: in case r = 0 the sequence is empty and w is the empty word, which will be written 1. Of course two words are to be considered equal if and only if they have the same elements in corresponding positions. ■

Remark 3.1.1 *The product of two words $w = x_1^{\varepsilon_1} \cdots x_r^{\varepsilon_r}$ and $v = y_1^{n_1} \cdots y_s^{n_s}$ is formed by juxtaposition: thus*

$$wv = x_1^{\varepsilon_1} \cdots x_r^{\varepsilon_r} y_1^{n_1} \cdots y_s^{n_s}$$

with the convention that $w1 = w = 1w$. The inverse of w is the word $w^{-1} = x_1^{-\varepsilon_r} \cdots x_r^{-\varepsilon_1}$ and $1^{-1} = 1$.

3.1.2 Reduced words

Definition 3.1.2 *Let us examine the construction just described with a view to obtaining a convenient description of the elements of the free group F.*
A word w in X is called reduced if it contains no pair of consecutive symbols of the form xx^{-1} or $x^{-1}x$, ($x \in X$). By convention the empty word is reduced. If w is an arbitrary word, we can delete from w all consecutive pairs xx^{-1} or $x^{-1}x$ to obtain an equivalent word. By repeating this procedure a finite number of times we shall eventually reach a reduced word which is equivalent to w. Thus each equivalence class of words contains a reduced word. The important point to establish is that there is just one reduced word in a class.

3.1.3 Rank of groups

Definition 3.1.3 *The rank of a group G, denoted rank(G) can refer to the smallest cardinality of a generating set for G, that is:*

$$rank(G) = min\{|X| : X \subseteq G, \langle X \rangle = G\}$$

If G is a finitely generated group, then the rank of G is a non negative integer.
The rank of a group is also often defined in such a way as to ensure subgroups have rank less than or equal to the whole group.

3.1.4 Example rank of groups

- *For a non trivial group G, we have rank(G) = 1 if and only if G is a cyclic group.*
  The trivial group 1 has rank(1) = 0, since the minimal generating set of 1 is the empty set.
- *If X is a set and G = F(X) is the free group with free basis X then:*

$$rank(G) = |X|.$$

- *If a group H is a homomorphic image of a group G then:*

$$rank(H) \leq rank(G)$$

- *For a free abelian group $\mathbb{Z}^n$ we have $rank(\mathbb{Z}^n = n)$.*

3.2 The class of minimax groups is countably recognizable

In his paper Francesco de Giovanni and Marco Trombetti proved that the class of minimax groups is countably recognizable.

Definition 3.2.1 *A group class X is said to be countably recognizable if, whenever all countable subgroups of a group G belong to X, then G itself is an X-group.*

Definition 3.2.2 *If $\mathfrak{X}$ is a class of groups, a group G is said to be minimal non-$\mathfrak{X}$ if it is not an $\mathfrak{X}$ -group or an $\mathfrak{X}$-critical ($G \notin \mathfrak{X}$) but all its proper subgroups belong to X.*

3.2.1 Minimal and maximal condition

Definition 3.2.3 *Recall that a group G satisfies the minimal condition on subgroups if there are no infinite descending chains of subgroups, and G satisfies the maximal condition on subgroups if it admits no infinite ascending chains of subgroups. It is almost obvious that both the class of groups satisfying the minimal condition and that of groups satisfying the maximal condition on subgroups are countably recognizable [6].*

Definition 3.2.4 *A group G is said to satisfy the weak minimal condition on subgroups if it has no infinite descending chains of subgroups [6] :*

$$X_1 > X_2 > \cdots > X_n > \cdots$$

such that the index $|X_n : X_{n+1}|$ is infinite for all n. The weak maximal condition on subgroups is defined replacing descending chains by ascending chains.

3.2.2 Minimax groups

Definition 3.2.5 *A group G is called minimax if it has a series of finite length :*

$$\{1\} = G_0 < G_1 < \cdots < G_n = G$$

each of whose factors satisfies either the minimal or the maximal condition on subgroups. The structure of soluble minimax groups has been described by Robinson (see [11] Part 2, Chapter 10).

Remark 3.2.1

1) *The class of soluble groups of finite rank is countably recognizable, and all soluble groups of finite rank are countable, it follows that any group whose countable subgroups are soluble and minimax is countable, and so also minimax.*

2) *The class of soluble minimax groups is countably recognizable. However, the situation is much more complicated in the insoluble case.*

3) *It was independently proved by Baer [2] and Zaicev [14] that for soluble groups the weak minimal condition, the weak maximal condition and the property of being minimax are equivalent. It turns out that also the class of groups satisfying the weak minimal condition and that of group satisfying the weak maximal condition on subgroups are countably recognizable.*
   Most of our notation can be found in [11].

Corollary 3.2.1 *It is clear that minimal non-$\mathfrak{X}$ groups are countable for every countably recognizable group class $\mathfrak{X}$. Thus it follows from the above theorem that any minimal non-minimax group is countable [11].*

Theorem 3.2.2 *The class of minimax groups is countably recognizable.*

Lemma 3.2.1 *Let G be a group, and let X be a subgroup of G.*

(a) If Y is a countable subgroup of G and $Y \leq X^{G,k}$, for some positive integer k, then there exists a countable subgroup U of G such that $Y \leq X^{U,k}$.

(b) If $X \neq X^{G,k}$ for some positive integer k, then there exists a countable subgroup U of G such that $X \neq X^{U,k}$.

Recall that if X is a class of groups, the residual of a group G with respect to X is the intersection of all normal subgroups N of G such that $\dfrac{G}{N}$ belongs to X [6].

Lemma 3.2.2 *Let G be a σ-minimax group for some $\sigma = (\sigma_1, \ldots, \sigma_n)$, where $n \geq 2$ and $\sigma_1 = \vee$. Then G contains a normal subgroup N satisfying the minimal condition and such that the factor group $\dfrac{G}{N}$ is $(\sigma_2, \ldots, \sigma_n)$-minimax [6].*

Proof. *Let :*

$$\{1\} = G_0 \leq G_1 \leq \cdots \leq G_n = G.$$

be a σ-series of G. As the statement is obvious if $n = 2$, we may suppose $n \geq 3$. It can be assumed by induction on n that G_{n-1} has a normal subgroup K with the minimal condition and such that $\dfrac{G_{n-1}}{K}$ is σ'-minimax, where $\sigma' = (\sigma_2, \ldots, \sigma_{n-1})$. As K contains the residual R of G_{n-1} with respect to the class of σ'-minimax groups, it follows that also the group $\dfrac{G_{n-1}}{R}$ is σ'-minimax. Clearly, R is a normal subgroup of G and $\dfrac{G}{R}$ is a $(\sigma_2, \ldots, \sigma_n)$-minimax group, and so the proof is complete.
Next lemma is the crucial point in the proof of our theorem. ■

Lemma 3.2.3 *Let G be a group whose countable subgroups are σ-minimax for a fixed minimax type $\sigma = (\sigma_1, \ldots, \sigma_n)$. Then G is minimax [6].*

Proof. *Assume for a contradiction that the statement is false, and choose a counterexample for which the minimax type σ has shortest length n. Then $n > 1$, because the class of groups with the minimal condition and that of groups satisfying the maximal condition are countably recognizable.*
Put $\sigma' = (\sigma_2, \cdots, \sigma_n)$, and suppose first $\sigma_1 = \vee$. Let C be the set of all countable subgroups of G, and for each element X of **C** *denote by X_0 the residual of X with respect to class of σ'-minimax groups. Write*

$$G_0 = \bigcup_{X \in C} X_0$$

.

If X and Y are arbitrary elements of C, we have

$$\langle X_0, Y_0 \rangle \leq \langle X, Y \rangle_0$$

and hence G_0 is a subgroup of G, which is obviously normal. Let H be any countable subgroup of G_0, and for each element h of H choose a countable subgroup $X(h)$ of G such that h belongs to $X(h)_0$. Then

$$K = \langle X(h) \mid h \in H \rangle$$

is a countable subgroup of G and

$$H \leq \langle X(h)_0 \mid h \in H \rangle \leq K_0.$$

Moreover, since K is σ-minimax, it follows from Lemma (3.2.2) that K_0 satisfies the minimal condition on subgroups, and hence also H has the minimal condition. Therefore G_0 satisfies the minimal condition on subgroups. Let V/G_0 be any countable subgroup of G_0, and let W be a countable subgroup of G such that $V = G_0W$. ThenV/G_0 is a *homomorphic image of W/W_0, and so it is σ'-minimax group by Lemma (3.2.2) It follows now from the minimal assumption on n that the factor group G/G_0 is minimax, so that G itself is minimax, and this contradiction shows that $\sigma_1 = \wedge$.*

Let K be any countable subgroup of G, and let $\varepsilon(K)$ be the set of all σ-subgroups of K. Clearly, $\varepsilon(K)$ is countable, because all its elements are finitely generated. For each element E of$\varepsilon(K)$, we will define a suitable countable subgroup $U_1(E)$ of G containing K.

If E is not subnormal in G of defect at most $n-1$, it follows from part (b) of Lemma (3.2.1) that there exists a countable subgroup V of G containing K such that $E^{V,n-1} \neq E$, and in this case we put $U_1(E) = V$. Suppose now that E is subnormal in G of defect at most $n-1$, so that $E^{G,n-1} \neq E$. As the group G is not minimax, there is a non-negative integer $i < n-1$ such that $E^{G,i}/E^{G,i+1}$ is not minimax, and so the minimal assumption on n yields that $E^{G,i}$ contains a countable subgroup X such that $XE^{G,i+1}/E^{G,i+1}$ is not σ'-minimax. In this situation, part (a) of Lemma (3.2.1) can be applied to obtain a countable subgroup W containing K such that X lies in $E^{W,i}$. Note that the group $E^{W,i}/E^{W,i+1}$ is not σ'-minimax, because it admits a section isomorphic to $XE^{G,i+1}/E^{G,i+1}$. In this second case, we put $U_1(E) = W$.

As the subgroup

$$U_1 = \langle U_1(E) | E \in \varepsilon(K) \rangle$$

is clearly countable, the above argument can be iterated to construt an ascending sequence $(U_n)_{n \in N}$ of countable subgroups of G. Then

$$U_\infty = \bigcup_{n \in N} U_n$$

is a countable subgroup of G, so that it is σ-minimax and we may consider an element E_∞ in the set $\varepsilon(U_\infty)$. In particular, E_∞ is a finitely generated subgroup of U_∞, and hence it is contained in U_m for some positive integer m. Moreover, E_∞ is subnormal *in U_∞ of defect $\leq n-1$, and so it follows from the definition of U_{m+1} that E_∞ must be even subnormal in G of defect at most $n-1$. Therefore the group*

$$E_\infty^{U_{m+1}(E_\infty),i}/E_\infty^{U_{m+1}(E_\infty),i+1}$$

is not σ'-minimax for some i, which is impossible because E_∞ belongs to the set $\varepsilon(U_{m+1}(E_\infty))$. This last contradiction completes the proof of the lemma. ■

Proof. *(proof of theorem 3.2.2) Denote by $\vee$ and $\wedge$ the minimal and the maximal condition on subgroups, respectively, and for a positive integer n let $\sigma = (\sigma_2, \dots, \sigma_n)$ be any n-tuple whose entries belong to the set $\{\vee, \wedge\}$. We shall say that a group G is minimax of type σ (or σ-minimax) if it has a σ-series, i.e. a finite series :*

$$\{1\} = G_0 \leq G_1 \leq \cdots \leq G_n = G$$

of length n such that the factor group $\dfrac{G_i}{G_{i-1}}$ satisfies the condition σ_i for each positive integer $i \leq n$. Clearly, σ-minimax groups are minimax and every minimax group is σ-minimax for some σ, but for a minimax group the minimax type is not uniquely determined. Note also that any abelian minimax group is $(\wedge, \vee)$-minimax.We point out finally that the class of σ-minimax groups is closed with respect to subgroups and homomorphic images, and that if H and K are normal subgroups of a group G such that both $\dfrac{G}{H}$ and $\dfrac{G}{k}$ are σ-minimax, then also the factor group $\dfrac{G}{H \cap K}$ is σ-minimax.

Let G be a minimax group of type $\sigma = (\sigma_1, \dots, \sigma_n)$. A subnormal subgroup X of G is called a σ-subgroup if it satisfies σ_1 and there exists a series :

$$X = X_1 \leq \cdots \leq X_n = G$$

such that $\dfrac{X_i}{X_{i-1}}$ satisfies σ_i for each $i = 2, . . ., n$. Of course, a normal subgroup N of a group G is a σ-subgroup if and only if it satisfies σ_1 and the factor group $\dfrac{G}{N}$ is σ'-minimax, where $\sigma' = (\sigma_2, \cdots, \sigma_n)$.

Let G be a group, and let X be a subgroup of G. Recall that the series of normal closures $\{X^{G,n}\}_{n \in \mathbb{N}_0}$ of X in G is defined by putting $X^{G,0} = G$ and

$$X^{G,n+1} = X^{X^{G,n}}$$

for each non-negative integer n. In particular, $X \leq X^{G,n}$ *for all n, and* $X^{G,1} = X^G$*, the normal closure of X in G. Note that X is subnormal in G of defect at most k if and only if* $X^{G,k} = X$*. The following result has been proved in [5].*

Let G be a group whose countable subgroups are minimax, and assume for a contradiction that G is not minimax. Then it follows from Lemma (3.2.3) that for each minimax type σ there exists a countable subgroup G_σ *of G which is not σ-minimax. As the set Σ of all minimax types is obviously countable, the subgroup*

$$G_\infty = \langle G_\sigma | \sigma \in \Sigma \rangle$$

is countable and it cannot be minimax. This contradiction proves the theorem. ■

The last result of the paper shows that also the weak minimal and the weak maximal conditions can be detected from the behaviour of countable subgroups.

Proposition 3.2.1 *The class of groups satisfying the weak minimal condition and that of groups satisfying the weak maximal condition are countably recognizable.*

Proof. *Let G be a group whose countable subgroups satisfy the weak minimal contradiction, and assume for a contradiction that G admits an infinite descending chain of subgroups*

$$X_1 > X_2 > \cdots > X_n > \cdots$$

such that the index $|X_n : X_{n+1}|$ *is infinite for all positive integers n. Then for each n we can choose a countably infinite subset* U_n *of* X_n *such that* $uX_{n+1} \neq vX_{n+1}$ *whenever u and v are elements of* U_n *and* $u \neq v$*. Then :*

$$U = \langle U_n | n \in \mathbb{N} \rangle$$

is a countable subgroup of G and U_n *lies in* $U \cap X_n$ *for all n. It follows that :*

$$U \cap X_1 > U \cap X_2 > \cdots > U \cap X_n > \cdots$$

is an infinite descending chain of subgroups of U and the index $|U \cap X_n : U \cap X_{n+1}|$ is infinite for each n. This contradiction shows that the class of groups satisfying the weak minimal condition is countably recognizable. A similar argument proves that also the class of groups satisfying the weak maximal condition is countably recognizable. ■

CONCLUSION

In this work we are interested in the class of minimax groups and we have seen that this class is countably recognizable and also we have seen that the class of groups satisfying the weak maximal and minimal conditions are countably recognizable this property is very important as it helps us to deduce properties of classes of groups by showing theme in countable subgroups of the group with out going to the whole group.

BIBLIOGRAPHY

[1] *R. Baer,* Abzählbar erkennbare gruppentheoretische Eigenschaften. Math. Z.*79, 344-363 (1962)*

[2] *R. Baer,* Polyminimaxgruppen. Math. Ann. *175, 1-43 (1968)*

[3] *M.R. Dixon, M.J. Evans, H.Smith,* Some countably recognizable classes of groups. J. Group Theory 10, *641-653 (2007)*

[4] *F. de Giovanni, M. Trombetti,* Countably recognizable classes of groups with restricted conjugacy classes. Int. J. Group Theory (to appear)

[5] *F. de Giovanni, M. Trombetti,* Countable recognizability and nilpotency properties of groups. Rend.Circ. Mat. Palermo., *(2016), doi:10.1007/s12215-016-0261-y*

[6] *F. de Giovanni, M. Trombetti,* The class of minimax groups is countably recognizable. *Monatshefte fur Mathematik, 185 (2016), 81-86. doi : 10.1007/s00605-016-0978-x*

[7] *G. Higman,* Almost free groups. Proc. Lond. Math. Soc., *1, 284-290 (1951)*

[8] *V.N. Obraztsov,* An embedding theorem for groups and its corollaries. Math. USSR Sb. 66, 541-553 (1990)

[9] *R.E. Phillips,* f -Systems in infinite groups. Arch. Math. (Basel) 20, 345-355 (1969)

[10] *R.E. Phillips*, Countably recognizable classes of groups. Rocky Mt. J. Math. 1, 489-497 (1971)

[11] *D.J.S. Robinson*, Finiteness conditions and generalized soluble groups. Springer, Berlin (1972)

[12] *D.J.S Robinson,* A Course in the Theory of Groups*, 2 nd ed Springer-Verlag New York, (1995)*

[13] *H. Smith,* More countably recognizable classes of groups. J. Pure Appl. Algebra 213, 1320-1324(2009)

[14] *D.I. Zaicev,* Groups which satisfy aweak minimality condition. Dokl.Akad. Nauk SSSR 178, 780-782(1968)